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# Existence of positive solution to a coupled system of singular fractional difference equations via fractional sum boundary value conditions

Chanon Promsakon[1], Saowaluck Chasreechai[1] and Thanin Sitthiwirattham[2*]

*Correspondence:
thanin_sit@dusit.ac.th
[2]Mathematics Department, Faculty of Science and Technology, Suan Dusit University, Bangkok, Thailand
Full list of author information is available at the end of the article

**Abstract**

In this article, we study a coupled system of singular fractional difference equations with fractional sum boundary conditions. A sufficient condition of the existence of positive solutions is established by employing the upper and lower solutions of the system and using Schauder's fixed point theorem. Finally, we provide an example to illustrate our results.

**MSC:** 39A05; 39A12

**Keywords:** Positive solution; Singularity; Fractional difference equations; Boundary value problem

## 1 Introduction

Fractional difference calculus is a powerful tool for studying problems in many fields such as biology, mechanics, control systems, ecology, electrical networks and other areas (see [1–10] and the references therein). Particularly, this calculus can be used to study stability of discrete fractional systems [11] and impulsive fractional difference equations [12]. Recently, fractional differences have been utilized in several research works such as a study of fuzzy fractional discrete-time diffusion equation [13], and a study of an image encryption technique based on the fractional chaotic maps [14]. The study of approximating solutions of fractional equations is an important topic in this area. Recently, many researchers presented the method to find approximating solutions of some fractional integro-differential equations (see [15–20]).

Basic definitions and properties of fractional difference calculus were presented by Goodrich and Peterson [21]. In addition, there are other research works dealing with fractional difference boundary value problems which have helped to build up some of the basic theory of this area (see [22–47] and references cited therein).

The boundary value problems for systems of fractional difference equations have been studied by some researchers; see [48–53] and references cited therein. For example, Pan

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et al. [48] proposed the system of fractional difference equations

$$\begin{cases} -\Delta^{\nu} y_1(t) = f(y_1(t+\nu_1), y_2(t+\mu-1)), \\ -\Delta^{\mu} y_2(t) = g(y_1(t+\nu_1), y_2(t+\mu-1)), \end{cases} \tag{1.1}$$

for $t \in \mathbb{N}_{0,b+1} := \{0, 1, 2, \ldots, b+1\}$ where $b \in \mathbb{N}_0$, with the difference boundary conditions

$$\begin{cases} y_1(\nu-2) = \Delta y_1(\nu+b) = 0, \\ y_2(\mu-2) = \Delta y_2(\mu+b) = 0, \end{cases} \tag{1.2}$$

where $1 < \mu, \nu \leq 2$, $0 < \beta \leq 1$, and $f, g : \mathbb{R} \to \mathcal{R}$ are continuous functions.

Goodrich [51] studied the coupled system of fractional difference equations

$$\begin{cases} -\Delta^{-\nu} x(t) = \lambda_1 f(t+\nu-1, y(t+\mu-1)), \\ -\Delta^{-\mu} y(t) = \lambda_2 g(t+\mu-1, y(t+\nu-1)), \end{cases} \tag{1.3}$$

for $t \in \mathbb{N}_{0,b+1}$, with the nonlinearities satisfying no growth conditions

$$\begin{cases} x(\nu-2) = H_1(\sum_{i=1}^{n} a_i y(\xi_i)), \quad x(\nu+b+1) = 0, \\ y(\mu-2) = H_2(\sum_{j=1}^{m} b_i x(\zeta_i)), \quad x(\mu+b+1) = 0, \end{cases} \tag{1.4}$$

where $1 < \nu \leq 2$, $1 < \mu \leq 2$, $\lambda_1, \lambda_2 > 0$, and $H_1, H_2$ are continuous functions.

In this paper, we aim to study the coupled system of singular fractional difference equations

$$\begin{cases} -\Delta^{\alpha_1} u_1(t) = F_1(t+\alpha_1-1, t+\alpha_2-1, \Delta^{\beta_1} u_1(t+\alpha_1-\beta_1), u_2(t+\alpha_2-1)), \\ -\Delta^{\alpha_2} u_2(t) = F_2(t+\alpha_1-1, t+\alpha_2-1, u_1(t+\alpha_1-1), \Delta^{\beta_2} u_2(t+\alpha_2-\beta_2)), \end{cases} \tag{1.5}$$

with fractional sum boundary conditions

$$\begin{cases} u_1(\alpha_1-2) = 0, \qquad u_1(T+\alpha_1) = \lambda_2 \Delta^{-\theta_2} g_2(T+\alpha_2+\theta_2) u_2(T+\alpha_2+\theta_2), \\ u_2(\alpha_2-2) = 0, \qquad u_2(T+\alpha_2) = \lambda_1 \Delta^{-\theta_1} g_1(T+\alpha_1+\theta_1) u_1(T+\alpha_1+\theta_1), \end{cases} \tag{1.6}$$

where $t \in \mathbb{N}_{0,T} := \{0, 1, \ldots, T\}$, $0 < \lambda_i < \frac{\Gamma(\alpha_i)}{(T+\alpha_i)^{\underline{\alpha_i-1}} \sum_{s=0}^{T} (T+\theta_i+1-\sigma(s))^{\underline{\theta_i-1}} g_i(s+\alpha_1-1)(s+\alpha_i-1)^{\underline{\alpha_i-1}}}$, $\alpha_i \in (1,2]$, $\beta_i, \theta_i \in (0,1]$, $g_i \in C(\mathbb{N}_{\alpha_i-2,T+\alpha_i}, \mathbb{R}^+)$ are given functions, $F_i : \mathbb{N}_{\alpha_1-1,T+\alpha_1-1} \times \mathbb{N}_{\alpha_2-1,T+\alpha_1} \times (0,+\infty) \times (0,+\infty) \to [0,+\infty)$ are are continuous and may be singular at $u_i = 0$ and $t = \alpha_i - 2, T + \alpha_i$ where $i = 1, 2$.

This paper is organized as follows. In the next section, we present some definitions and basic lemmas. In Sect. 3, we prove the existence of solutions of the boundary value problem (1.5)–(1.6) by employing the upper and lower solutions of the system and Schauder's fixed point theorem. An example and application of our results are presented in the last section.

## 2 Preliminaries

As the following, we provide some notations, definitions, and lemmas which are used in the main results.

**Definition 2.1** The generalized falling function is defined by $t^{\underline{\alpha}} := \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)}$, for any $t$ and $\alpha$ for which the right-hand side is defined. If $t+1-\alpha$ is a pole of the Gamma function and $t+1$ is not a pole, then $t^{\underline{\alpha}} = 0$.

**Theorem 2.1** ([22]) *Assume the following factorial functions are well defined. If $t \le r$, then $t^{\underline{\alpha}} \le r^{\underline{\alpha}}$ for any $\alpha > 0$.*

**Definition 2.2** For $\alpha > 0$ and $f$ defined on $\mathbb{N}_a$, the $\alpha$-order fractional sum of $f$ is defined by

$$\Delta^{-\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} \bigl(t-\sigma(s)\bigr)^{\underline{\alpha-1}} f(s),$$

where $t \in \mathbb{N}_{a+\alpha}$ and $\sigma(s) = s+1$.

**Definition 2.3** For $\alpha > 0$ and $f$ defined on $\mathbb{N}_a$, the $\alpha$-order Riemann–Liouville fractional difference of $f$ is defined by

$$\Delta^{\alpha} f(t) := \Delta^{N} \Delta^{-(N-\alpha)} f(t) = \frac{1}{\Gamma(-\alpha)} \sum_{s=a}^{t+\alpha} \bigl(t-\sigma(s)\bigr)^{\underline{-\alpha-1}} f(s),$$

where $t \in \mathbb{N}_{a+N-\alpha}$ and $N \in \mathbb{N}$ is chosen so that $0 \le N-1 < \alpha \le N$.

**Theorem 2.2** ([22]) *Let $0 \le N-1 < \alpha \le N$. Then*

$$\Delta^{-\alpha} \Delta^{\alpha} y(t) = y(t) + C_1 t^{\underline{\alpha-1}} + C_2 t^{\underline{\alpha-2}} + \cdots + C_N t^{\underline{\alpha-N}},$$

*for some $C_i \in \mathbb{R}$, with $1 \le i \le N$.*

We next propose a lemma dealing with a solution of a linear variant of the boundary value problem (1.5).

**Lemma 2.1** *For $i,j \in \{1,2\}$ and $i \ne j$, let $0 < \Lambda < 1, \mathcal{P}(h_1,h_2), \mathcal{Q}(h_1,h_2) \ge \Lambda$, $0 < \lambda_i < \frac{\Gamma(\alpha_i)}{(T+\alpha_i)^{\underline{\alpha_i-1}} \sum_{s=0}^{T} (T+\theta_i+1-\sigma(s))^{\underline{\theta_i-1}} g_i(s+\alpha_1-1)(s+\alpha_i-1)^{\underline{\alpha_i-1}}}$, $\alpha_i \in (1,2], \theta_i \in (0,1]$ be given constants, $h_i \in C(\mathbb{N}_{\alpha_i-1,T+\alpha_i-1},\mathbb{R})$, $g_i \in C(\mathbb{N}_{\alpha_i-2,T+\alpha_i},\mathbb{R}^+)$ given functions, and let $\phi_i(u_1,u_2)$ be given functionals. The problem*

$$-\Delta^{\alpha_i} u_i(t) = h_i(t+\alpha_i-1), \quad t \in \mathbb{N}_{0,T}, \tag{2.1}$$

$$u_i(\alpha_i-2) = 0, \tag{2.2}$$

$$u_i(T+\alpha_i) = \lambda_j \Delta^{-\theta_j} g_j(T+\alpha_j+\theta_j) u_j(T+\alpha_j+\theta_j) \tag{2.3}$$

*has the unique solution*

$$u_1(t) = t^{\underline{\alpha_1-1}} \Biggl\{ \frac{\lambda_1}{\Lambda \Gamma(\theta_1)} \sum_{s=0}^{T+1} \bigl(T+\theta_1+1-\sigma(s)\bigr)^{\underline{\theta_1-1}} g_1(s+\alpha_1-1)(s+\alpha_1-1)^{\underline{\alpha_1-1}} \mathcal{P}(h_1,h_2)$$

$$+\frac{\lambda_2}{\Lambda\Gamma(\theta_2)}\sum_{s=0}^{T+1}\bigl(T+\theta_2+1-\sigma(s)\bigr)^{\underline{\theta_2-1}}g_2(s+\alpha_2-1)(s+\alpha_2-1)^{\underline{\alpha_2-1}}\mathcal{Q}(h_1,h_2)\Bigg\}$$

$$-\frac{1}{\Gamma(\alpha_1)}\sum_{s=0}^{t-\alpha_1}\bigl(t-\sigma(s)\bigr)^{\underline{\alpha_1-1}}h_1(s+\alpha_1-1),\quad t\in\mathbb{N}_{\alpha_1-2,T+\alpha_1},\tag{2.4}$$

$$u_2(t)=t^{\underline{\alpha_2-1}}\Bigg\{\frac{(T+\alpha_2)^{\underline{\alpha_2-1}}}{\Lambda}\mathcal{P}(h_1,h_2)+\frac{(T+\alpha_1)^{\underline{\alpha_1-1}}}{\Lambda}\mathcal{Q}(h_1,h_2)\Bigg\}$$

$$-\frac{1}{\Gamma(\alpha_2)}\sum_{s=0}^{t-\alpha_2}\bigl(t-\sigma(s)\bigr)^{\underline{\alpha_2-1}}h_2(s+\alpha_2-1),\quad t\in\mathbb{N}_{\alpha_2-2,T+\alpha_2},\tag{2.5}$$

*where*

$$\Lambda=\frac{\lambda_2(T+\alpha_2)^{\underline{\alpha_2-1}}}{\Gamma(\alpha_2)}\sum_{s=0}^{T+1}\bigl(T+\theta_1+1-\sigma(s)\bigr)^{\underline{\theta_1-1}}g_1(s+\alpha_1-1)(s+\alpha_1-1)^{\underline{\alpha_1-1}}$$

$$-\frac{\lambda_1(T+\alpha_1)^{\underline{\alpha_1-1}}}{\Gamma(\alpha_1)}\sum_{s=0}^{T+1}\bigl(T+\theta_2+1-\sigma(s)\bigr)^{\underline{\theta_2-1}}g_2(s+\alpha_2-1)(s+\alpha_2-1)^{\underline{\alpha_2-1}},\tag{2.6}$$

$$\mathcal{P}(h_1,h_2)=\frac{1}{\Gamma(\alpha_1)}\sum_{s=0}^{T}\bigl(T+\alpha_1-\sigma(s)\bigr)^{\underline{\alpha_1-1}}h_1(s+\alpha_1-1)-\frac{\lambda_2}{\Gamma(\alpha_2)\Gamma(\theta_2)}$$

$$\times\sum_{s=0}^{T}\sum_{\xi=s}^{T}\bigl(T+\theta_2-\sigma(\xi)\bigr)^{\underline{\theta_2-1}}\bigl(\xi+\alpha_2-\sigma(s)\bigr)^{\underline{\alpha_2-1}}$$

$$\times g_2(s+\alpha_2-1)h_2(s+\alpha_2-1),\tag{2.7}$$

$$\mathcal{Q}(h_1,h_2)=-\frac{1}{\Gamma(\alpha_2)}\sum_{s=0}^{T}\bigl(T+\alpha_2-\sigma(s)\bigr)^{\underline{\alpha_2-1}}h_2(s+\alpha_2-1)+\frac{\lambda_1}{\Gamma(\alpha_1)\Gamma(\theta_1)}$$

$$\times\sum_{s=0}^{T}\sum_{\xi=s}^{T}\bigl(T+\theta_1-\sigma(\xi)\bigr)^{\underline{\theta_1-1}}\bigl(\xi+\alpha_1-\sigma(s)\bigr)^{\underline{\alpha_1-1}}$$

$$\times g_1(s+\alpha_1-1)h_1(s+\alpha_1-1).\tag{2.8}$$

*Proof* For $i,j\in\{1,2\}$ where $i\neq j$, using Lemma 2.2 and the fractional sum of order $\alpha\in(1,2]$ for (2.1), we obtain

$$u_i(t)=C_{1i}t^{\underline{\alpha_i-1}}+C_{2i}t^{\underline{\alpha_i-2}}-\frac{1}{\Gamma(\alpha_i)}\sum_{s=0}^{t-\alpha_i}\bigl(t-\sigma(s)\bigr)^{\underline{\alpha_i-1}}h_i(s+\alpha_i-1),\tag{2.9}$$

for $t\in\mathbb{N}_{\alpha_i-2,T+\alpha_i}$. Using the boundary condition (2.2), this implies that

$$C_{2i}=0.\tag{2.10}$$

Then, we have

$$u_i(t)=C_{1i}t^{\underline{\alpha_i-1}}-\frac{1}{\Gamma(\alpha_i)}\sum_{s=0}^{t-\alpha_i}\bigl(t-\sigma(s)\bigr)^{\underline{\alpha_i-1}}h_i(s+\alpha_i-1).\tag{2.11}$$

Taking the fractional sum of order $0 < \theta_i \leq 1$ for (2.11), we obtain

$$\begin{aligned}
&\Delta^{-\theta_i} u(t) \\
&\quad = \frac{C_{1i}}{\Gamma(\theta_i)} \sum_{s=\alpha_i-1}^{t-\theta_i} \bigl(t-\sigma(s)\bigr)^{\underline{\theta_i-1}} g_i(s) s^{\underline{\alpha_i-1}} \\
&\qquad - \frac{1}{\Gamma(\theta_i)\Gamma(\alpha_i)} \sum_{\xi=\alpha_i}^{t-\theta_i} \sum_{s=0}^{\xi-\alpha_i} \bigl(t-\sigma(\xi)\bigr)^{\underline{\theta_i-1}} \bigl(\xi-\sigma(s)\bigr)^{\underline{\alpha_i-1}} \\
&\qquad \times g_i(s+\alpha_i-1) h_i(s+\alpha_i-1),
\end{aligned} \tag{2.12}$$

for $t \in \mathbb{N}_{\alpha_i+\theta_i-2, T+\alpha_i+\theta_i}$. From the boundary condition (2.3), we find that

$$\begin{aligned}
&C_{11}(T+\alpha_1)^{\underline{\alpha_1-1}} - \frac{1}{\Gamma(\alpha_1)} \sum_{s=0}^{T} \bigl(T+\alpha_1-\sigma(s)\bigr)^{\underline{\alpha_1-1}} h_1(s+\alpha_1-1) \\
&\quad = \frac{\lambda_2 C_{12}}{\Gamma(\theta_2)} \sum_{s=\alpha_2-1}^{T+\alpha_2} \bigl(T+\alpha_2+\theta_2-\sigma(s)\bigr)^{\underline{\theta_2-1}} g_2(s) s^{\underline{\alpha_2-1}} - \frac{\lambda_2}{\Gamma(\alpha_2)\Gamma(\theta_2)} \\
&\qquad \times \sum_{\xi=\alpha_2}^{T+\alpha_2} \sum_{s=0}^{\xi-\alpha_2} \bigl(T+\alpha_2+\theta_2-\sigma(\xi)\bigr)^{\underline{\theta_2-1}} \bigl(\xi-\sigma(s)\bigr)^{\underline{\alpha_2-1}} \\
&\qquad \times g_2(s+\alpha_2-1) h_2(s+\alpha_2-1),
\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}
&C_{12}(T+\alpha_2)^{\underline{\alpha_2-1}} - \frac{1}{\Gamma(\alpha_2)} \sum_{s=0}^{T} \bigl(T+\alpha_2-\sigma(s)\bigr)^{\underline{\alpha_2-1}} h_2(s+\alpha_2-1) \\
&\quad = \frac{\lambda_1 C_{11}}{\Gamma(\theta_1)} \sum_{s=\alpha_1-1}^{T+\alpha_1} \bigl(T+\alpha_1+\theta_1-\sigma(s)\bigr)^{\underline{\theta_1-1}} g_1(s) s^{\underline{\alpha_1-1}} - \frac{\lambda_1}{\Gamma(\alpha_1)\Gamma(\theta_1)} \\
&\qquad \times \sum_{\xi=\alpha_1}^{T+\alpha_1} \sum_{s=0}^{\xi-\alpha_1} \bigl(T+\alpha_1+\theta_1-\sigma(\xi)\bigr)^{\underline{\theta_1-1}} \bigl(\xi-\sigma(s)\bigr)^{\underline{\alpha_1-1}} \\
&\qquad \times g_1(s+\alpha_1-1) h_1(s+\alpha_1-1).
\end{aligned} \tag{2.14}$$

After solving the system of equations (2.13) and (2.14), we have

$$\begin{aligned}
C_{11} &= \frac{\lambda_1}{\Lambda\Gamma(\theta_1)} \sum_{s=0}^{T+1} \bigl(T+\theta_1+1-\sigma(s)\bigr)^{\underline{\theta_1-1}} g_1(s+\alpha_1-1)(s+\alpha_1-1)^{\underline{\alpha_1-1}} \mathcal{P}(h_1,h_2) \\
&\quad + \frac{\lambda_2}{\Lambda\Gamma(\theta_2)} \sum_{s=0}^{T+1} \bigl(T+\theta_2+1-\sigma(s)\bigr)^{\underline{\theta_2-1}} \\
&\quad \times g_2(s+\alpha_2-1)(s+\alpha_2-1)^{\underline{\alpha_2-1}} \mathcal{Q}(h_1,h_2),
\end{aligned} \tag{2.15}$$

and

$$C_{12} = \frac{(T+\alpha_2)^{\underline{\alpha_2-1}}}{\Lambda}\mathcal{P}(h_1,h_2) + \frac{(T+\alpha_1)^{\underline{\alpha_1-1}}}{\Lambda}\mathcal{Q}(h_1,h_2), \tag{2.16}$$

where $\Lambda, \mathcal{P}(h_1,h_2)$ and $\mathcal{Q}(h_1,h_2)$ are defined in (2.6)–(2.8), respectively.

Finally, substituting $C_{11}$ and $C_{12}$ into (2.11), we obtain (2.4) and (2.5). The proof of this lemma is complete. □

**Corollary 2.1** *Problem* (2.1)–(2.3) *has the unique solution which is of the from*

$$\begin{aligned} u_i(t_i) = {} & \sum_{s=0}^{T} G_{i1}(t_i,s)g_1(s+\alpha_1-1)h_1(s+\alpha_1-1) \\ & - \sum_{s=0}^{T} G_{i2}(t_i,s)g_2(s+\alpha_2-1)h_2(s+\alpha_2-1) \end{aligned} \tag{2.17}$$

*for* $t_i \in \mathbb{N}_{\alpha_i-2,T+\alpha_i}$, *where*

$$G_{11}(t_1,s) = \frac{t_1^{\underline{\alpha_1-1}}}{\Lambda}\sum_{\xi=0}^{T}\mathcal{H}_{11}(\xi-\alpha_1-1,s) + \mathcal{K}_1(t_1,s), \tag{2.18}$$

$$G_{12}(t_1,s) = \frac{t_1^{\underline{\alpha_1-1}}}{\Lambda}\sum_{\xi=0}^{T}\mathcal{H}_{12}(\xi-\alpha_2-1,s), \tag{2.19}$$

$$G_{21}(t_2,s) = \frac{t_2^{\underline{\alpha_2-1}}}{\Lambda}\sum_{\xi=0}^{T}\mathcal{H}_{21}(\xi-\alpha_1-1,s), \tag{2.20}$$

$$G_{22}(t_2,s) = \frac{t_2^{\underline{\alpha_2-1}}}{\Lambda}\sum_{\xi=0}^{T}\mathcal{H}_{22}(\xi-\alpha_2-1,s) + \mathcal{K}_2(t_2,s), \tag{2.21}$$

*with*

$$\mathcal{K}_1(t_1,s) = \frac{1}{\Gamma(\alpha_1)}\begin{cases} (\frac{\lambda_1\mathcal{A}_1(T+\alpha_1-\sigma(s))^{\underline{\alpha_1-1}}}{\Lambda\Gamma(\theta_1)})t_1^{\underline{\alpha_1-1}} - (t_1-\sigma(s))^{\underline{\alpha_1-1}}, & s\in\mathbb{N}_{0,t_1-\alpha_1}, \\ (\frac{\lambda_1\mathcal{A}_1(T+\alpha_1-\sigma(s))^{\underline{\alpha_1-1}}}{\Lambda\Gamma(\theta_1)})t_1^{\underline{\alpha_1-1}}, & s\in\mathbb{N}_{t_1-\alpha_1+1,T}, \end{cases} \tag{2.22}$$

$$\mathcal{K}_2(t_2,s) = \frac{1}{\Gamma(\alpha_2)}\begin{cases} [(T+\alpha_1)^{\underline{\alpha_1-1}}(T+\alpha_2-\sigma(s))^{\underline{\alpha_2-1}}]t_2^{\underline{\alpha_2-1}} \\ \quad + (t_2-\sigma(s))^{\underline{\alpha_2-1}}, & s\in\mathbb{N}_{0,t_2-\alpha_2}, \\ [(T+\alpha_1)^{\underline{\alpha_1-1}}(T+\alpha_2-\sigma(s))^{\underline{\alpha_2-1}}]t_2^{\underline{\alpha_2-1}}, & s\in\mathbb{N}_{t_2-\alpha_2+1,T}, \end{cases} \tag{2.23}$$

$$\begin{aligned} & \mathcal{H}_{11}(\xi+\alpha_1-1,s) \\ & = \frac{\lambda_1}{\Gamma(\alpha_1)\Gamma(\theta_1)}\begin{cases} (T+\theta_1-\sigma(\xi))^{\underline{\theta_1-1}}[(\Lambda+1)(T+\alpha_1-\sigma(s))^{\underline{\alpha_1-1}} \\ \quad \times(\xi-\alpha_1+1)^{\underline{\alpha_1-1}} + \frac{\lambda_2\mathcal{A}_2}{\Gamma(\theta_2)}(\xi+\alpha_1-\sigma(s))^{\underline{\alpha_1-1}}], \\ \quad s\in\mathbb{N}_{0,\xi}, \\ (T+\theta_1-\sigma(\xi))^{\underline{\theta_1-1}}(\Lambda+1)(T+\alpha_1-\sigma(s))^{\underline{\alpha_1-1}} \\ \quad \times(\xi-\alpha_1+1)^{\underline{\alpha_1-1}}, \\ \quad s\in\mathbb{N}_{\xi+1,T}, \end{cases} \end{aligned} \tag{2.24}$$

$$\mathcal{H}_{12}(\xi+\alpha_2-1,s)$$
$$=\frac{\lambda_2}{\Gamma(\alpha_2)\Gamma(\theta_2)}\begin{cases}(T+\theta_2-\sigma(\xi))^{\underline{\theta_2-1}}[(T+\alpha_2-\sigma(s))^{\underline{\alpha_2-1}}(\xi-\alpha_2+1)^{\underline{\alpha_2-1}} \\ \quad +\frac{\lambda_1\mathcal{A}_1}{\Lambda\Gamma(\theta_1)}(\xi+\alpha_2-\sigma(s))^{\underline{\alpha_2-1}}], \\ \quad s\in\mathbb{N}_{0,\xi} \\ (T+\theta_2-\sigma(\xi))^{\underline{\theta_2-1}}[(T+\alpha_2-\sigma(s))^{\underline{\alpha_2-1}}(\xi-\alpha_2+1)^{\underline{\alpha_2-1}}, \\ \quad s\in\mathbb{N}_{\xi+1,T}\end{cases} \tag{2.25}$$

$$\mathcal{H}_{21}(\xi+\alpha_1-1,s)$$
$$=\frac{1}{\Gamma(\alpha_1)}\begin{cases}(T+\alpha_2)^{\underline{\alpha_2-1}}(T+\alpha_1-\sigma(s))^{\underline{\alpha_1-1}} \\ \quad +\frac{\lambda_1(T+\alpha_1)^{\underline{\alpha_1-1}}}{\Gamma(\theta_1)}(T+\theta_1-\sigma(\xi))^{\underline{\theta_1-1}}(\xi+\alpha_1-\sigma(s))^{\underline{\alpha_1-1}}, \\ \quad s\in\mathbb{N}_{0,\xi}, \\ (T+\alpha_2)^{\underline{\alpha_2-1}}(T+\alpha_1-\sigma(s))^{\underline{\alpha_1-1}}, \\ \quad s\in\mathbb{N}_{\xi+1,T},\end{cases} \tag{2.26}$$

$$\mathcal{H}_{22}(\xi+\alpha_2-1,s)$$
$$=\frac{1}{\Gamma(\alpha_2)}\begin{cases}(1-\Lambda)(T+\alpha_1)^{\underline{\alpha_1-1}}(T+\alpha_2-\sigma(s))^{\underline{\alpha_2-1}} \\ \quad +\frac{\lambda_2(T+\alpha_2)^{\underline{\alpha_2-1}}}{\Gamma(\theta_2)}(T+\theta_2-\sigma(\xi))^{\underline{\theta_2-1}}(\xi+\alpha_2-\sigma(s))^{\underline{\alpha_2-1}}, \\ \quad s\in\mathbb{N}_{0,\xi}, \\ (1-\Lambda)(T+\alpha_1)^{\underline{\alpha_1-1}}(T+\alpha_2-\sigma(s))^{\underline{\alpha_2-1}}, \\ \quad s\in\mathbb{N}_{\xi+1,T},\end{cases} \tag{2.27}$$

*and*

$$\mathcal{A}_i=\bigl(T+\theta_i+1-\sigma(s)\bigr)^{\underline{\theta_i-1}}g_i(s+\alpha_i-1)(s+\alpha_i-1)^{\underline{\alpha_i-1}}. \tag{2.28}$$

**Lemma 2.2** *For $i,j\in\{1,2\}$, $i\neq j$ and letting $0<\Lambda<1$, $\mathcal{P}(h_1,h_2),\mathcal{Q}(h_1,h_2)\geq\Lambda$, $0<\lambda_i<\frac{\Gamma(\alpha_i)}{(T+\alpha_i)^{\underline{\alpha_i-1}}\sum_{s=0}^{T}(T+\theta_i+1-\sigma(s))^{\underline{\theta_i-1}}g_i(s+\alpha_1-1)(s+\alpha_i-1)^{\underline{\alpha_i-1}}}$, the Green's functions are defined by* (2.18)–(2.20) *and satisfy*:

($X1$) *$G_{i1}(t_i),G_{i2}(t_i)>0$ for all $t_i\in\mathbb{N}_{\alpha_i-2,T+\alpha_i}$;*

($X2$) *There exist two constants $\omega_{i1},\omega_{i2}$ such that for all $(t_i,s)\in\mathbb{N}_{\alpha_i-2,T+\alpha_i}\times\mathbb{N}_{0,T}$,*

$$\frac{t_i^{\underline{\alpha_i-1}}}{\Lambda}\sum_{\xi=0}^{T}\mathcal{H}_{ii}(\xi+\alpha_i-1,s)\leq G_{ii}(t_i,s)\leq\omega_{ii}t_i^{\underline{\alpha_i-1}}, \tag{2.29}$$

$$\frac{t_1^{\underline{\alpha_1-1}}\lambda_2\Gamma(\alpha_2)}{\Lambda}\leq G_{12}(t_1,s)\leq\omega_{12}t_1^{\underline{\alpha_1-1}}, \tag{2.30}$$

$$\frac{t_2^{\underline{\alpha_2-1}}(T+\alpha_2)^{\underline{\alpha_2-1}}}{\Lambda}\leq G_{21}(t_2,s)\leq\omega_{21}t_2^{\underline{\alpha_2-1}}; \tag{2.31}$$

($X3$) *$u_i(t_i)\geq 0$ for all $t_i\in\mathbb{N}_{\alpha_i-2,T+\alpha_i}$.*

*Proof* ($X1$) is obvious. Here we only prove ($X2$)–($X3$).

Since $0 < \Lambda < 1$ and from the fact that $\mathcal{H}_{i1}(\xi + \alpha_1 - 1, s), \mathcal{H}_{i2}(\xi + \alpha_2 - 1, s) \geq 0$ for all $(t_i, s) \in \mathbb{N}_{\alpha_i-2,T+\alpha_1} \times \mathbb{N}_{0,T}$, we have

$$\frac{t_i^{\underline{\alpha_i-1}}}{\Lambda} \sum_{\xi=0}^{T} \mathcal{H}_{ii}(\xi + \alpha_i - 1, s) \leq G_{ii}(t_i, s), \tag{2.32}$$

$$\frac{t_1^{\underline{\alpha_1-1}} \lambda_2 \Gamma(\alpha_2)}{\Lambda} \leq G_{12}(t_1, s), \tag{2.33}$$

$$\frac{t_2^{\underline{\alpha_2-1}} (T + \alpha_2)^{\underline{\alpha_2-1}}}{\Lambda} \leq G_{21}(t_2, s). \tag{2.34}$$

By the definition of $\mathcal{K}_i(t_i, s)$, we find that

$$\mathcal{K}_1(t_1, s) \leq \frac{\lambda_1 t_1^{\underline{\alpha_1-1}}}{\Gamma(\alpha_1)\Gamma(\theta_1)} \left[(T + \alpha_1 - 1)^{\underline{\alpha_1-1}}\right]^2 (T + \theta_1 - 1)^{\underline{\theta_1-1}} \quad \text{and} \tag{2.35}$$

$$\mathcal{K}_2(t_2, s) \leq \frac{t_2^{\underline{\alpha_2-1}}}{\Gamma(\alpha_1)} (T + \alpha_2 - 1)^{\underline{\alpha_2-1}} (T + \alpha_1)^{\underline{\alpha_1-1}}. \tag{2.36}$$

Letting

$$\begin{aligned} \omega_{11} = {} & \frac{\lambda_1}{\Gamma(\alpha_1)\Gamma(\theta_1)} \Bigg[ \left[(T + \alpha_1 - 1)^{\underline{\alpha_1-1}}\right]^2 (T + \theta_1 - 1)^{\underline{\theta_1-1}} \\ & + \max_{0 \leq \xi \leq T} \frac{\mathcal{H}_{11}(\xi + \alpha_1 - 1, s)}{\Lambda} \Bigg], \end{aligned} \tag{2.37}$$

$$\omega_{12} = \frac{\lambda_2}{\Gamma(\alpha_2)\Gamma(\theta_2)} \max_{0 \leq \xi \leq T} \frac{\mathcal{H}_{12}(\xi + \alpha_2 - 1, s)}{\Lambda}, \tag{2.38}$$

$$\begin{aligned} \omega_{22} = {} & \frac{1}{\Gamma(\alpha_2)} \Bigg[ 1 + (T + \alpha_1)^{\underline{\alpha_1-1}} (T + \alpha_2 - 1)^{\underline{\alpha_2-1}} \\ & + \max_{0 \leq \xi \leq T} \frac{\mathcal{H}_{22}(\xi + \alpha_2 - 1, s)}{\Lambda} \Bigg], \end{aligned} \tag{2.39}$$

$$\omega_{21} = \frac{1}{\Gamma(\alpha_1)} \max_{0 \leq \xi \leq T} \frac{\mathcal{H}_{21}(\xi + \alpha_1 - 1, s)}{\Lambda}, \tag{2.40}$$

we obtain, for all $(t_i, s) \in \mathbb{N}_{\alpha_i-2,T+\alpha_1} \times \mathbb{N}_{0,T}$,

$$G_{i1}(t_i, s) \leq \omega_{i1} t_i^{\underline{\alpha_i-1}} \quad \text{and} \quad G_{i2}(t_i, s) \leq \omega_{i2} t_i^{\underline{\alpha_i-1}}. \tag{2.41}$$

Consequently, by (2.32)–(2.34) and (2.41), this implies that $(X2)$ holds.

Next, we claim that $(X3)$ holds. By (2.4)–(2.5) with the conditions $0 < \Lambda < 1$, $\mathcal{P}(h_1, h_2)$, $\mathcal{Q}(h_1, h_2) \geq \Lambda$ and $0 < \lambda_i < \frac{\Gamma(\alpha_i)}{(T+\alpha_i)^{\underline{\alpha_i-1}} \sum_{s=0}^{T} (T+\theta_i+1-\sigma(s))^{\underline{\theta_i-1}} g_i(s+\alpha_1-1)(s+\alpha_i-1)^{\underline{\alpha_i-1}}}$, we have

$$\begin{aligned} u_1(t_1) = {} & \sum_{s=0}^{T} G_{11}(t_1, s) g_1(s + \alpha_1 - 1) h_1(s + \alpha_1 - 1) \\ & - \sum_{s=0}^{T} G_{12}(t_2, s) g_2(s + \alpha_2 - 1) h_2(s + \alpha_2 - 1) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(\alpha_1)}\left[\frac{2\lambda_1 t_1^{\underline{\alpha_1-1}}}{\Gamma(\theta_1)}\sum_{s=0}^{T}\sum_{\xi=0}^{T}\bigl(T+\alpha_1-\sigma(s)\bigr)^{\underline{\alpha_1-1}}\bigl(T+\theta_1-\sigma(\xi)\bigr)^{\underline{\theta_1-1}}\right.\\
&\quad\left.\times(\xi+\alpha_1-1)^{\underline{\alpha_1-1}}-\bigl(t_1-\sigma(s)\bigr)^{\underline{\alpha_1-1}}\right]+\frac{t_1^{\underline{\alpha_1-1}}}{\Lambda}\mathcal{P}(h_1,h_2)\\
&\geq\bigl[t_1^{\underline{\alpha_1-1}}-\bigl(t_1-\sigma(s)\bigr)^{\underline{\alpha_1-1}}\bigr]+\frac{t_1^{\underline{\alpha_1-1}}}{\Lambda}\mathcal{P}(h_1,h_2)\geq 0,
\end{aligned}
\tag{2.42}
$$

and

$$
\begin{aligned}
u_2(t_2) &= \sum_{s=0}^{T}G_{21}(t_1,s)g_1(s+\alpha_1-1)h_1(s+\alpha_1-1)\\
&\quad-\sum_{s=0}^{T}G_{22}(t_2,s)g_2(s+\alpha_2-1)h_2(s+\alpha_2-1)\\
&= \frac{t_2^{\underline{\alpha_2-1}}}{\Lambda}\mathcal{Q}(h_1,h_2)-\frac{(t_2-\sigma(s))^{\underline{\alpha_2-1}}}{\Gamma(\alpha_2)}\\
&\geq t_2^{\underline{\alpha_2-1}}-\bigl(t_2-\sigma(s)\bigr)^{\underline{\alpha_2-1}}\geq 0,
\end{aligned}
\tag{2.43}
$$

so $(X3)$ holds. The proof is complete. □

The following theorems [54] are provided to study the existence of positive solution to the boundary value problem (1.5) in the next section.

**Theorem 2.3** (Arzelá–Ascoli theorem) *A set of functions in $C[a,b]$ with the sup norm is relatively compact if and only it is uniformly bounded and equicontinuous on $[a,b]$.*

**Theorem 2.4** *If a set is closed and relatively compact, then it is compact.*

**Theorem 2.5** (Schauder's fixed point theorem) *Let $T$ be a continuous and compact mapping of a Banach space $E$ into itself such that the set*

$$\{x\in E: x=\eta Tx, \textit{for some } 0\leq\eta\leq 1\}$$

*is bounded. Then $T$ has a fixed point.*

## 3 Main results

In this section, we aim to establish the existence result for problem (1.5)–(1.6). For each $i,j\in\{1,2\}$ where $i\neq j$, we let $E_i: C(\mathbb{N}_{\alpha_i-2,T+\alpha_i},\mathbb{R})$ be the Banach space for all functions on $\mathbb{N}_{\alpha_i-2,T+\alpha_i}$.

Therefore, the product space $\mathcal{C}=E_1\times E_2$ is a Banach space. We consider the spaces

$$\mathcal{C}_i=\bigl\{(u_1,u_2)\in\mathcal{C}:\Delta^{\beta_i}u_i(t_i-\beta_i+1)\in\mathcal{C}\bigr\},$$

for $t_i\in\mathbb{N}_{\alpha_i-2,T+\alpha_i}$, and define the norm by

$$\bigl\|(u_1,u_2)\bigr\|_{\mathcal{C}_i}=\bigl\|\Delta^{\beta_i}u_i\bigr\|_{E_i}+\|u_j\|_{E_j},$$

where

$$\|\Delta^{\beta_i} u_i\|_{E_i} = \max_{t_i \in \mathbb{N}_{\alpha_i-2,T+\alpha_i}} \left|\Delta^{\beta_i} u_i(t_i - \beta_i + 1)\right| \quad \text{and}$$

$$\|u_j\|_{E_j} = \max_{t_j \in \mathbb{N}_{\alpha_i-2,T+\alpha_i}} \left|u_j(t_j)\right|.$$

Let $\mathcal{U} = \mathcal{C}_1 \cap \mathcal{C}_2$. Obviously, the space $(\mathcal{U}, \|(u_1,u_2)\|_{\mathcal{U}})$ is also a Banach space with the norm

$$\|(u_1,u_2)\|_{\mathcal{U}} = \max\{\|(u_1,u_2)\|_{\mathcal{C}_1}, \|(u_1,u_2)\|_{\mathcal{C}_2}\}.$$

A positive solution of problem (1.5)–(1.6) is a pair of functions $(x_1,x_2) \in \mathcal{U}$ satisfying (1.5)–(1.6) with $x_i(t_i) \geq 0$ for all $t_i \in \mathbb{N}_{\alpha_i-2,T+\alpha_i}$ and $(x_1,x_2) \neq (0,0)$.

From Lemmas 2.1 and 2.2, we obtain the following lemma.

**Lemma 3.1** *For $t_i \in \mathbb{N}_{\alpha_i-2,T+\alpha_i}$, $i,j \in \{1,2\}$ and $i \neq j$. If $(u_1,u_2) \in \mathcal{U}$ satisfy*

(i) $u_i(\alpha_i - 2) = 0, u_i(T+\alpha_i) = \lambda_j \Delta^{-\theta_j} g_j(T+\alpha_j+\theta_j) u_j(T+\alpha_j+\theta_j)$;

(ii) $\Delta^{\alpha_i} u_i(t_i) \leq 0$ *for all* $t_i \in \mathbb{N}_{\alpha_i-2,T+\alpha_i}$.

*Then* $u_i(t_i) \geq 0$.

In what follows, we give the definitions of the lower and upper solution of problem (1.5)–(1.6).

**Definition 3.1** A pair of functions $(\chi_1^*(t_1), \chi_2^*(t_2)) \in \mathcal{U}$ is called a lower solution of problem (1.5)–(1.6) if it satisfies

$$-\Delta^{\alpha_i} \chi_i^*(t_i) \leq F_i\big(t_1,t_2,\chi_1^*(t_1),\chi_2^*(t_2)\big),$$

$$\chi_i^*(\alpha_i - 2) \geq 0,$$

$$\chi_i^*(T+\alpha_i) \geq \lambda_j \Delta^{-\theta_j} g_j(T+\alpha_j+\theta_j) \chi_j^*(T+\alpha_j+\theta_j).$$

**Definition 3.2** A pair of functions $(\bar{\chi}_1^*(t_1), \bar{\chi}_2^*(t_2)) \in \mathcal{U}$ is called an upper solution of problem (1.5)–(1.6), if it satisfies

$$-\Delta^{\alpha_i} \bar{\chi}_i^*(t_i) \geq F_i\big(t_1,t_2,\bar{\chi}_1^*(t_1),\bar{\chi}_2^*(t_2)\big),$$

$$\bar{\chi}_i^*(\alpha_i - 2) \leq 0,$$

$$\bar{\chi}_i^*(T+\alpha_i) \leq \lambda_j \Delta^{-\theta_j} g_j(T+\alpha_j+\theta_j) \bar{\chi}_j^*(T+\alpha_j+\theta_j).$$

The following assumptions are set throughout this paper: for $i,j \in \{1,2\}$ and $i \neq j$,

$(H1)$ $0 < \Lambda < 1$ and $\sum_{\xi=0}^{T} \mathcal{H}_{i,j}(\xi+\alpha_i-1,s) \geq 0$ for all $s \in \mathbb{N}_{0,T}$.

$(H2)$ $F_i \in C(\mathbb{N}_{\alpha_1-1,T+\alpha_1-1} \times \mathbb{N}_{\alpha_2-1,T+\alpha_2-1} \times (0,+\infty) \times (0,+\infty), [0,+\infty))$ are decreasing in third and fourth variables, and

$$F_i\big(t_i,t_j,t_i^{\alpha_i-1},t_j^{\alpha_j-1}\big) \in l^1.$$

($H3$) For all $\ell \in (0,1)$, there exist constants $0 < \rho_i < 1$ such that, for any $(t_1, t_2, v_1, v_2) \in \mathbb{N}_{\alpha_1-1,T+\alpha_1-1} \times \mathbb{N}_{\alpha_2-1,T+\alpha_2-1} \times (0,+\infty) \times (0,+\infty)$,

$$F_i(t_1, t_2, \ell v_1, \ell v_2) \le \ell^{-\rho_i} F_i(t_1, t_2, v_1, v_2).$$

($H4$) $\varsigma_i \le g_i(t_i) \le \mathcal{G}_i$ for all $t_i \in \mathbb{N}_{\alpha_i-2,T+\alpha_i}$.

*Remark* Conditions ($H2$)–($H3$) imply that $F_i$ have a power singularity at $u_i = 0$ for $i = 1, 2$.

**Theorem 3.1** *Suppose that* ($H1$)–($H4$) *hold. Then problem* (1.5)–(1.6) *has at least one positive solution* $(u_1^*, u_2^*)$, *which satisfies*

$$\left(\varsigma \mathcal{L}^{-1} t_1^{\underline{\alpha_1-1}}, \varsigma \mathcal{L}^{-1} t_2^{\underline{\alpha_2-1}}\right) \le \left(u_1^*, u_2^*\right) \le \left(\mathcal{G}\mathcal{L} t_1^{\underline{\alpha_1-1}}, \mathcal{G}\mathcal{L} t_2^{\underline{\alpha_2-1}}\right), \tag{3.1}$$

*where* $\mathcal{G} := \max\{\mathcal{G}_1, \mathcal{G}_2\}$, $\varsigma := \max\{\varsigma_1, \varsigma_2\}$, $\mathcal{L}^\rho := \max\{\mathcal{L}_1^{\rho_1}, \mathcal{L}_2^{\rho_2}\}$,

$$\begin{aligned}\mathcal{L} := \max\Bigg\{ & 1, \left|\omega_{i1} \sum_{s=0}^{T} \tilde{\mathcal{F}}_1(s) - \omega_{i2} \sum_{s=0}^{T} \tilde{\mathcal{F}}_2(s)\right|^{\frac{1}{1-\rho}}, \\ & \left|\frac{\Lambda\Gamma(T+2)}{\sum_{s=0}^{T}\sum_{\xi=0}^{T} \mathcal{H}_{11}\tilde{\mathcal{F}}_1(s) - \sum_{s=0}^{T}\sum_{\xi=0}^{T} \mathcal{H}_{12}\tilde{\mathcal{F}}_2(s)}\right|^{\frac{1}{1-\rho}}, \\ & \left|\frac{\Lambda\Gamma(T+2)}{\sum_{s=0}^{T}\sum_{\xi=0}^{T} \mathcal{H}_{21}\tilde{\mathcal{F}}_1(s) - \sum_{s=0}^{T}\sum_{\xi=0}^{T} \mathcal{H}_{22}\tilde{\mathcal{F}}_2(s)}\right|^{\frac{1}{1-\rho}}\Bigg\},\end{aligned} \tag{3.2}$$

*with*

$$\begin{aligned}\tilde{\mathcal{F}}_1(s) &:= F_1\left(s+\alpha_1-1, \alpha_2-1, (s+\alpha_1-1)^{\underline{\alpha_1-1}}, (\alpha_2-1)^{\underline{\alpha_2-1}}\right), \\ \tilde{\mathcal{F}}_2(s) &:= F_2\left(\alpha_1-1, s+\alpha_2-1, (\alpha_1-1)^{\underline{\alpha_1-1}}, (s+\alpha_2-1)^{\underline{\alpha_2-1}}\right),\end{aligned}$$

*and* $\omega_{i,j}, \mathcal{H}_{i,j}, i, j = 1, 2$ *are defined in the previous section. In particular, if* $\mathcal{L} = 1$, *then* $(t_1^{\underline{\alpha_1-1}}, t_2^{\underline{\alpha_2-1}})$ *is a positive solution of problem* (1.5)–(1.6).

*Proof* Define the cone

$$\begin{aligned}\mathcal{P} = \Big\{(u_1, u_2) \in \mathcal{U} : & \mathcal{L}_i^{-1} t_i^{\underline{\alpha_i-1}} \le \Delta^{\beta_i} u_i(t_i - \beta_i + 1) \le \mathcal{L}_i t_i^{\underline{\alpha_i-1}} \\ & \text{and } \mathcal{L}_i^{-1} t_j^{\underline{\alpha_j-1}} \le u_j(t_j) \le \mathcal{L}_i t_j^{\underline{\alpha_j-1}} \text{ for } i, j \in \{1, 2\} \text{ and } i \ne j\Big\},\end{aligned}$$

and the operator $\mathcal{T} : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ by

$$\left(\mathcal{T}(u_1, u_2)\right)(t_1, t_2) = \left(\left(\mathcal{T}_1(u_1, u_2)\right)(t_1, t_2), \left(\mathcal{T}_2(u_1, u_2)\right)(t_1, t_2)\right), \tag{3.3}$$

for all $(u_1, u_2) \in \mathcal{P}$ and

$$\begin{aligned}&\left(\mathcal{T}_i(u_1, u_2)\right)(t_1, t_2) \\ &\quad = \sum_{s=0}^{T} G_{i1}(t_i, s) g_1(s+\alpha_1-1) F_1\left(s+\alpha_1-1, t_2, \Delta^{\beta_1} u_1(s+\alpha_1-\beta_1), u_2(t_2)\right)\end{aligned}$$

$$-\sum_{s=0}^{T} G_{i2}(t_i,s)g_2(s+\alpha_2-1)F_2\big(t_1,s+\alpha_2-1,u_1(t_1),\Delta^{\beta_2}u_2(s+\alpha_2-\beta_2)\big), \tag{3.4}$$

where $G_{i1}(t_i,s)$ and $G_{i2}(t_i,s)$ are defined in (2.17)–(2.20).

Firstly, we claim that $\mathcal{T}$ is well defined and $\mathcal{T}(\mathcal{P})\subset\mathcal{P}$. By Lemma 2.1 and $(H1)$–$(H4)$, we obtain

$$\begin{aligned}
&\big|\big(\mathcal{T}_i(u_1,u_2)\big)(t_1,t_2)\big| \\
&\quad= \Bigg|\sum_{s=0}^{T} G_{i1}(t_i,s)g_1(s+\alpha_1-1)F_1\big(s+\alpha_1-1,t_2,\Delta^{\beta_1}u_1(s+\alpha_1-\beta_1),u_2(t_2)\big) \\
&\qquad -\sum_{s=0}^{T} G_{i2}(t_i,s)g_2(s+\alpha_2-1)F_2\big(t_1,s+\alpha_2-1,u_1(t_1),u_2(s+\alpha_2-\beta_2)\big)\Bigg| \\
&\quad\le \Bigg|\omega_{i1}t_i^{\underline{\alpha_i-1}}\mathcal{G}_1\sum_{s=0}^{T} F_1\big(s+\alpha_1-1,t_2,\mathcal{L}_1^{-1}(s+\alpha_1-1)^{\underline{\alpha_1-1}},\mathcal{L}_1^{-1}t_2^{\underline{\alpha_2-1}}\big) \\
&\qquad -\omega_{i2}t_i^{\underline{\alpha_i-1}}\mathcal{G}_2\sum_{s=0}^{T} F_2\big(t_1,s+\alpha_2-1,\mathcal{L}_2^{-1}t_1^{\underline{\alpha_1-1}},\mathcal{L}_2^{-1}(s+\alpha_2-1)^{\underline{\alpha_2-1}}\big)\Bigg| \\
&\quad\le t_i^{\underline{\alpha_i-1}}\mathcal{G}\Bigg|\omega_{i1}\sum_{s=0}^{T}\mathcal{L}_1^{\rho_1}F_1\big(s+\alpha_1-1,t_2,(s+\alpha_1-1)^{\underline{\alpha_1-1}},t_2^{\underline{\alpha_2-1}}\big) \\
&\qquad -\omega_{i2}\sum_{s=0}^{T}\mathcal{L}_2^{\rho_2}F_2\big(t_1,s+\alpha_2-1,t_1^{\underline{\alpha_1-1}},(s+\alpha_2-1)^{\underline{\alpha_2-1}}\big)\Bigg| \\
&\quad\le t_i^{\underline{\alpha_i-1}}\mathcal{G}\mathcal{L}^{\rho}\Bigg|\omega_{i1}\sum_{s=0}^{T} F_1\big(s+\alpha_1-1,\alpha_2-1,(s+\alpha_1-1)^{\underline{\alpha_1-1}},(\alpha_2-1)^{\underline{\alpha_2-1}}\big) \\
&\qquad -\omega_{i2}\sum_{s=0}^{T} F_2\big(\alpha_1-1,s+\alpha_2-1,(\alpha_1-1)^{\underline{\alpha_1-1}},(s+\alpha_2-1)^{\underline{\alpha_2-1}}\big)\Bigg| \\
&\quad= t_i^{\underline{\alpha_i-1}}\mathcal{G}\mathcal{L}.
\end{aligned} \tag{3.5}$$

On the other hand, we have

$$\begin{aligned}
&\big|\big(\mathcal{T}_1(u_1,u_2)\big)(t_1,t_2)\big| \\
&\quad\ge \frac{t_1^{\underline{\alpha_1-1}}}{\Lambda}\Bigg|\sum_{s=0}^{T}\sum_{\xi=0}^{T}\mathcal{H}_{11}(\xi+\alpha_1-1,s)g_1(s+\alpha_1-1) \\
&\qquad \times F_1\big(s+\alpha_1-1,t_2,\mathcal{L}_1(s+\alpha_1-1)^{\underline{\alpha_1-1}},\mathcal{L}_1t_2^{\underline{\alpha_2-1}}\big) \\
&\qquad -\sum_{s=0}^{T}\sum_{\xi=0}^{T}\mathcal{H}_{12}(\xi+\alpha_2-1,s)g_1(s+\alpha_1-1) \\
&\qquad \times F_2\big(t_1,s+\alpha_2-1,\mathcal{L}_2t_1^{\underline{\alpha_1-1}},\mathcal{L}_2(s+\alpha_2-1)^{\underline{\alpha_2-1}}\big)\Bigg|
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{t_1^{\alpha_1-1}}{\Lambda}\Bigg|\mathcal{L}_1^{-\rho_1}\varsigma_1\sum_{s=0}^{T}\sum_{\xi=0}^{T}\mathcal{H}_{11}(\xi+\alpha_1-1,s)g_1(s+\alpha_1-1)\\
&\quad\times F_1\big(s+\alpha_1-1,T+\alpha_2-1,(s+\alpha_1-1)^{\underline{\alpha_1-1}},(T+\alpha_2-1)^{\underline{\alpha_2-1}}\big)\\
&\quad-\mathcal{L}_2^{-\rho_2}\varsigma_2\sum_{s=0}^{T}\sum_{\xi=0}^{T}\mathcal{H}_{12}(\xi+\alpha_2-1,s)g_2(s+\alpha_2-1)\\
&\quad\times F_2\big(T+\alpha_1-1,s+\alpha_2-1,(T+\alpha_1-1)^{\underline{\alpha_1-1}},(s+\alpha_2-1)^{\underline{\alpha_2-1}}\big)\Bigg|\\
&\geq t_1^{\alpha_1-1}\cdot\frac{\varsigma\mathcal{L}^{-1}}{\Gamma(T+2)}
\end{aligned}
\tag{3.6}
$$

and

$$
\begin{aligned}
&\big|\big(\mathcal{T}_2(u_1,u_2)\big)(t_1,t_2)\big|\\
&\quad\geq \frac{t_2^{\alpha_2-1}}{\Lambda}\Bigg|\mathcal{L}_1^{-\rho_1}\varsigma_1\sum_{s=0}^{T}\sum_{\xi=0}^{T}\mathcal{H}_{21}(\xi+\alpha_1-1,s)\\
&\quad\times F_1\big(s+\alpha_1-1,T+\alpha_2-1,(s+\alpha_1-1)^{\underline{\alpha_1-1}},(T+\alpha_2-1)^{\underline{\alpha_2-1}}\big)\\
&\quad-\mathcal{L}_2^{-\rho_2}\varsigma_2\sum_{s=0}^{T}\sum_{\xi=0}^{T}\mathcal{H}_{22}(\xi+\alpha_2-1,s)\\
&\quad\times F_2\big(T+\alpha_1-1,s+\alpha_2-1,(T+\alpha_1-1)^{\underline{\alpha_1-1}},(s+\alpha_2-1)^{\underline{\alpha_2-1}}\big)\Bigg|\\
&\quad\geq t_2^{\alpha_2-1}\cdot\frac{\varsigma\mathcal{L}^{-1}}{\Gamma(T+2)}.
\end{aligned}
\tag{3.7}
$$

Next, taking the fractional difference of order $0<\beta_i\leq 1$ for (3.4), we have

$$
\begin{aligned}
&\Delta^{\beta_i}\big(\mathcal{T}_i(u_1,u_2)\big)(t_i-\beta_i+1,t_j)\\
&\quad=\sum_{s=0}^{T}\big[\Delta^{\beta_i}G_{i1}(t_i,s)\big]g_1(s+\alpha_1-1)F_1\big(s+\alpha_1-1,t_2,\Delta^{\beta_1}u_1(s+\alpha_1-\beta_1),u_2(t_2)\big)\\
&\quad\quad-\sum_{s=0}^{T}\big[\Delta^{\beta_i}G_{i2}(t_i,s)\big]g_2(s+\alpha_2-1)\\
&\quad\quad\times F_2\big(t_1,s+\alpha_2-1,u_1(t_1),\Delta^{\beta_2}u_2(s+\alpha_2-\beta_2)\big).
\end{aligned}
\tag{3.8}
$$

By the same arguments as before and since $\Delta^{\beta_i}\mathcal{K}_i(t_i,s)\leq\mathcal{K}_i(t_i,s)$, we obtain

$$
\begin{aligned}
&\big|\Delta^{\beta_1}\big(\mathcal{T}_1(u_1,u_2)\big)(t_1-\beta_1+1,t_2)\big|\\
&\quad=\Bigg|\sum_{s=0}^{T}\Bigg[\Delta^{\beta_1}\mathcal{K}_1(t_1,s)+\frac{1}{\Lambda\Gamma(-\beta_1)}\Bigg(\sum_{p=\alpha_1-1}^{t_1+1}\big(t_1-\beta_1+1-\sigma(p)\big)^{\underline{-\beta_1-1}}p^{\underline{\alpha_1-1}}\Bigg)\\
&\quad\quad\times\sum_{\xi=0}^{T}\mathcal{H}_{11}(\xi+\alpha_1-1,s)\Bigg]g_1(s+\alpha_1-1)F_1\big(s+\alpha_1-1,t_2,\Delta^{\beta_1}u_1(s+\alpha_1-\beta_1),u_2(t_2)\big)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{\Lambda\Gamma(-\beta_1)}\left(\sum_{p=\alpha_1-1}^{t_1+1}\bigl(t_1-\beta_1+1-\sigma(p)\bigr)^{\underline{-\beta_1-1}}p^{\underline{\alpha_1-1}}\right)\sum_{s=0}^{T}\sum_{\xi=0}^{T}\mathcal{H}_{12}(\xi+\alpha_2-1,s)] \\
&\times g_2(s+\alpha_2-1)F_2\bigl(t_1,s+\alpha_2-1,u_1(t_1),\Delta^{\beta_2}u_2(s+\alpha_2-\beta_2)\bigr)\Bigg| \\
\le{}& \Bigg|\sum_{s=0}^{T}\left[\mathcal{K}_1(t_1,s)+\frac{(t_1+1)^{\underline{\alpha_1-1}}(t_1+\alpha_1-\beta_1+2)^{\underline{-\beta_1}}}{\Lambda\Gamma(1-\beta_1)}\sum_{\xi=0}^{T}\mathcal{H}_{11}(\xi+\alpha_1-1,s)\right] \\
&\times g_1(s+\alpha_1-1)F_1\bigl(s+\alpha_1-1,t_2,\Delta^{\beta_1}u_1(s+\alpha_1-\beta_1),u_2(t_2)\bigr) \\
&- \frac{(t_1+1)^{\underline{\alpha_1-1}}(t_1+\alpha_1-\beta_1+2)^{\underline{-\beta_1}}}{\Lambda\Gamma(1-\beta_1)}\sum_{s=0}^{T}\sum_{\xi=0}^{T}\mathcal{H}_{12}(\xi+\alpha_2-1,s)] \\
&\times g_2(s+\alpha_2-1)F_2\bigl(t_1,s+\alpha_2-1,u_1(t_1),\Delta^{\beta_2}u_2(s+\alpha_2-\beta_2)\bigr)\Bigg| \\
\le{}& \Bigg|\mathcal{G}_1\sum_{s=0}^{T}G_{11}(t_1,s)F_1\bigl(s+\alpha_1-1,t_2,\Delta^{\beta_1}u_1(s+\alpha_1-\beta_1),u_2(t_2)\bigr) \\
&- \mathcal{G}_2\sum_{s=0}^{T}G_{12}(t_1,s)F_2\bigl(t_1,s+\alpha_2-1,u_1(t_1),\Delta^{\beta_2}u_2(s+\alpha_2-\beta_2)\bigr)\Bigg| \\
\le{}& \mathcal{L}^*\mathcal{G}t_1^{\underline{\alpha_1-1}},
\end{aligned}
\tag{3.9}
$$

and

$$
\begin{aligned}
&\bigl|\Delta^{\beta_2}\bigl(\mathcal{T}_2(u_1,u_2)\bigr)(t_1,t_2-\beta_2+1)\bigr| \\
&\quad\le \Bigg|\frac{(t_2+1)^{\underline{\alpha_2-1}}(t_2+\alpha_2-\beta_2+2)^{\underline{-\beta_2}}}{\Gamma(1-\beta_1)}\sum_{s=0}^{T}\sum_{\xi=0}^{T}\mathcal{H}_{21}(\xi+\alpha_1-1,s) \\
&\qquad\times g_1(s+\alpha_1-1)F_1\bigl(s+\alpha_1-1,t_2,\Delta^{\beta_1}u_1(s+\alpha_1-\beta_1),u_2(t_2)\bigr) \\
&\qquad- \sum_{s=0}^{T}\left[\mathcal{K}_2(t_2,s)+\frac{(t_2+1)^{\underline{\alpha_2-1}}(t_2+\alpha_2-\beta_2+2)^{\underline{-\beta_2}}}{\Lambda\Gamma(1-\beta_1)}\sum_{\xi=0}^{T}\mathcal{H}_{22}(\xi+\alpha_2-1,s)\right] \\
&\qquad\times g_2(s+\alpha_2-1)F_2\bigl(t_1,s+\alpha_2-1,u_1(t_1),\Delta^{\beta_2}u_2(s+\alpha_2-\beta_2)\bigr)\Bigg| \\
&\quad\le \Bigg|\mathcal{G}_1\sum_{s=0}^{T}G_{21}(t,s)F_1\bigl(s+\alpha_1-1,t_2,\Delta^{\beta_1}u_1(s+\alpha_1-\beta_1),u_2(t_2)\bigr) \\
&\qquad- \mathcal{G}_2\sum_{s=0}^{T}G_{22}(t,s)F_2\bigl(t_1,s+\alpha_2-1,u_1(t_1),\Delta^{\beta_2}u_2(s+\alpha_2-\beta_2)\bigr)\Bigg| \\
&\quad\le \mathcal{L}\mathcal{G}t_2^{\underline{\alpha_2-1}}.
\end{aligned}
\tag{3.10}
$$

On the other hand, we have

$$
\bigl|\Delta^{\beta_1}\bigl(\mathcal{T}_1(u_1,u_2)\bigr)(t_1-\beta_1+1,t_2)\bigr|
$$

$$
\begin{aligned}
&\geq \frac{\Gamma(\alpha_1)(t_1+\alpha_1-\beta_1+2)^{\underline{-\beta_1}}}{\Lambda\Gamma(1-\beta_1)}\Bigg|\sum_{s=0}^{T}\sum_{\xi=0}^{T}\mathcal{H}_{11}(\xi+\alpha_1-1,s)\\
&\quad\times g_1(s+\alpha_1-1)F_1\bigl(s+\alpha_1-1,t_2,\Delta^{\beta_1}u_1(s+\alpha_1-\beta_1),u_2(t_2)\bigr)\\
&\quad-\sum_{s=0}^{T}\sum_{\xi=0}^{T}\mathcal{H}_{12}g_2(s+\alpha_2-1)F_2\bigl(t_1,s+\alpha_2-1,u_1(t_1),\Delta^{\beta_2}u_2(s+\alpha_2-\beta_2)\bigr)\Bigg|\\
&\geq \frac{t_1^{\alpha_1-1}}{\Lambda\Gamma(T+2)}\Bigg|\varsigma_1\sum_{s=0}^{T}\sum_{\xi=0}^{T}\mathcal{H}_{11}(\xi+\alpha_1-1,s)\\
&\quad\times F_1\bigl(s+\alpha_1-1,t_2,\Delta^{\beta_1}u_1(s+\alpha_1-\beta_1),u_2(t_2)\bigr)\\
&\quad-\varsigma_2\sum_{s=0}^{T}\sum_{\xi=0}^{T}\mathcal{H}_{12}F_2\bigl(t_1,s+\alpha_2-1,u_1(t_1),\Delta^{\beta_2}u_2(s+\alpha_2-\beta_2)\bigr)\Bigg|\\
&\geq t_1^{\alpha_1-1}\varsigma\mathcal{L}^{-1},
\end{aligned}
\tag{3.11}
$$

and

$$
\begin{aligned}
&\bigl|\Delta^{\beta_2}\bigl(\mathcal{T}_2(u_1,u_2)\bigr)(t_1,t_2-\beta_2+1)\bigr|\\
&\quad\geq \frac{t_2^{\alpha_2-1}}{\Lambda\Gamma(T+2)}\Bigg|\varsigma_1\sum_{s=0}^{T}\sum_{\xi=0}^{T}\mathcal{H}_{21}(\xi+\alpha_1-1,s)\\
&\quad\times F_1\bigl(s+\alpha_1-1,t_2,\Delta^{\beta_1}u_1(s+\alpha_1-\beta_1),u_2(t_2)\bigr)\\
&\quad-\varsigma_2\sum_{s=0}^{T}\sum_{\xi=0}^{T}\mathcal{H}_{22}(\xi+\alpha_2-1,s)F_2\bigl(t_1,s+\alpha_2-1,u_1(t_1),\Delta^{\beta_2}u_2(s+\alpha_2-\beta_2)\bigr)\Bigg|\\
&\quad\geq t_2^{\alpha_2-1}\varsigma\mathcal{L}^{-1}.
\end{aligned}
\tag{3.12}
$$

Thus it follows from (2.5)–(2.7) and (2.9)–(2.12) that $\mathcal{T}$ is well defined and $\mathcal{T}(\mathcal{P})\subset\mathcal{P}$. Furthermore, by Lemma 2.2, we obtain

$$
\begin{aligned}
&-\Delta^{\alpha_i}\mathcal{T}_i(u_1,u_2)(t_1,t_2)=F_i\bigl(t_1,t_2,\Delta^{\beta_i}\mathcal{T}_i(u_1,u_2)(t_1,t_2),\mathcal{T}_i(u_1,u_2)(t_1,t_2)\bigr),\\
&\mathcal{T}_i(u_1,u_2)(\alpha_i-2,t_j)=0,\\
&\mathcal{T}_i(u_1,u_2)(T+\alpha_i,t_j)=\lambda_j\Delta^{-\theta_j}g_j(T+\alpha_j+\theta_j)\mathcal{T}_j(u_1,u_2)(t_i,T+\alpha_j+\theta_j).
\end{aligned}
\tag{3.13}
$$

We let

$$
\chi_i(t_i)=\min\bigl\{t_i^{\underline{\alpha_i-1}},\mathcal{T}_i\bigl(t_i^{\underline{\alpha_i-1}},t_j^{\underline{\alpha_j-1}}\bigr)\bigr\},
\tag{3.14}
$$

$$
\bar{\chi}_i(t_i)=\max\bigl\{t_i^{\underline{\alpha_i-1}},\mathcal{T}_i\bigl(t_i^{\underline{\alpha_i-1}},t_j^{\underline{\alpha_j-1}}\bigr)\bigr\}.
\tag{3.15}
$$

Since $(t_1^{\underline{\alpha_1-1}},t_2^{\underline{\alpha_2-1}}),(\mathcal{T}_1(t_1^{\underline{\alpha_1-1}},t_2^{\underline{\alpha_2-1}}),\mathcal{T}_2(t_1^{\underline{\alpha_1-1}},t_2^{\underline{\alpha_2-1}}))\in\mathcal{P}$, we have

$$
\begin{aligned}
&(\chi_1,\chi_2),(\bar{\chi}_1,\bar{\chi}_2)\in\mathcal{P},\\
&\chi_1\leq t_1^{\underline{\alpha_1-1}}\leq\bar{\chi}_1\quad\text{and}\quad\chi_2\leq t_2^{\underline{\alpha_2-1}}\leq\bar{\chi}_2.
\end{aligned}
\tag{3.16}
$$

Let

$$\left(\chi_1^*,\chi_2^*\right)=\left(\mathcal{T}_1(\chi_1,\chi_2),\mathcal{T}_2(\chi_1,\chi_2)\right)\quad\text{and}\quad\left(\bar{\chi}_1^*,\bar{\chi}_2^*\right)=\left(\mathcal{T}_1(\bar{\chi}_1,\bar{\chi}_2),\mathcal{T}_2(\bar{\chi}_1,\bar{\chi}_2)\right).\tag{3.17}$$

Then, by (3.14)–(3.17) and $(H3)$, we obtain

$$\begin{aligned}\left(\bar{\chi}_1^*,\bar{\chi}_2^*\right)&\le\left(\mathcal{T}_1\left(t_1^{\alpha_1-1},t_2^{\alpha_2-1}\right),\mathcal{T}_2\left(t_1^{\alpha_1-1},t_2^{\alpha_2-1}\right)\right)\\&\le\left(\mathcal{T}_1(\chi_1,\chi_2),\mathcal{T}_2(\chi_1,\chi_2)\right)=\left(\chi_1^*,\chi_2^*\right)\le(\bar{\chi}_1,\bar{\chi}_2),\end{aligned}\tag{3.18}$$

$$\left(\chi_1^*,\chi_2^*\right)\ge\left(\mathcal{T}_1\left(t_1^{\alpha_1-1},t_2^{\alpha_2-1}\right),\mathcal{T}_2\left(t_1^{\alpha_1-1},t_2^{\alpha_2-1}\right)\right)\ge(\chi_1,\chi_2).\tag{3.19}$$

So, it follows from (3.13) and (3.16)–(3.19) that

$$\begin{aligned}&\Delta^{\alpha_i}\chi_i^*(t_i)+F_i\left(t_1,t_2,\chi_1^*(t_1),\chi_2^*(t_2)\right)\\&\quad=\Delta^{\alpha_i}\mathcal{T}_i(\chi_1,\chi_2)(t_1,t_2)+F_i\left(t_1,t_2,\chi_1^*(t_1),\chi_2^*(t_2)\right)\\&\quad=-F_i\left(t_1,t_2,\chi_1(t_1),\chi_2(t_2)\right)+F_i\left(t_1,t_2,\chi_1^*(t_1),\chi_2^*(t_2)\right)\\&\quad\le-F_i(t_1,t_2,\chi_1,\chi_2)+F_i\left(t_1,t_2,\chi_1(t_1),\chi_2(t_2)\right)=0,\\&\chi_i^*(\alpha_i-2)=0,\\&\chi_i^*(T+\alpha_i)=\lambda_j\Delta^{-\theta_j}g_j(T+\alpha_j+\theta_j)\chi_j^*(T+\alpha_j+\theta_j),\end{aligned}\tag{3.20}$$

and

$$\begin{aligned}&\Delta^{\alpha_i}\bar{\chi}_i^*(t_1,t_2)+F_i\left(t_1,t_2,\bar{\chi}_1^*(t_1),\bar{\chi}_2^*(t_2)\right)\\&\quad=\Delta^{\alpha_i}\mathcal{T}_i(\bar{\chi}_1,\bar{\chi}_2)(t_1,t_2)+F_i\left(t_1,t_2,\bar{\chi}_1^*(t_1),\bar{\chi}_2^*(t_2)\right)\\&\quad=-F_i(t_1,t_2,\bar{\chi}_1,\bar{\chi}_2)+F_i\left(t_1,t_2,\bar{\chi}_1^*(t_1),\bar{\chi}_2^*(t_2)\right)\\&\quad\ge-F_i(t_1,t_2,\bar{\chi}_1,\bar{\chi}_2)+F_i\left(t_1,t_2,\bar{\chi}_1(t_1),\bar{\chi}_2(t_2)\right)=0,\\&\bar{\chi}_i^*(\alpha_i-2)=0,\\&\bar{\chi}_i^*(T+\alpha_i)=\lambda_j\Delta^{-\theta_j}g_j(T+\alpha_j+\theta_j)\bar{\chi}_j^*(T+\alpha_j+\theta_j).\end{aligned}\tag{3.21}$$

Thus, it follows from (3.18)–(3.21) that $(\bar{\chi}_1^*,\bar{\chi}_2^*),(\chi_1^*,\chi_2^*)$ are lower and upper solutions of problem (1.5)–(1.6), and $(\bar{\chi}_1^*,\bar{\chi}_2^*),(\chi_1^*,\chi_2^*)\in\mathcal{P}$.

Define the function $\mathcal{F}_i^*$ and the operator $\mathcal{T}^*$ in $\mathcal{U}$ by

$$\mathcal{F}_i^*(t_1,t_2,x,y)=\begin{cases}F_i(t_1,t_2,\bar{\chi}_1^*,\bar{\chi}_2^*), & (x,y)<(\bar{\chi}_1^*,\bar{\chi}_2^*),\\F_i(t_1,t_2,x,y), & (\bar{\chi}_1^*,\bar{\chi}_2^*)\le(x,y)\le(\chi_1^*,\chi_2^*),\\F_i(t_1,t_2,\chi_1^*,\chi_2^*), & (x,y)>(\chi_1^*,\chi_2^*),\end{cases}\tag{3.22}$$

$$\mathcal{T}^*(u_1,u_2)(t_1,t_2)=\left(\mathcal{T}_1^*(u_1,u_2)(t_1,t_2),\mathcal{T}_2^*(u_1,u_2)(t_1,t_2)\right),\tag{3.23}$$

where

$$\mathcal{T}_i^*(u_1,u_2)(t_1,t_2)$$

$$= \sum_{s=0}^{T} G_{i1}(t_i,s)g_1(s+\alpha_1-1)\mathcal{F}_1^*\big(s+\alpha_1-1,t_2,\Delta^{\beta_1}u_1(s+\alpha_1-\beta_1),u_2(t_2)\big)$$
$$- \sum_{s=0}^{T} G_{i2}(t_i,s)g_2(s+\alpha_2-1)\mathcal{F}_2^*\big(t_1,s+\alpha_2-1,u_1(t_1),\Delta^{\beta_2}u_2(s+\alpha_2-\beta_2)\big). \tag{3.24}$$

It follows from the assumption that $\mathcal{F}_i^*:\mathbb{N}_{\alpha_1-1,T+\alpha_1-1}\times\mathbb{N}_{\alpha_2-1,T+\alpha_2-1}\times[0,\infty)\times[0,\infty)\to[0,\infty)$ are continuous. Consider the following problem:

$$\begin{aligned}
&-\Delta^{\alpha_i}u_i(t_i) = \mathcal{F}_i^*\big(t_1,t_2,\Delta^{\beta_i}u_i(s+\alpha_i-\beta_i),u_j(t_j)\big),\\
&u_i(\alpha_i-2)=0,\\
&u_i(T+\alpha_i)=\lambda_j\Delta^{-\theta_j}g_j(T+\alpha_j+\theta_j)u_j(T+\alpha_j+\theta_j).
\end{aligned}\tag{3.25}$$

For $i,j\in\{1,2\}, i\neq j$ and for all $(u_1,u_2)\in\mathcal{U}$, by (3.22) we obtain

$$\begin{aligned}
&\big|\mathcal{T}_i^*(u_1,u_2)(t_1,t_2)\big|\\
&\quad\le t_i^{\underline{\alpha_i-1}}\Bigg|\omega_{i1}\mathcal{G}_1\sum_{s=0}^{T}\mathcal{F}_1^*\big(s+\alpha_1-1,t_2,\Delta^{\beta_1}u_1,u_2\big)\\
&\qquad-\omega_{i2}\mathcal{G}_2\sum_{s=0}^{T}\mathcal{F}_2^*\big(t_1,s+\alpha_2-1,u_1,\Delta^{\beta_2}u_2\big)\Bigg|\\
&\quad\le (T+\alpha_i)^{\underline{\alpha_i-1}}\mathcal{G}\Bigg|\omega_{i1}\sum_{s=0}^{T}\mathcal{F}_1^*\big(s+\alpha_1-1,t_2,\bar{\chi}_1^*,\bar{\chi}_2^*\big)-\omega_{i2}\sum_{s=0}^{T}\mathcal{F}_2^*\big(t_1,s+\alpha_2-1,\bar{\chi}_1^*,\bar{\chi}_2^*\big)\Bigg|\\
&\quad\le (T+\alpha_i)^{\underline{\alpha_i-1}}\mathcal{G}\Bigg|\omega_{i1}\sum_{s=0}^{T}\mathcal{F}_1^*\big(s+\alpha_1-1,t_2,\mathcal{L}_1^{-1}(s+\alpha_1-1)^{\underline{\alpha_1-1}},\mathcal{L}_2^{-1}t_2^{\underline{\alpha_2-1}}\big)\\
&\qquad-\omega_{i2}\sum_{s=0}^{T}\mathcal{F}_2^*\big(t_1,s+\alpha_2-1,\mathcal{L}_1^{-1}t_1^{\underline{\alpha_1-1}},\mathcal{L}_2^{-1}(s+\alpha_2-1)^{\underline{\alpha_2-1}}\big)\Bigg|\\
&\quad\le (T+\alpha_i)^{\underline{\alpha_i-1}}\mathcal{G}\Bigg|\omega_{i1}\mathcal{L}_1^{\rho_1}\sum_{s=0}^{T}\mathcal{F}_1^*\big(s+\alpha_1-1,\alpha_2-1,(s+\alpha_1-1)^{\underline{\alpha_1-1}},(\alpha_2-1)^{\underline{\alpha_2-1}}\big)\\
&\qquad-\omega_{i2}\mathcal{L}_2^{\rho_1}\sum_{s=0}^{T}\mathcal{F}_2^*\big(\alpha_1-1,s+\alpha_2-1,(\alpha_1-1)^{\underline{\alpha_1-1}},(s+\alpha_2-1)^{\underline{\alpha_2-1}}\big)\Bigg|\\
&\quad\le (T+\alpha_i)^{\underline{\alpha_i-1}}\mathcal{G}\mathcal{L}.
\end{aligned}\tag{3.26}$$

By the same argument, we obtain $|\Delta^{\beta_i}\mathcal{T}_i^*(u_1,u_2)(t_1,t_2)|\le(T+\alpha_i)^{\underline{\alpha_i-1}}\mathcal{G}\mathcal{L}$.

Thus,

$$\|\mathcal{T}^*\|_{\mathcal{C}_i}=\|\Delta^{\beta_i}\mathcal{T}_i^*\|_{E_i}+\|\mathcal{T}_j^*\|_{E_j}\le\big[(T+\alpha_1)^{\underline{\alpha_1-1}}+(T+\alpha_2)^{\underline{\alpha_2-1}}\big]\mathcal{G}\mathcal{L}:=\mathcal{M}.$$

Consequently, we have

$$\|\mathcal{T}^*\|_{\mathcal{U}}=\max\big\{\|\mathcal{T}^*\|_{\mathcal{C}_1},\|\mathcal{T}^*\|_{\mathcal{C}_2}\big\}\le\mathcal{M},$$

which implies that $\mathcal{T}^*$ is uniformly bounded. Moreover, it follows from the continuity of $\mathcal{F}_i^*$ and the uniform continuity of $G_{i1}(t_i,s), G_{i2}(t_i,s)$ and $(H2)$ that $\mathcal{T}^*:\mathcal{U}\times\mathcal{U}\to\mathcal{U}$ is continuous.

Let $\mathcal{E}\subset\mathcal{U}\times\mathcal{U}$ be bounded. By the Arzelá–Ascoli theorem and Theorem 2.4, we easily know that $\mathcal{T}^*(\mathcal{E})$ is equicontinuous. Therefore $\mathcal{T}^*$ is completely continuous. Hence, by using Schauder's fixed point theorem, $\mathcal{T}^*$ has at least one fixed point $(u_1^*,u_2^*)$ such that $(u_1^*,u_2^*)=\mathcal{T}^*(u_1^*,u_2^*)$.

Next, we will show that

$$\bigl(\bar{\chi}_1^*(t_1),\bar{\chi}_2^*(t_2)\bigr)\le\bigl(u_1^*(t_1),u_2^*(t_2)\bigr)\le\bigl(\chi_1^*(t_1),\chi_2^*(t_2)\bigr),\quad t_i\in\mathbb{N}_{\alpha_i-2,T+\alpha_i}. \tag{3.27}$$

Firstly, we will prove that $(u_1^*,u_2^*)\le(\chi_1^*,\chi_2^*)$. Suppose $(u_1^*,u_2^*)>(\chi_1^*,\chi_2^*)$. According to the definition of $\mathcal{F}_i^*$, we have

$$-\Delta^{\alpha_i}u_i^*(t_i)=\mathcal{F}_i^*\bigl(t_1,t_2,u_1^*(t_1),u_2^*(t_2)\bigr)=\mathcal{F}_i\bigl(t_1,t_2,\chi_1^*(t_1),\chi_2^*(t_2)\bigr). \tag{3.28}$$

On the other hand, since $(\chi_1^*,\chi_2^*)$ is an upper solution of problem (1.5), we have

$$-\Delta^{\alpha_i}\chi_i^*(t_i)\ge\mathcal{F}_i\bigl(t_1,t_2,\chi_1^*(t_1),\chi_2^*(t_2)\bigr). \tag{3.29}$$

Letting $z_i(t_i)=\chi_i^*(t_i)-u_i^*(t_i)$, and from (3.28)–(3.29), it implies that

$$\Delta^{\alpha_i}z_i(t_i)=\Delta^{\alpha_i}\chi_i^*(t_i)-\Delta^{\alpha_i}u_i^*(t_i)\le 0. \tag{3.30}$$

Furthermore, since $(\chi_1^*,\chi_2^*)$ is an upper solution of problem (1.5) and $(u_1^*,u_2^*)$ is a fixed point of $\mathcal{T}^*$, we have

$$z_i(\alpha_i-2)=0,\qquad z_i(T+\alpha_i)=\lambda_j\Delta^{-\theta_j}g_j(T+\alpha_j+\theta_j)z_j(T+\alpha_j+\theta_j). \tag{3.31}$$

By Lemma 3.1, we have

$$z_i(t_1,t_2)\ge 0.$$

So, $(u_1^*(t_1),u_2^*(t_2))\le(\chi_1^*(t_1),\chi_2^*(t_2))$ for all $t_i\in\mathbb{N}_{\alpha_i-2,T+\alpha_i}$, which contradicts $(u_1^*,u_2^*)>(\chi_1^*,\chi_2^*)$. Therefore we have $(u_1^*,u_2^*)\le(\chi_1^*,\chi_2^*)$ for all $t_i\in\mathbb{N}_{\alpha_i-2,T+\alpha_i}$.

In the same argument, we have $(u_1^*(t_1),u_2^*(t_2))\ge(\bar{\chi}_1^*(t_1),\bar{\chi}_2^*(t_2))$ for all $t_i\in\mathbb{N}_{\alpha_i-2,T+\alpha_i}$.

Thus (3.27) holds. Hence $(u_1^*(t_1),u_2^*(t_2))$ is a positive solution of problem (1.5)–(1.6). From $((\bar{\chi}_1^*,\bar{\chi}_2^*),(\chi_1^*,\chi_2^*))\in\mathcal{P}$ and (3.27), we obtain

$$\bigl(\varsigma\mathcal{L}^{-1}t_1^{\frac{\alpha_1-1}{}},\varsigma\mathcal{L}^{-1}t_2^{\frac{\alpha_2-1}{}}\bigr)\le\bigl(u_1^*,u_2^*\bigr)\le\bigl(\mathcal{G}\mathcal{L}t_1^{\frac{\alpha_1-1}{}},\mathcal{G}\mathcal{L}t_2^{\frac{\alpha_2-1}{}}\bigr).$$

This completes the proof. □

## 4 An example

In this section, in order to illustrate our result, we consider the coupled system of singular fractional difference equations with fractional sum boundary conditions

$$-\Delta^{\frac{4}{3}}u_1(t)=F_1\left(t+\frac{1}{3},t+\frac{1}{2},\Delta^{\frac{1}{2}}u_1\left(t+\frac{5}{6}\right),u_2\left(t+\frac{1}{2}\right)\right),$$

$$
\begin{aligned}
&-\Delta^{\frac{3}{2}}u_2(t) = F_2\left(t+\frac{1}{3}, t+\frac{1}{2}, u_1\left(t+\frac{1}{3}\right)\Delta^{\frac{1}{3}}u_1\left(t+\frac{7}{6}\right)\right), \quad t \in \mathbb{N}_{0,10},\\
&u_1\left(-\frac{2}{3}\right) = 0, \qquad u_1\left(\frac{34}{3}\right) = 9\Delta^{-\frac{3}{4}}(g_2u_2)\left(\frac{49}{4}\right),\\
&u_2\left(-\frac{1}{2}\right) = 0, \qquad u_2\left(\frac{23}{2}\right) = 4\Delta^{-\frac{2}{3}}(g_1u_1)(12),
\end{aligned}
$$

where $a_i, b_i, x_i, y_i > 0$, $0 < x_i + \frac{1}{3}a_i < 1, 0 < y_i + \frac{1}{2}b_i < 1, i = 1,2$, and, for $t_1 \in \mathbb{N}_{-\frac{2}{3},\frac{34}{3}}, t_2 \in \mathbb{N}_{-\frac{1}{2},\frac{23}{2}}$,

$$
\begin{aligned}
&F_1\left(t_1, t_2, \Delta^{\frac{1}{2}}u_1, u_2\right) = t_1^{-x_1}\left(\Delta^{\frac{1}{2}}u_1\right)^{-a_1} + t_2^{-y_1}u_2^{-b_1},\\
&F_2\left(t_1, t_2, u_1, \Delta^{\frac{1}{3}}u_2\right) = t_1^{-x_2}u_1^{-a_2} + t_2^{-y_2}\left(\Delta^{\frac{1}{3}}u_2\right)^{-b_2},\\
&g_1(t_1) = \frac{1}{200e + 10\sin^2 2\pi t_1} \quad \text{and} \quad g_2(t_2) = \frac{1}{100\pi + 20\cos^2 2\pi t_2}.
\end{aligned}
$$

Here $\alpha_1 = \frac{4}{3}, \alpha_2 = \frac{3}{2}, \beta_1 = \frac{1}{2}, \beta_2 = \frac{1}{3}, \theta_1 = \frac{2}{3}, \theta_2 = \frac{3}{4}, T = 10$. We can find that

$$
0 < \lambda_1 < 24.524, \qquad 0 < \lambda_2 < 9.651, \qquad \Lambda = 0.248 < 1,
$$

Clearly, $\sum_{\xi=0}^{T} = \mathcal{H}_{ij}(\xi + \alpha_i - 1, s) \geq 0$ for all $s \in \mathbb{N}_{0,10}$. So, $(H1)$ holds.

For $t_1 \in \mathbb{N}_{-\frac{2}{3},\frac{34}{3}}, t_2 \in \mathbb{N}_{-\frac{1}{2},\frac{23}{2}}$, we obtain that $F_1, F_2$ are decreasing in $u_i, \Delta^{\alpha_i}u_i$, and

$$
\begin{aligned}
&F_1\left(s+\alpha_1-1, s+\alpha_2-1, (s+\alpha_1-1)^{\underline{\alpha_1-1}}, (s+\alpha_2-1)^{\underline{\alpha_2-1}}\right)\\
&\quad = \left(s+\frac{1}{3}\right)^{-x_1}\left[\left(s+\frac{1}{3}\right)^{\underline{\frac{1}{3}}}\right]^{-a_1} + \left(s+\frac{1}{2}\right)^{-y_1}\left[\left(s+\frac{1}{2}\right)^{\underline{\frac{1}{2}}}\right]^{-b_1}\\
&\quad \leq \left(s+\frac{1}{3}\right)^{-(x_1+\frac{1}{3}a_1)} + \left(s+\frac{1}{2}\right)^{-(y_1+\frac{1}{2}b_1)} \in l^1,\\
&F_2\left(s+\alpha_1-1, s+\alpha_2-1, (s+\alpha_1-1)^{\underline{\alpha_1-1}}, (s+\alpha_2-1)^{\underline{\alpha_2-1}}\right)\\
&\quad = \left(s+\frac{1}{3}\right)^{-x_2}\left[\left(s+\frac{1}{3}\right)^{\underline{\frac{1}{3}}}\right]^{-a_2} + \left(s+\frac{1}{2}\right)^{-y_2}\left[\left(s+\frac{1}{2}\right)^{\underline{\frac{1}{2}}}\right]^{-b_2}\\
&\quad \leq \left(s+\frac{1}{3}\right)^{-(x_2+\frac{1}{3}a_2)} + \left(s+\frac{1}{2}\right)^{-(y_2+\frac{1}{2}b_2)} \in l^1.
\end{aligned}
$$

Therefore, $(H2)$ holds.

For all $\ell \in (0,1)$ and $(t_1, t_2, \nu_1, \nu_2) \in \mathbb{N}_{-\frac{2}{3},\frac{34}{3}} \times \mathbb{N}_{-\frac{1}{2},\frac{23}{2}} \times (0,\infty) \times (0,\infty)$, we have

$$
F_i(t_1, t_2, \ell\nu_1, \ell\nu_2) \leq \ell^{-\max\{a_i,b_i\}}F_i(t_1, t_2, \nu_1, \nu_2),
$$

Thus, $(H3)$ holds. Also, $(H4)$ holds for all $t_i \in \mathbb{N}_{\alpha_i-2,T+\alpha_i}$ where

$$
\begin{aligned}
&\varsigma_1 = 0.00181 \leq g_1(t_1) \leq 0.00184 = \mathcal{G}_i \quad \text{and}\\
&\varsigma_2 = 0.00299 \leq g_2(t_2) \leq 0.00318 = \mathcal{G}_2.
\end{aligned}
$$

Hence, by Theorem 3.1, this problem has at least one positive solution $(u_1^*, u_2^*)$.

For a numerical example to show the existence of a positive solution, we give

$$F_1(t_1) = \frac{1}{2}t_1^{-\frac{1}{2}} \quad \text{and} \quad F_2(t_2) = \frac{1}{500}t_2^{-\frac{1}{3}}.$$

We can find that $\lambda_1 = 2.1, \lambda_2 = 3.8, \Lambda = 0.041, \mathcal{P}(F_1, F_2) = 6.697$ and $\mathcal{Q}(F_1, F_2) = 0.054$, then we have

$$u_1(t_1) = \frac{7.054\Gamma(t_1+1)}{\Gamma}(t_1 + 0.666) - 1.120\sum_{s=0}^{t_1-\frac{4}{3}} \frac{\Gamma(t_1-s)}{2(s+\frac{1}{3})(\frac{1}{2})\Gamma(t_1-s-\frac{1}{3})},$$

$$u_2(t_2) = \frac{573.023\Gamma(t_2+1)}{\Gamma}(t_2 + 0.500) - 1.128\sum_{s=0}^{t_2-\frac{3}{2}} \frac{\Gamma(t_2-s)}{(s+\frac{1}{2})(\frac{1}{3})\Gamma(t_2-s-\frac{1}{2})}$$

for $t_1 \in \mathbb{N}_{-\frac{2}{3},\frac{34}{3}}, t_2 \in \mathbb{N}_{-\frac{1}{2},\frac{23}{2}}$. Therefore, we obtain

$$u_1\left(-\frac{2}{3}\right) = 0, \quad u_1\left(\frac{1}{3}\right) = 6.299, \quad u_1\left(\frac{4}{3}\right) = 7.533, \quad u_1\left(\frac{7}{3}\right) = 8.211,$$
$$u_1\left(\frac{10}{3}\right) = 8.636, \quad u_1\left(\frac{13}{3}\right) = 8.914, \quad u_1\left(\frac{16}{3}\right) = 9.097, \quad u_1\left(\frac{19}{3}\right) = 9.211,$$
$$u_1\left(\frac{22}{3}\right) = 9.275, \quad u_1\left(\frac{25}{3}\right) = 9.299, \quad u_1\left(\frac{28}{3}\right) = 9.291, \quad u_1\left(\frac{31}{3}\right) = 9.257,$$
$$u_1\left(\frac{34}{3}\right) = 9.201, \quad \text{and}$$
$$u_2\left(-\frac{1}{2}\right) = 0, \quad u_2\left(\frac{1}{2}\right) = 507.828, \quad u_2\left(\frac{3}{2}\right) = 761.740, \quad u_2\left(\frac{5}{2}\right) = 952.173,$$
$$u_2\left(\frac{7}{2}\right) = 1110.866, \quad u_2\left(\frac{9}{2}\right) = 1249.722, \quad u_2\left(\frac{111}{2}\right) = 1347.692,$$

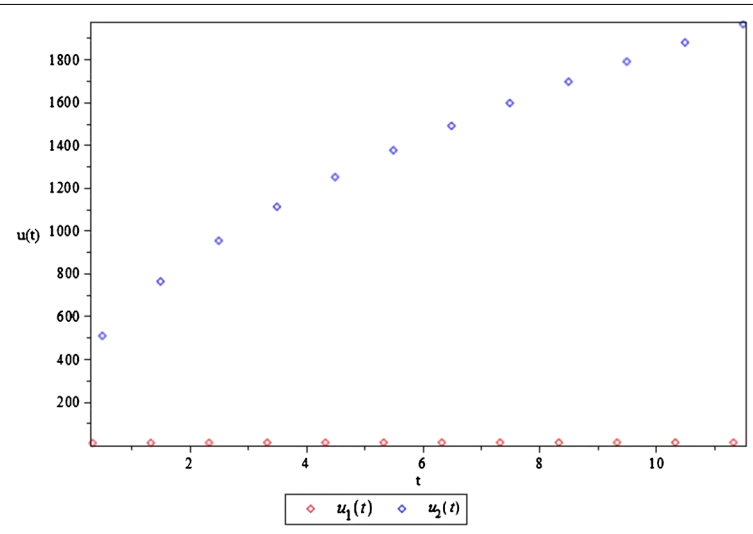


**Figure 1** The graph of $u_1(t_1)$ where $t_1 \in \mathbb{N}_{\frac{1}{3},\frac{34}{3}}$ and $u_2(t_2)$ where $t_2 \in \mathbb{N}_{\frac{1}{2},\frac{23}{2}}$

$$u_2\left(\frac{13}{2}\right) = 1489.247, \qquad u_2\left(\frac{15}{2}\right) = 1595.619, \qquad u_2\left(\frac{17}{2}\right) = 1695.343,$$
$$u_2\left(\frac{19}{2}\right) = 1789.526, \qquad u_2\left(\frac{21}{2}\right) = 51878.999, \qquad u_2\left(\frac{23}{2}\right) = 1964.406.$$

In Fig. 1, the graphs of solutions $u_1$ and $u_2$ are plotted in a two-dimensional space.

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**Author details**
[1]Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok, Thailand. [2]Mathematics Department, Faculty of Science and Technology, Suan Dusit University, Bangkok, Thailand.

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