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Existence results for fractional order differential equation with nonlocal Erdélyi–Kober and generalized Riemann–Liouville type integral boundary conditions at resonance

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Abstract

In this paper, we discuss a nonlinear fractional order boundary value problem with nonlocal Erdélyi–Kober and generalized Riemann–Liouville type integral boundary conditions. By using Mawhin continuation theorem, we investigate the existence of solutions of this boundary value problem at resonance. It is shown that the boundary value problem

$$
\begin{aligned}
&{}^cD^q x(t) = f(t, x(t), x'(t)), \quad t \in [0,T], 1 < q \le 2, \\
&x(0) = \alpha I_\eta^{\gamma,\delta} x(\zeta), \qquad x(T) = \beta^\rho I^p x(\xi),
\end{aligned}
$$

has at least one solution under some suitable conditions, where $\alpha, \beta \in \mathbb{R}$, $0 < \zeta, \xi < T$.

Keywords: Boundary value problem; Resonance; Integral conditions

1 Introduction

In this paper, we intend to discuss the following boundary value problem at resonance:

$$
\begin{cases}
{}^cD^q x(t) = f(t, x(t), x'(t)), \quad t \in [0,T], \\
x(0) = \alpha I_\eta^{\gamma,\delta} x(\zeta), \qquad x(T) = \beta^\rho I^p x(\xi), \quad 0 < \zeta, \eta \le T,
\end{cases}
\tag{1}
$$

where ${}^cD^q$ is the Caputo fractional derivative of order $1 < q \le 2$, $I_\eta^{\gamma,\delta}$ is a Erdélyi–Kober type integral of order $\delta > 0$ with $\eta > 0$ and $\gamma \in \mathbb{R}$, ${}^\rho I^p$ denotes the generalized Riemann–Liouville type integral of order $p > 0$, $\rho > 0$, and $\alpha, \beta \in \mathbb{R}$.

Boundary value problems at resonance have aroused people's interest these days (see [5, 6, 8, 9, 14, 16, 17, 19–21, 25, 29–33, 35, 40, 41, 43]). For instance, in [17], Jiang and Qiu studied the existence of solutions for the following $(k, n-k)$ conjugate boundary value problem at resonance:

$$
(-1)^{n-k} y^{(n)}(t) = f\big(t, y(t), y'(t), \dots, y^{(n-1)}(t)\big), \quad t \in [0,1],
$$

$$y^{(i)}(0) = y^{(j)}(1) = 0, \quad 0 \le i \le k-1, 0 \le j \le n-k-2,$$
$$y^{(n-1)}(1) = \sum_{i=1}^{m} \alpha_i y^{(n-1)}(\xi_i),$$

where $1 \le k \le n-1$, $0 < \xi_1 < \xi_2 < \cdots < \xi_m < 1$. Integral boundary value problems have also gained many people's attention and have been applied to many fields, such as physics, chemistry, and engineering, see [11, 13, 22, 29–31, 35]. Besides, the subject of fractional differential equations has attracted much attention, see [1–5, 7, 10, 12, 15, 23, 24, 27, 28, 32–34, 36–39, 42]. For example, in [5], Zhang and Bai investigated the existence of solutions for the following m-point boundary value problems:

$$D_{0^+}^{\alpha} u(t) = f\bigl(t, u(t), D_{0^+}^{\alpha-1} u(t)\bigr) + e(t), \quad t \in (0,1), \alpha \in (1,2],$$
$$I_{0^+}^{\alpha} u(t)\big|_{t=0} = 0, \qquad D_{0^+}^{\alpha-1} u(1) = \sum_{i=1}^{m-2} \beta_i D_{0^+}^{\alpha-1} u(\eta_i)$$

by using the coincidence degree theory of Mawhin. Very recently, in [2], the authors considered boundary value problem (1) under the nonresonance condition $\nu_1\nu_4 + \nu_2\nu_3 \neq 0$. They established the existence and uniqueness results of BVP (1) by using the standard fixed point theorems.

Inspired by the work above, in this paper, we intend to discuss the boundary value problem (1) under the resonance condition $\nu_1\nu_4 + \nu_2\nu_3 = 0$. We shall study resonant BVP (1) in three different cases of $\dim\ker L = 1$. Different from the above results, the boundary conditions we study are nonlocal Erdélyi–Kober type integral and generalized Riemann–Liouville type integral. To the best of our knowledge, it is innovative to study the boundary value problem with the nonlocal Erdélyi–Kober type integral and generalized Riemann–Liouville type integral by using the method of Mawhin continuation theorem.

The organization of this paper is as follows. In Sect. 2, we provide some definitions, lemmas, and Mawhin continuation theorem which will be used to prove the main results. In Sect. 3, we will give our main results and the proof, some lemmas will also be given to prove the solvability of BVP (1).

2 Preliminaries

Firstly, for the convenience of the reader, we recall some definitions and lemmas.

Definition 2.1 ([2, 18]) The fractional integral of order q with the lower limit zero for a function f is defined by

$$J^q f(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{f(s)}{(t-s)^{1-q}}\, ds, \quad t > 0, q > 0,$$

provided the right-hand side is point-wise defined on $[0,\infty)$, $\Gamma(\cdot)$ is the gamma function.

Definition 2.2 ([2]) The generalized fractional integral of order $q > 0$ and $\rho > 0$ for a function $f(t)$ is defined by

$${}^{\rho}I^q f(t) = \frac{\rho^{1-q}}{\Gamma(q)} \int_0^t \frac{s^{\rho-1} f(s)}{(t^\rho - s^\rho)^{1-q}}\, ds, \quad t \in (0,\infty),$$

provided the right-hand side is point-wise defined on $(0,\infty)$.

Definition 2.3 ([2]) The Erdélyi–Kober fractional integral of order $\delta > 0$ with $\eta > 0$ and $\gamma \in \mathbb{R}$ of a continuous function $f(t)$ is defined as

$$I_\eta^{\gamma,\delta} f(t) = \frac{\eta t^{-\eta(\delta+\gamma)}}{\Gamma(\delta)} \int_0^t \frac{s^{\eta\gamma+\eta-1} f(s)}{(t^\eta - s^\eta)^{1-\delta}}\, ds, \quad t \in (0,\infty),$$

provided the right-hand side is point-wise defined on $\mathbb{R}_+$.

Definition 2.4 ([2, 18]) The Riemann–Liouville fractional derivative of order $q > 0$, $n-1 < q < n$, $n \in \mathbb{N}$ can be written as

$$D_{0^+}^q f(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-q-1} f(s)\, ds,$$

where the function $f(t)$ has absolutely continuous derivative up to order $(n-1)$.

Definition 2.5 ([2, 18]) The Caputo derivative of order q for a function $f : [0,\infty) \to \mathbb{R}$ is defined as

$${}^c D^q f(t) = D_{0^+}^q \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right), \quad t > 0, n-1 < q < n.$$

Lemma 2.1 ([18]) *Given that* $x \in C^1[0,1]$ *with a fractional derivative of order* q $(1 < q < 2)$ *that belongs to* $C(0,1) \cap L(0,1)$, *then*

$$J^q D^q x(t) = x(t) - x(0) - x'(0)t.$$

Lemma 2.2 ([2]) *Let* $\delta, \eta > 0$, $\gamma, q \in \mathbb{R}$, *then we can get*

$$I_\eta^{\gamma,\delta} t^q = \frac{t^q \Gamma(\gamma + \frac{q}{\eta} + 1)}{\Gamma(\gamma + \frac{q}{\eta} + \delta + 1)}.$$

Lemma 2.3 ([2]) *Let* $q, p > 0$, *then we have*

$${}^\rho I^q t^p = \frac{\Gamma(\frac{p+\rho}{\rho})}{\Gamma(\frac{p+\rho q+\rho}{\rho})} \frac{t^{p+\rho q}}{\rho^q}.$$

Definition 2.6 ([26]) Assume that X and Y are real Banach spaces, $L : \operatorname{dom} L \subset X \to Y$ is a Fredholm operator of index zero if the following conditions hold:

(1) $\operatorname{Im} L$ is a closed subspace of Y;

(2) $\dim \operatorname{Ker} L = \operatorname{co}\dim \operatorname{Im} L < \infty$.

Let X, Y be real Banach spaces, $L : \operatorname{dom} L \subset X \to Y$ be a Fredholm operator of index zero, and $N : X \to Y$ be a nonlinear continuous map. $P : X \to X$, $Q : Y \to Y$ are continuous projectors such that

$$\operatorname{Im} P = \ker L, \qquad \ker Q = \operatorname{Im} L, \qquad X = \ker L \oplus \ker P, \qquad Y = \operatorname{Im} L \oplus \operatorname{Im} Q.$$

It follows that

$$L|_{\operatorname{dom} L\cap \operatorname{Ker} P} : \operatorname{dom} L \cap \ker P \to \operatorname{Im} L$$

is invertible, and the inverse of the mapping is denoted by K_P (generalized inverse operator of L). Let Ω be an open bounded subset of X with $\operatorname{dom} L \cap \Omega \neq \emptyset$, the mapping $N : X \to Y$ will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N : \overline{\Omega} \to X$ is compact.

Theorem 2.1 (Mawhin continuation theorem [26]) *Let $L : \operatorname{dom} L \subset X \to Y$ be a Fredholm operator of index zero and N be L-compact on $\overline{\Omega}$. The equation $L\varphi = N\varphi$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$ if the following conditions are satisfied*:

(1) $Lx \neq \lambda Nx$ *for every* $(x, \lambda) \in [(\operatorname{dom} L \setminus \ker L) \cap \partial\Omega] \times (0, 1)$;
(2) $Nx \notin \operatorname{Im} L$ *for every* $x \in \ker L \cap \partial\Omega$;
(3) $\deg(QN|_{\ker L}, \Omega \cap \ker L, 0) \neq 0$, *where* $Q : Y \to Y$ *is a projection such that* $\operatorname{Im} L = \ker Q$.

Let $Y = C[0, T]$ with the norm $\|x\|_\infty = \max_{t\in[0,1]} |x(t)|$ and $X = C^1[0, T]$ with the norm $\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty\}$. Obviously, X and Y are Banach spaces.

An operator L is defined as $L : Lx(t) = {}^cD^q x(t)$ with

$$\operatorname{dom} L = \bigl\{x \in X : {}^cD^q x \in Y, x(0) = \alpha I_\eta^{\gamma,\delta} x(\zeta), x(T) = \beta^\rho I^p x(\xi)\bigr\}.$$

Define the operator $N : X \to Y$ as follows:

$$(Nx)(t) = f\bigl(t, x(t), x'(t)\bigr).$$

So problem (1) becomes $Lx = Nx$.

Let

$$\nu_1 = 1 - \alpha\frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + \delta + 1)}, \qquad \nu_2 = \alpha\zeta\frac{\Gamma(\gamma + \frac{1}{\eta} + 1)}{\Gamma(\gamma + \frac{1}{\eta} + \delta + 1)},$$

$$\nu_3 = 1 - \beta\frac{\xi^{\rho q}}{\rho^q}\frac{1}{\Gamma(q + 1)}, \qquad \nu_4 = T - \beta\frac{\xi^{\rho q+1}}{\rho^q}\frac{\Gamma(\frac{1+\rho}{\rho})}{\Gamma(\frac{1+\rho q+\rho}{\rho})},$$

then we consider the following three resonant conditions:

$(A1)$ $\nu_1 = \nu_3 = 0$, $\nu_2 \neq 0$, $\nu_4 \neq 0$;
$(A2)$ $\nu_2 = \nu_4 = 0$, $\nu_1 \neq 0$, $\nu_3 \neq 0$;
$(A3)$ $\nu_i \neq 0$ $(i = 1, 2, 3, 4)$, $\nu_1\nu_4 + \nu_2\nu_3 = 0$.

Lemma 2.4 *Assume that* $(A1)$ *holds. Then there exists* $z \in Y$ *such that*

$$\nu_2\bigl(\beta^\rho I^p J^q z(\xi) - J^q z(T)\bigr) + \alpha\nu_4 I_\eta^{\gamma,\delta} J^q z(\zeta) = 1. \tag{2}$$

Proof We define two linear functionals $B_1, B_2 : X \to \mathbb{R}$ as follows:

$$B_1 x = x(0) - \alpha I_\eta^{\gamma,\delta} x(\zeta), \quad x \in X,$$

$$B_2 x = x(T) - \beta^\rho I^p x(\xi), \quad x \in X.$$

Let $\varphi(t) = 1$, $\psi(t) = t$. It follows from $(A1)$ and Lemmas 2.2 and 2.3 that

$$\begin{aligned} &B_1\varphi = \varphi(0) - \alpha I_\eta^{\gamma,\delta}\varphi(\zeta) = \nu_1 = 0, \qquad B_1\psi = \psi(0) - \alpha I_\eta^{\gamma,\delta}\psi(\zeta) = -\nu_2, \\ &B_2\varphi = \varphi(T) - \beta^\rho I^p\varphi(\xi) = \nu_3 = 0, \qquad B_2\psi = \psi(T) - \beta^\rho I^p\psi(\xi) = \nu_4. \end{aligned} \tag{3}$$

So, (2) can be rewritten by

$$B_1\psi \cdot B_2\big(J^q z\big) - B_2\psi \cdot B_1\big(J^q z\big) = 1.$$

For convenience, set

$$Bx = B_1\psi \cdot B_2(x) - B_2\psi \cdot B_1(x). \tag{4}$$

If there is $\widetilde{z} \in Y$ such that $B\widetilde{z} \neq 0$ and, as a result, $z = \frac{\widetilde{z}}{B\widetilde{z}} \in Y$ with $Bz = 1$. Assume the contrary. Then $B(J^q z) = 0$ for all $z \in Y$, and, in particular, for integer $n \geq 2$,

$$\frac{\Gamma(n+1)}{\Gamma(n-q+1)}B\big(J^q t^{n-q}\big) = B\big(t^n\big) = 0.$$

By (3), $B(1) = B(t) = 0$. Therefore, $B(g) = 0$ for every polynomial g. Note that $B \neq 0$ on all of X, there exists $x_0 \in X$ such that $Bx_0 \neq 0$. Thus, there exists a sequence of polynomials g_n such that $\|x_0 - g_n\|_X < \frac{1}{n}$. So, we deduce that

$$0 \neq |Bx_0| = \big|B(x_0 - g_n) + Bg_n\big| = \big|B(x_0 - g_n)\big| \leq \|B\|\|x_0 - g_n\|_X < \frac{\|B\|}{n}$$

for all integer n, which is a contradiction. Thus, there exists $z \in Y$ satisfying (2). Thus the lemma holds. □

Similar to the proof of Lemma 2.4, we also get the following lemmas.

Lemma 2.5 *Assume that* $(A2)$ *holds. Then there exists* $z_1 \in Y$ *such that*

$$J^q z_1(T) - \beta^\rho I^p J^q z_1(\xi) = 1.$$

Lemma 2.6 *Assume that* $(A3)$ *holds. Then there exists* $z_2 \in Y$ *such that*

$$\nu_2\big(\beta^\rho I^p J^q z_2(\xi) - J^q z_2(T)\big) + \alpha\nu_4 I_\eta^{\gamma,\delta} J^q z_2(\zeta) = 1.$$

Remark 2.1 The main idea of Lemmas 2.4, 2.5, and 2.6 comes from [16, 19, 20].

3 Main results

Assume that the following conditions hold in this paper:

(H1) $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function.

(H2) There exist nonnegative functions $u, v, w \in C[0,T]$ such that

$$\big|f(t,x_1,x_2)\big| \leq u(t)|x_1| + v(t)|x_2| + w(t), \quad t \in [0,T], x_1, x_2 \in \mathbb{R}.$$

(H3) There exists a constant $M > 0$ such that if $|x(t)| + |x'(t)| > M$ for all $t \in [0, T]$, then

$$\nu_2\bigl(\beta^{\rho} I^{p} J^{q} Nx(\xi) - J^{q} Nx(T)\bigr) + \alpha \nu_4 I_{\eta}^{\gamma,\delta} J^{q} Nx(\zeta) \neq 0.$$

(H3′) There exists a constant $M > 0$ such that if $|x'(t)| > M$ for all $t \in [0, T]$, then

$$\beta^{\rho} I^{p} J^{q} Nx(\xi) - J^{q} Nx(T) \neq 0.$$

(H4) There is a constant $D > 0$ such that either

$$c\nu_2\bigl(\beta^{\rho} I^{p} J^{q} N\phi_1(\xi) - J^{q} N\phi_1(T)\bigr) + c\alpha \nu_4 I_{\eta}^{\gamma,\delta} J^{q} N\phi_1(\zeta) > 0 \tag{5}$$

or

$$c\nu_2\bigl(\beta^{\rho} I^{p} J^{q} N\phi_1(\xi) - J^{q} N\phi_1(T)\bigr) + c\alpha \nu_4 I_{\eta}^{\gamma,\delta} J^{q} N\phi_1(\zeta) < 0 \tag{6}$$

holds if $|c| > D$, where $\phi_1(t) = c$.

(H4′) There is a constant $D > 0$ such that either

$$c\beta^{\rho} I^{p} J^{q} N\phi_2(\xi) - cJ^{q} N\phi_2(T) > 0$$

or

$$c\beta^{\rho} I^{p} J^{q} N\phi_2(\xi) - cJ^{q} N\phi_2(T) < 0$$

holds if $|c| > D$, where $\phi_2(t) = ct$.

(H4″) There is a constant $D > 0$ such that either

$$c\nu_2\bigl(\beta^{\rho} I^{p} J^{q} N\phi_3(\xi) - J^{q} N\phi_3(T)\bigr) + c\alpha \nu_4 I_{\eta}^{\gamma,\delta} J^{q} N\phi_3(\zeta) > 0$$

or

$$c\nu_2\bigl(\beta^{\rho} I^{p} J^{q} N\phi_3(\xi) - J^{q} N\phi_3(T)\bigr) + c\alpha \nu_4 I_{\eta}^{\gamma,\delta} J^{q} N\phi_3(\zeta) < 0$$

holds if $|c| > D$, where $\phi_3(t) = c(1 + kt)$, $k = \frac{\nu_1}{\nu_2}$.

Then we can present the following theorem.

Theorem 3.1 *Suppose that* $(A1)$ *and* (H1)–(H4) *are satisfied, then there must be at least one solution of problem* (1) *in* X *provided that* $2T^{q}\|u\|_{\infty} + 2T^{q-1}\|v\|_{\infty} < \Gamma(q)$.

To prove the theorem, we need the following lemmas.

Lemma 3.1 *Assume that* $(A1)$ *holds, then* $L : \operatorname{dom} L \subset X \to Y$ *is a Fredholm operator with index zero. And a linear continuous projector* $P : X \to X$ *can be defined by*

$$(Px)(t) = x(0).$$

Furthermore, define the linear operator $K_p : \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$ *as follows*:

$$(K_p y)(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y(s)\,ds - \frac{\alpha I_\eta^{\gamma,\delta} J^q y(\zeta)}{\nu_2} \cdot t$$

such that $K_p = (L|_{\operatorname{dom} L \cap \ker P})^{-1}$.

Proof Let $\varphi(t) = 1$, $\psi(t) = t$. From $(A1)$ and Lemma 2.4, we can easily get

$$\ker L = \{c, c \in \mathbb{R}\}.$$

Moreover, we can obtain that

$$\operatorname{Im} L = \bigl\{ y \in Y : \nu_2 \bigl(\beta^\rho I^p J^q y(\xi) - J^q y(T)\bigr) + \alpha \nu_4 I_\eta^{\gamma,\delta} J^q y(\zeta) = 0 \bigr\}.$$

On the one hand, suppose $y \in \operatorname{Im} L$, then there exists $x \in \operatorname{dom} L$ such that

$$y = Lx \in Y.$$

Then we have

$$x(t) = J^q y(t) + c_0 + c_1 t = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y(s)\,ds + c_0 + c_1 t, \tag{7}$$

where $c_0, c_1 \in \mathbb{R}$. Furthermore, for $x \in \operatorname{dom} L$,

$$\begin{aligned} x(0) = \alpha I_\eta^{\gamma,\delta} x(\zeta) &= \alpha I_\eta^{\gamma,\delta} J^q y(\zeta) + c_0 \alpha I_\eta^{\gamma,\delta} \varphi(\zeta) + c_1 \alpha I_\eta^{\gamma,\delta} \psi(\zeta) \\ &= \alpha I_\eta^{\gamma,\delta} J^q y(\zeta) + c_0 + c_1 \nu_2, \end{aligned}$$

and

$$x(0) = J^q y(t) + c_0 + c_1 t\big|_{t=0} = c_0.$$

The above two equalities imply that

$$\alpha I_\eta^{\gamma,\delta} J^q y(\zeta) + c_1 \nu_2 = 0. \tag{8}$$

Using (3) and (7), we get the system

$$\begin{aligned} x(T) = \beta^\rho I^p x(\xi) &= \beta^\rho I^p J^q y(\xi) + c_0 \beta^\rho I^p \varphi(\xi) + c_1 \beta^\rho I^p \psi(\xi) \\ &= \beta^\rho I^p J^q y(\xi) + c_0 + c_1 (T - \nu_4), \\ x(T) = J^q y(t) + c_0 + c_1 t\big|_{t=T} &= J^q y(T) + c_0 + c_1 T. \end{aligned}$$

From this together with the second boundary value condition of (1), we can get

$$\beta^\rho I^p J^q y(\xi) - J^q y(T) = c_1 \nu_4. \tag{9}$$

By using the eliminated element method, equalities (8) and (9) are changed into the equality

$$\nu_2\big(\beta^\rho I^p J^q y(\xi) - J^q y(T)\big) + \alpha\nu_4 I_\eta^{\gamma,\delta} J^q y(\zeta) = 0.$$

So we obtain that

$$\operatorname{Im} L \subset \big\{y \in Y : \nu_2\big(\beta^\rho I^p J^q y(\xi) - J^q y(T)\big) + \alpha\nu_4 I_\eta^{\gamma,\delta} J^q y(\zeta) = 0\big\}.$$

On the other hand, if $y \in Y$ satisfies $\nu_2(\beta^\rho I^p J^q y(\xi) - J^q y(T)) + \alpha\nu_4 I_\eta^{\gamma,\delta} J^q y(\zeta) = 0$, we let

$$x(t) = J^q y(t) - \frac{\alpha I_\eta^{\gamma,\delta} J^q y(\zeta)}{\nu_2} \cdot t.$$

Then we conclude that

$$Lx(t) = {}^c D^q x(t) = y(t),$$

and

$$\begin{aligned} &x(0) = 0, \\ &\alpha I_\eta^{\gamma,\delta} x(\zeta) = \alpha I_\eta^{\gamma,\delta} J^q y(\zeta) - \frac{\alpha I_\eta^{\gamma,\delta} J^q y(\zeta)}{\nu_2} \cdot \alpha I_\eta^{\gamma,\delta} \psi(\zeta) \\ &\qquad = \alpha I_\eta^{\gamma,\delta} J^q y(\zeta) - \frac{\alpha I_\eta^{\gamma,\delta} J^q y(\zeta)}{\nu_2} \cdot \nu_2 = 0. \end{aligned}$$

Besides,

$$\begin{aligned} &x(T) = J^q y(T) - \frac{\alpha I_\eta^{\gamma,\delta} J^q y(\zeta)}{\nu_2} \cdot T, \\ &\beta^\rho I^p x(\xi) = \beta^\rho I^p J^q y(\xi) - \frac{\alpha I_\eta^{\gamma,\delta} J^q y(\zeta)}{\nu_2} \cdot \beta^\rho I^p \psi(\xi) \\ &\qquad = \beta^\rho I^p J^q y(\xi) - \frac{\alpha I_\eta^{\gamma,\delta} J^q y(\zeta)}{\nu_2} \cdot (T - \nu_4) \\ &\qquad = -\frac{\alpha\nu_4 I_\eta^{\gamma,\delta} J^q y(\zeta)}{\nu_2} + J^q y(T) - \frac{\alpha I_\eta^{\gamma,\delta} J^q y(\zeta)}{\nu_2} \cdot (T - \nu_4) \\ &\qquad = J^q y(T) - \frac{\alpha I_\eta^{\gamma,\delta} J^q y(\zeta)}{\nu_2} \cdot T, \end{aligned}$$

therefore

$$x(0) = \alpha I_\eta^{\gamma,\delta} x(\zeta), \qquad x(T) = \beta^\rho I^p x(\xi).$$

That is, $x \in \operatorname{dom} L$, then $y \in \operatorname{Im} L$. In conclusion,

$$\operatorname{Im} L = \big\{y \in Y : \nu_2\big(\beta^\rho I^p J^q y(\xi) - J^q y(T)\big) + \alpha\nu_4 I_\eta^{\gamma,\delta} J^q y(\zeta) = 0\big\}.$$

We define the linear operator $P : X \to X$ as

$$(Px)(t) = x(0).$$

It is obvious that $P^2 x = Px$ and $\operatorname{Im} P = \ker L$. For any $x \in X$, together with $x = (x - Px) + Px$, we have $X = \ker P + \ker L$. It is easy to obtain that $\ker L \cap \ker P = \emptyset$, which implies

$$X = \ker P \oplus \ker L.$$

Next the operator $Q : Y \to Y$ is defined as follows:

$$\begin{aligned}(Qy)(t) &= \bigl(\nu_2\bigl(\beta^\rho I^p J^q y(\xi) - J^q y(T)\bigr) + \alpha \nu_4 I_\eta^{\gamma,\delta} J^q y(\zeta)\bigr) z(t) \\ &= B\bigl(J^q y\bigr) z(t),\end{aligned}$$

where B is given by (4) and $z \in Y$ satisfying $B(J^q z) = 1$.

Obviously, Q is a projection operator such that $\ker Q = \operatorname{Im} L$ and $\operatorname{Im} L = \{cz(t) : c \in \mathbb{R}\}$. For any $y \in Y$, because $y = (y - Qy) + Qy$, we have $Y = \operatorname{Im} L + \operatorname{Im} Q$. Moreover, by a simple calculation, we can get $\operatorname{Im} Q \cap \operatorname{Im} L = \emptyset$. Above all, $Y = \operatorname{Im} L \oplus \operatorname{Im} Q$.

To sum up, we can get that $\operatorname{Im} L$ is a closed subspace of Y; $\dim \ker L = \operatorname{co}\dim \operatorname{Im} L < +\infty$; that is, L is a Fredholm operator of index zero.

We now define the operator $K_p y : Y \to X$ as follows:

$$\begin{aligned}(K_p y)(t) &= J^q y(t) - \frac{\alpha I_\eta^{\gamma,\delta} J^q y(\zeta)}{\nu_2} \cdot t \\ &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y(s)\, ds - \frac{\alpha I_\eta^{\gamma,\delta} J^q y(\zeta)}{\nu_2} \cdot t.\end{aligned}$$

For any $y \in \operatorname{Im} L$, we have

$$(K_p y)(0) = \alpha I_\eta^{\gamma,\delta} (K_p y)(\zeta), \qquad (K_p y)(T) = \beta^\rho I^p (K_p y)(\xi),$$

then $(K_p y)(t) \in \operatorname{dom} L$. In addition, $(K_p y)(0) = 0$, which means $K_p y \in \ker P$. Therefore

$$K_p y \in \operatorname{dom} L \cap \ker P, \quad y \in \operatorname{Im} L.$$

Next we will prove that K_p is the inverse of $L|_{\operatorname{dom} L \cap \ker P}$. It is clear that

$$(LK_p y)(t) = y(t), \quad y \in \operatorname{Im} L.$$

By Lemma 2.1, for each $x \in \operatorname{dom} L \cap \ker P$, we have $x(0) = 0$ and

$$\begin{aligned}(K_p Lx)(t) &= J^q D^q x(t) - \frac{\alpha I_\eta^{\gamma,\delta} J^q D^q x(\zeta)}{\nu_2} \cdot t \\ &= x(t) - x(0) - x'(0)t - \frac{\alpha I_\eta^{\gamma,\delta} x(\zeta) - x(0)\alpha I_\eta^{\gamma,\delta} \varphi(\zeta) - x'(0)\alpha I_\eta^{\gamma,\delta} \psi(\zeta)}{\nu_2} \cdot t \\ &= x(t) - x(0) - x'(0)t - \frac{x(0) - x(0)\alpha I_\eta^{\gamma,\delta} \varphi(\zeta) - x'(0)\alpha I_\eta^{\gamma,\delta} \psi(\zeta)}{\nu_2} \cdot t\end{aligned}$$

$$
\begin{aligned}
&= x(t) - x'(0)t + x'(0)t \\
&= x(t).
\end{aligned}
$$

This implies that $K_pLx = x$. So $K_p = (L|_{\operatorname{dom} L\cap \ker P})^{-1}$. Thus the lemma holds. □

Lemma 3.2 *N is L-compact on $\overline{\Omega}$ if $\operatorname{dom} L \cap \overline{\Omega} \neq \emptyset$, where Ω is a bounded open subset of X.*

Proof It follows from the continuity of f in condition (H1) and $z \in Y$ that $(I - Q)N(\overline{\Omega})$ is bounded. In addition,

$$\left\{\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}y(s)\,ds - \frac{\alpha I_\eta^{\gamma,\delta}J^q y(\zeta)}{\nu_2}\cdot t : y \in (I-Q)N(\overline{\Omega})\right\}$$

and

$$\left\{\frac{1}{\Gamma(q-1)}\int_0^t (t-s)^{q-2}y(s)\,ds - \frac{\alpha I_\eta^{\gamma,\delta}J^q y(\zeta)}{\nu_2} : y \in (I-Q)N(\overline{\Omega})\right\}$$

are equi-continuous and uniformly bounded. By Ascoli–Arzela theorem, we get $K_p(I - Q)N : \overline{\Omega} \to X$ is compact. Thus, N is L-compact. The proof is completed. □

Lemma 3.3 *The set $\Omega_1 = \{x \in \operatorname{dom} L \setminus \ker L : Lx = \lambda Nx, \lambda \in [0,1]\}$ is bounded if* (H1)–(H3) *are satisfied.*

Proof Take $x \in \Omega_1$, then $x \notin \ker L$, so $\lambda \neq 0$ and $Nx \in \operatorname{Im} L$. Thus we have

$$\left(\nu_2\left(\beta^\rho I^p J^q Nx(\xi) - J^q Nx(T)\right) + \alpha\nu_4 I_\eta^{\gamma,\delta}J^q Nx(\zeta)\right)z(t) = 0,$$

where $z \in Y$ satisfying $B(J^q z) = 1$. So we get

$$\nu_2\left(\beta^\rho I^p J^q Nx(\xi) - J^q Nx(T)\right) + \alpha\nu_4 I_\eta^{\gamma,\delta}J^q Nx(\zeta) = 0. \tag{10}$$

According to (H3), there exists at least a point $t_0 \in [0, T]$ such that

$$\left|x(t_0)\right| + \left|x'(t_0)\right| \leq M.$$

Using the Newton–Leibnitz formula, we have

$$\|x\|_\infty = \max_{t\in[0,T]}\left|x(t)\right| = \max_{t\in[0,T]}\left|x(t_0) + \int_{t_0}^t x'(s)\,ds\right| \leq M + T\left\|x'\right\|_\infty. \tag{11}$$

In addition, for $Lx = \lambda Nx$ and $x \in \operatorname{dom} L$, we have

$$x(t) = \frac{\lambda}{\Gamma(q)}\int_0^t (t-s)^{q-1}f\left(s, x(s), x'(s)\right)ds + x(0) + x'(0)t$$

and

$$x'(t) = \frac{\lambda}{\Gamma(q-1)}\int_0^t (t-s)^{q-2}f\left(s, x(s), x'(s)\right)ds + x'(0). \tag{12}$$

Take $t = t_0$ in (12), we get

$$x'(t_0) = \frac{\lambda}{\Gamma(q-1)} \int_0^{t_0} (t_0 - s)^{q-2} f\bigl(s, x(s), x'(s)\bigr)\,ds + x'(0).$$

This together with $|x'(t_0)| \le M$ and (11) implies that

$$\begin{aligned}
\bigl|x'(0)\bigr| &\le \bigl|x'(t_0)\bigr| + \frac{\lambda}{\Gamma(q-1)} \int_0^{t_0} (t_0 - s)^{q-2} \bigl|f\bigl(s, x(s), x'(s)\bigr)\bigr|\,ds \\
&\le M + \frac{\lambda}{\Gamma(q-1)} \int_0^{t_0} (t_0 - s)^{q-2} \bigl[u(s)\bigl|x(s)\bigr| + v(s)\bigl|x'(s)\bigr| + w(s)\bigr]\,ds \\
&\le M + \frac{T^{q-1}}{\Gamma(q)} \bigl(\|u\|_\infty \|x\|_\infty + \|v\|_\infty \bigl\|x'\bigr\|_\infty + \|w\|_\infty\bigr) \\
&\le M + \frac{T^{q-1}}{\Gamma(q)} \bigl(\|u\|_\infty \bigl(M + T\bigl\|x'\bigr\|_\infty\bigr) + \|v\|_\infty \bigl\|x'\bigr\|_\infty + \|w\|_\infty\bigr) \\
&= M + \frac{T^{q-1}}{\Gamma(q)} \bigl(M\|u\|_\infty + \|w\|_\infty\bigr) + \frac{T^q\|u\|_\infty + T^{q-1}\|v\|_\infty}{\Gamma(q)} \bigl\|x'\bigr\|_\infty.
\end{aligned}$$

Then we conclude that

$$\begin{aligned}
\bigl|x'(t)\bigr| &\le \frac{\lambda}{\Gamma(q-1)} \int_0^{t} (t - s)^{q-2} \bigl|f\bigl(s, x(s), x'(s)\bigr)\bigr|\,ds + \bigl|x'(0)\bigr| \\
&\le \frac{T^{q-1}}{\Gamma(q)} \bigl(\|u\|_\infty \|x\|_\infty + \|v\|_\infty \bigl\|x'\bigr\|_\infty + \|w\|_\infty\bigr) \\
&\quad + M + \frac{T^{q-1}}{\Gamma(q)} \bigl(M\|u\|_\infty + \|w\|_\infty\bigr) + \frac{T^q\|u\|_\infty + T^{q-1}\|v\|_\infty}{\Gamma(q)} \bigl\|x'\bigr\|_\infty \\
&\le \frac{T^{q-1}}{\Gamma(q)} \bigl(\|u\|_\infty \bigl(M + T\bigl\|x'\bigr\|_\infty\bigr) + \|v\|_\infty \bigl\|x'\bigr\|_\infty + \|w\|_\infty\bigr) \\
&\quad + M + \frac{T^{q-1}}{\Gamma(q)} \bigl(M\|u\|_\infty + \|w\|_\infty\bigr) + \frac{T^q\|u\|_\infty + T^{q-1}\|v\|_\infty}{\Gamma(q)} \bigl\|x'\bigr\|_\infty \\
&= M + 2\frac{T^{q-1}}{\Gamma(q)} \bigl(M\|u\|_\infty + \|w\|_\infty\bigr) + 2\frac{T^q\|u\|_\infty + T^{q-1}\|v\|_\infty}{\Gamma(q)} \bigl\|x'\bigr\|_\infty.
\end{aligned}$$

Therefore, we can obtain that

$$\bigl\|x'\bigr\|_\infty \le \frac{M\Gamma(q) + 2MT^{q-1}\|u\|_\infty + 2T^{q-1}\|w\|_\infty}{\Gamma(q) - 2T^q\|u\|_\infty - 2T^{q-1}\|v\|_\infty} = M_1.$$

Combining this with (11), we have

$$\|x\|_\infty \le M + T\bigl\|x'\bigr\|_\infty \le M + TM_1.$$

Then Ω_1 is bounded. The proof of the lemma is completed. □

Lemma 3.4 *The set* $\Omega_2 = \{x : x \in \ker L, Nx \in \operatorname{Im} L\}$ *is bounded if* (H1), (H4) *hold.*

Proof Let $x \in \Omega_2$, then $x(t) \equiv c$ and $Nx \in \operatorname{Im} L$, so we can get

$$\nu_2\bigl(\beta^\rho I^p J^q Nx(\xi) - J^q Nx(T)\bigr) + \alpha\nu_4 I_\eta^{\gamma,\delta} J^q Nx(\zeta) = 0.$$

According to (H4), we have $|c| \le D$, that is to say, Ω_2 is bounded. We complete the proof. □

Lemma 3.5 *The set* $\Omega_3 = \{x \in \ker L : \lambda x + \alpha(1-\lambda)JQNx = 0, \lambda \in [0,1]\}$ *is bounded if conditions* (H1), (H4) *are satisfied, where* $J : \operatorname{Im} Q \to \ker L$ *is a linear isomorphism defined by*

$$J(cz_1) = c, \quad c \in \mathbb{R},$$

and

$$\alpha = \begin{cases} -1, & \textit{if (5) holds;} \\ 1, & \textit{if (6) holds,} \end{cases}$$

where z_1 *is introduced in Lemma* 2.4.

Proof Suppose that $x \in \Omega_3$, we have $x(t) = c$ and

$$\lambda x + \alpha(1-\lambda)JQNx = 0,$$

thus we have

$$\lambda c = -\alpha(1-\lambda)\bigl(\nu_2\bigl(\beta^\rho I^p J^q Nx(\xi) - J^q Nx(T)\bigr) + \alpha\nu_4 I_\eta^{\gamma,\delta} J^q Nx(\zeta)\bigr).$$

If $\lambda = 0$, by condition (H4) we have $|c| \le D$. If $\lambda = 1$, then $c = 0$. If $\lambda \in (0,1)$, we suppose $|c| > D$, then

$$\lambda c^2 = -\alpha(1-\lambda)c\bigl(\nu_2\bigl(\beta^\rho I^p J^q Nx(\xi) - J^q Nx(T)\bigr) + \alpha\nu_4 I_\eta^{\gamma,\delta} J^q Nx(\zeta)\bigr) < 0$$

which contradicts with $\lambda c^2 > 0$, so $|c| \le D$. Then the lemma holds. □

Theorem 3.1 can be proved now.

Proof of Theorem 3.1 Suppose that $\Omega \supset \bigcup_{i=1}^3 \overline{\Omega_i} \cup \{0\}$ is a bounded open subset of X, from Lemma 3.2 we know that N is L-compact on $\overline{\Omega}$. In view of Lemmas 3.3 and 3.4, we can get:

(i) $Lx \ne \lambda Nx$ for every $(x,\lambda) \in [(\operatorname{dom} L \setminus \ker L) \cap \partial\Omega] \times (0,1)$;

(ii) $Nx \notin \operatorname{Im} L$ for every $x \in \ker L \cap \partial\Omega$.

Set $H(x,\lambda) = \lambda Jx + \alpha(1-\lambda)QNx$. It follows from Lemma 3.5 that we have $H(x,\lambda) \ne 0$ for any $x \in \partial\Omega \cap \ker L$. So, by the homotopic property of degree, we have

$$\deg(JQN|_{\ker L}, \Omega \cap \ker L, 0) = \deg(\alpha I, \Omega \cap \ker L, 0) \ne 0.$$

All the conditions of Theorem 2.1 are satisfied. So there must be at least one solution of problem (1) in X. The proof of Theorem 3.1 is completed. □

Theorem 3.2 *Suppose that* $(A2)$ *and* (H1), (H2), (H3′), (H4′) *are satisfied, then there must be at least one solution of problem* (1) *in* X *provided that* $2T^q\|u\|_\infty + 2T^{q-1}\|v\|_\infty < \Gamma(q)$.

To prove the theorem, we need the following lemmas.

Lemma 3.6 *Assume that* $(A2)$ *holds, then* $L : \operatorname{dom} L \subset X \to Y$ *is a Fredholm operator with index zero. And the linear continuous projector* $P : X \to X$ *can be defined by*

$$(Px)(t) = \frac{x(T)}{T}t.$$

Furthermore, define the linear operator $K_p : \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$ *as follows*:

$$(K_p y)(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y(s)\, ds$$

such that $K_p = (L|_{\operatorname{dom} L \cap \ker P})^{-1}$.

Proof Let $\varphi(t) = 1$, $\psi(t) = t$. In view of $(A2)$ we know

$$\begin{aligned} &\alpha = 0, \\ &\varphi(0) - \alpha I_\eta^{\gamma,\delta}\varphi(\zeta) = 1, \qquad \psi(0) - \alpha I_\eta^{\gamma,\delta}\psi(\zeta) = 0, \\ &\varphi(T) - \beta^\rho I^p\varphi(\xi) = v_3, \qquad \psi(T) - \beta^\rho I^p\psi(\xi) = 0, \end{aligned} \tag{13}$$

and we get

$$\ker L = \{ct, c \in \mathbb{R}\}$$

and

$$\operatorname{Im} L = \bigl\{y \in Y : J^q y(T) = \beta^\rho I^p J^q y(\xi)\bigr\}.$$

Besides, operators $P : X \to X$, $Q : Y \to Y$ can be defined as follows:

$$(Px)(t) = \frac{x(T)}{T}t$$

and

$$(Qy)(t) = \bigl(J^q y(T) - \beta^\rho I^p J^q y(\xi)\bigr) z_1(t),$$

where $z_1 \in Y$ satisfying $J^q z_1(T) - \beta^\rho I^p J^q z_1(\xi) = 1$. In addition, for each $x \in \operatorname{dom} L \cap \ker P$, we have $x(0) = 0$, $x(T) = 0$, then we get the generalized inverse operator of L as follows:

$$(K_p y)(t) = J^q y(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y(s)\, ds.$$

The detailed proof of Lemma 3.6 is similar to that of Lemma 3.1, so we omit it. □

Proof of Theorem 3.2 The proof of Theorem 3.2 is similar to that of Theorem 3.1, we omit it. □

Theorem 3.3 *Suppose that* $(A3)$ *and* (H1), (H2), (H3), *and* (H4″) *are satisfied, then there must be at least one solution of problem* (1) *in* X *provided that* $2T^q\|u\|_\infty + 2T^{q-1}\|v\|_\infty < \Gamma(q)$.

To prove the theorem, we need the following lemmas.

Lemma 3.7 *Assume that* $(A3)$ *holds, then* $L : \operatorname{dom} L \subset X \to Y$ *is a Fredholm operator with index zero. And a linear continuous projector* $P : X \to X$ *can be defined by*

$$(Px)(t) = x(0)(1 + kt),$$

where $k = \frac{v_1}{v_2} = -\frac{v_3}{v_4}$. *Furthermore, define the linear operator* $K_p y : \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$ *as follows*:

$$(K_p y)(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y(s)\, ds$$

such that $K_p = (L|_{\operatorname{dom} L \cap \ker P})^{-1}$.

Proof Let $\varphi_1(t) = 1 + kt$. In view of $(A3)$ we know

$$\varphi_1(0) - \alpha I_\eta^{\gamma,\delta} \varphi_1(\zeta) = 0, \qquad \varphi_1(T) - \beta^\rho I^p \varphi_1(\xi) = 0,$$

we can easily get

$$\ker L = \{c(1 + kt) : c \in \mathbb{R}\}.$$

Moreover, we can obtain that

$$\operatorname{Im} L = \{y \in Y : v_1(\beta^\rho I^p J^q y(\xi) - J^q y(T)) = \alpha v_3 I_\eta^{\gamma,\delta} J^q y(\zeta)\}.$$

We define the linear operator $P : X \to X$ as

$$(Px)(t) = x(0)(1 + kt),$$

and the operator $Q : Y \to Y$ as

$$(Qy)(t) = \bigl(v_1(\beta^\rho I^p J^q y(\xi) - J^q y(T)) - \alpha v_3 I_\eta^{\gamma,\delta} J^q y(\zeta)\bigr) z_2(t),$$

where $z_2 \in Y$ satisfying $v_2(\beta^\rho I^p J^q z_2(\xi) - J^q z_2(T)) + \alpha v_4 I_\eta^{\gamma,\delta} J^q z_2(\zeta) = 1$.

In addition, for each $x \in \operatorname{dom} L \cap \ker P$, we have $x(0) = 0$, then we get the generalized inverse operator of L as follows:

$$\begin{aligned}(K_p y)(t) &= J^q y(t) - \frac{\alpha I_\eta^{\gamma,\delta} J^q y(\zeta)}{v_2} \cdot t \\ &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y(s)\, ds - \frac{\alpha I_\eta^{\gamma,\delta} J^q y(\zeta)}{v_2} \cdot t.\end{aligned}$$

The detailed proof of Lemma 3.7 is similar to that of Lemma 3.1, so we omit it. □

Proof of Theorem 3.3 The proof of Theorem 3.3 is similar to that of Theorem 3.1, we omit it. □

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The authors declare that they have no competing interests.

Authors' contributions
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