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Fractional differential equations for the generalized Mittag-Leffler function

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Abstract
In this paper, we establish some (presumably new) differential equation formulas for the extended Mittag-Leffler-type function by using the Saigo-Maeda fractional differential operators involving the Appell function $F_3(\cdot)$ and results in terms of the Wright generalized hypergeometric-type function ${}_{m+1}\psi_{n+1}^{(\{\kappa_l\}_{l\in\mathbb{N}_0})}(z;p)$ recently established by Agarwal. Some interesting special cases are also pointed out.

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Keywords: generalized Gamma function; generalized beta functions; generalized Mittag-Leffler function; generalized Wright hypergeometric function; fractional derivative operators

1 Introduction and preliminaries

Fractional calculus (derivative and integrals) is very old as the conventional calculus and has bern recently applied in various areas of engineering, science, finance, applied mathematics, and bio engineering (see, e.g., [1, 2]). Many differential equations involving various special functions have found significant importance and applications in various subfields of mathematical analysis. During the last few decades, a number of workers have studied, in depth, the properties, applications, and different extensions of various hypergeometric operators of fractional derivatives. A detailed account of such operators along with their properties and applications have been considered by several authors (see [3–22] and [23]).

A useful generalization of the hypergeometric fractional derivatives, including the Saigo operators [15–17], has been introduced by Marichev [13] as Mellin-type convolution operators with a special function $F_3(\cdot)$ in the kernel (for more details, see Samko et al. [20, p. 194, Eq. (10.47) and Section 10.3]) and later extended and studied by Saigo and Maeda [18, p. 393, Eqs. (4.12) and (4.13)]. Note that the generalized fractional derivative operators (for Saigo-Maeda operators, see [18]) are defined as follows:

$$\begin{aligned}\left(D_{0+}^{\tau,\tau',\upsilon,\upsilon',\sigma}f\right)(x) &= \left(I_{0+}^{-\tau',-\tau,-\upsilon',-\upsilon,-\sigma}f\right)(x)\\ &= \left(\frac{d}{dx}\right)^{k}\left(I_{0+}^{-\tau',-\tau,-\upsilon'+k,-\upsilon,-\sigma+k}f\right)(x), \qquad (1.1)\end{aligned}$$

$$\begin{aligned}\left(D_{-}^{\tau,\tau',\upsilon,\upsilon',\sigma}f\right)(x) &= \left(I_{-}^{-\tau',-\tau,-\upsilon',-\upsilon,-\sigma}f\right)(x)\\ &= \left(-\frac{d}{dx}\right)^{k}\left(I_{0-}^{-\tau',-\tau,-\upsilon'+k,-\upsilon,-\sigma+k}f\right)(x),\end{aligned} \tag{1.2}$$

where $\Re(\sigma) > 0$, $k = [\Re(\sigma)] + 1$, $\tau, \tau', \upsilon, \upsilon', \sigma \in \mathbb{C}$, $\mathbb{C}$ being the set of complex numbers, and $x > 0$.

Following Saigo et al. [18] and Saxena and Saigo [21], the left-hand sided and right-hand sided generalized differentiation for a power function are, respectively, given as follows (see, [24, p. 7, Eqs. (4.1) and (4.2)]):

$$\left(D_{0+}^{\tau,\tau',\upsilon,\upsilon',\sigma}t^{\varrho-1}\right)(x) = \frac{\Gamma(\varrho)\Gamma(\varrho-\sigma+\tau+\tau'+\upsilon')\Gamma(\varrho+\tau-\upsilon)}{\Gamma(\varrho-\upsilon)\Gamma(\varrho-\sigma+\tau+\tau')\Gamma(\varrho-\sigma+\tau+\upsilon')}x^{\tau+\tau'+\varrho-\sigma-1}, \tag{1.3}$$

where $\Re(\varrho) > \max\{0, \Re(-\tau+\upsilon), \Re(-\tau-\tau'-\upsilon'-\sigma)\}$, and

$$\begin{aligned}&\left(D_{0-}^{\tau,\tau',\upsilon,\upsilon',\sigma}t^{-\rho}\right)(x)\\ &\quad= \frac{\Gamma(\varrho+\sigma-\tau-\tau')\Gamma(\varrho-\tau'-\upsilon+\sigma)\Gamma(\varrho+\upsilon')}{\Gamma(\varrho)\Gamma(\varrho-\tau-\tau'-\upsilon+\sigma)\Gamma(\varrho-\tau'+\upsilon')}x^{-\varrho+\tau+\tau'-\sigma},\end{aligned} \tag{1.4}$$

where $\Re(\varrho) > \max\{\Re(-\upsilon'), \Re(\tau+\tau'-\sigma), \Re(\tau'+\upsilon-\sigma)+\Re(\sigma)+1\}$.

In certain areas of applied mathematics and mathematical physics, special functions and their generalizations are used for finding solutions of the initial or boundary value problems for partial differential equations and fractional differential equations. It is also important to mention here that the special and degenerated cases of hypergeometric functions; in particular, the Bessel, Mittage-Leffler, and Wright hypergeometric functions have an importance due to application point of view.

Recently, Parmar introduced the extended Mittag-Leffler type function in the following form [25, p. 1072, Eq. (16)]:

$$\begin{aligned}&E_{\xi,\zeta}^{(\{\kappa_l\}_{l\in\mathbb{N}_0};\lambda)}(z;p) = \sum_{n=0}^{\infty}\frac{\mathcal{B}^{(\{\kappa_l\}_{l\in\mathbb{N}_0})}(\lambda+n,1-\lambda;p)}{B(\lambda,1-\lambda)}\frac{z^n}{\Gamma(\xi n+\zeta)}\\ &\quad\left(z,\zeta,\lambda\in\mathbb{C},\Re(\xi)>0,\Re(\zeta)>0,\Re(\lambda)>1;p\geq 0\right),\end{aligned} \tag{1.5}$$

where $\mathcal{B}^{(\{\kappa_l\}_{l\in\mathbb{N}_0})}(\lambda+n,1-\lambda;p)$ is the extended beta function defined by [26]

$$\begin{aligned}&\mathcal{B}^{(\{\kappa_l\}_{l\in\mathbb{N}_0})}(\alpha,\beta;p) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\Theta\left(\{\kappa_l\}_{l\in\mathbb{N}_0};-\frac{p}{t(1-t)}\right)dt\\ &\quad\left(\min\left\{\Re(\alpha),\Re(\beta)\right\}>0;\Re(p)\geq 0\right),\end{aligned} \tag{1.6}$$

and $\Theta(\{\kappa_l\}_{l\in\mathbb{N}_0};z)$ is a function of an appropriately bounded sequence $\{\kappa_l\}_{l\in\mathbb{N}_0}$ of arbitrary real or complex numbers defined as follows [26, p. 243, Eq. (2.1)]:

$$\begin{aligned}&\Theta\left(\{\kappa_l\}_{l\in\mathbb{N}_0};z\right)\\ &\quad:= \begin{cases}\sum_{l=0}^{\infty}\{\kappa_l\}_{l\in\mathbb{N}_0}\frac{z^l}{l!}, & |z|<R; 0<R<\infty; \kappa_0 := 1,\\ \mathcal{M}_0 z^{\omega}\exp(z)[1+O(\frac{1}{z})], & \Re(z)\to\infty; \mathcal{M}_0>0; \omega\in\mathbb{C}.\end{cases}\end{aligned} \tag{1.7}$$

It can be easily seen that different selection of the sequence $\{\kappa_l\}_{l\in\mathbb{N}_0}$ would generate particular cases of (1.5) as explained in the following examples.

Example 1 If we set $\kappa_l = \frac{(\rho)_l}{(\sigma)_l}$ ($l \in \mathbb{N}_0$), then (1.5) results in to the extended generalized Mittag-Leffler function [25, p. 1072, Eq. (17)]

$$E_{\xi,\zeta}^{(\rho,\sigma);\lambda}(z;p) = \sum_{n=0}^{\infty} \frac{\mathcal{B}^{(\rho,\sigma)}(\lambda+n,1-\lambda;p)}{B(\lambda,1-\lambda)} \frac{z^n}{\Gamma(\xi n+\zeta)}$$
$$\bigl(z,\zeta,\lambda \in \mathbb{C}, \Re(\rho) > 0, \Re(\sigma) > 0, \Re(\xi) > 0, \Re(\zeta) > 0, \Re(\lambda) > 1; p \geq 0\bigr). \tag{1.8}$$

Example 2 Setting $\kappa_l = 1$ ($l \in \mathbb{N}$) in (1.5) (see [27] with $c = 1$), we get the function

$$E_{\xi,\zeta}^{\lambda}(z;p) = \sum_{n=0}^{\infty} \frac{B(\lambda+n,1-\lambda;p)}{B(\lambda,1-\lambda)} \frac{z^n}{\Gamma(\xi n+\zeta)}$$
$$\bigl(z,\zeta,\lambda \in \mathbb{C}, \Re(\xi) > 0, \Re(\zeta) > 0, \Re(\lambda) > 1; p \geq 0\bigr). \tag{1.9}$$

Example 3 Similarly, for $p = 0$, (1.9) immediately reduces to the Prabhakar-type [28] Mittage-Leffler function.

Example 4 For $\xi = \zeta = 1$, (1.5), (1.8), and (1.9) can be expressed, respectively, in terms of the extended confluent hypergeometric functions as follows:

$$E_{1,1}^{(\{\kappa_l\}_{l\in\mathbb{N}_0};\lambda)}(z;p) = \Phi_p^{(\{\kappa_l\}_{l\in\mathbb{N}_0})}(\lambda;1;z), \tag{1.10}$$
$$E_{1,1}^{(\rho,\sigma);\lambda}(z;p) = \Phi_p^{(\rho,\sigma)}(\lambda;1;z) \tag{1.11}$$

and

$$E_{1,1}^{\lambda}(z;p) = \Phi_p(\lambda;1;z). \tag{1.12}$$

Similarly, by (1.6) and (1.7) of the extended beta function $\mathcal{B}^{(\{\kappa_l\}_{l\in\mathbb{N}_0})}(\lambda+n,1-\lambda;p)$ and a function of an appropriately bounded sequence $\{\kappa_l\}_{l\in\mathbb{N}_0}$ of arbitrary real or complex numbers $\Theta(\{\kappa_l\}_{l\in\mathbb{N}_0};z)$, Agarwal [29] introduced and studied a further potentially useful extension of the Wright hypergeometric function as follows:

$$\begin{aligned}
&{}_{m+1}\psi_{n+1}^{(\{\kappa_l\}_{l\in\mathbb{N}_0})}(z;p) \\
&\quad = {}_{m+1}\psi_{n+1}^{(\{\kappa_l\}_{l\in\mathbb{N}_0})}\left[\begin{matrix} (a_i,\alpha_i)_{1,m}, & (\gamma,1) \\ (b_j,\beta_j)_{1,n}, & (c,1) \end{matrix}\middle| (z;p)\right] \\
&\quad = \frac{1}{\Gamma(c-\gamma)} \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{m}\Gamma(a_i+k\alpha_i)}{\prod_{j=1}^{n}\Gamma(b_j+k\beta_j)} \frac{\mathcal{B}^{(\{\kappa_l\}_{l\in\mathbb{N}_0})}(\gamma+k,c-\gamma;p)z^k}{k!}
\end{aligned}$$
$$\bigl(z,\gamma \in \mathbb{C}, \Re(c) > \Re(\gamma) > 0; p \geq 0\bigr). \tag{1.13}$$

In this paper, we aim to establish certain (presumably) new fractional differential equation formulas involving the extended generalized Mittag-Leffler type function (1.9) and extended Wright Generalized hypergeometric function (1.13) by using the fractional differential operators (1.1) and (1.2), respectively. Some particular cases of our main findings are also pointed out.

2 Main results

In this section, we establish the fractional differential formulas involving the Saigo-Meada fractional derivative operators (1.1) and (1.2). These formulas are given in the following theorems.

Theorem 1 *Let* $x > 0$, $\tau, \tau', \upsilon, \upsilon', \sigma, \mu, \xi, \zeta, \lambda, t, p \in \mathbb{C}$ ($\Re(\xi) > 0, \Re(\zeta) > 0, \Re(p) \geq 0$) *be such that*

$$\Re(\mu) > \max\bigl\{0, \Re(-\tau + \upsilon), \Re\bigl(-\tau - \tau' - \upsilon' - \sigma\bigr)\bigr\}. \tag{2.1}$$

Then the following formula holds:

$$\begin{aligned}
&\bigl(D_{0+}^{\tau,\tau',\upsilon,\upsilon',\sigma} t^{\mu-1} E_{\xi,\zeta}^{(\{\kappa_l\}_{l\in\mathbb{N}_0};\lambda)}(t;p)\bigr)(x) \\
&\quad= \frac{x^{\mu+\tau+\tau'-\sigma-1}}{\Gamma(\lambda)} \\
&\qquad\times {}_4\psi_4^{(\{\kappa_l\}_{l\in\mathbb{N}_0})}\left[\begin{matrix} (\mu,1), & (\mu-\upsilon+\tau,1), & (\mu+\tau+\tau'+\upsilon'-\sigma,1), & (\gamma,1) \\ (\mu-\upsilon,1), & (\mu+\tau+\tau'-\sigma,1), & (\mu-\sigma+\tau+\upsilon',1), & (\zeta,\xi) \end{matrix}\middle|(x;p)\right].
\end{aligned} \tag{2.2}$$

Proof Let $\mathcal{I}$ be the left-hand side of (2.2). Using (1.5) and changing the order of integration and summation, which is verified under the conditions of the theorem, we have

$$\begin{aligned}
\mathcal{I} = {}&\sum_{n=0}^{\infty} \frac{\mathcal{B}^{(\{\kappa_l\}_{l\in\mathbb{N}_0})}(\lambda+n, 1-\lambda;p)}{B(\lambda,1-\lambda)} \frac{1}{\Gamma(\xi n+\zeta)} \\
&\times \bigl(D_{0+}^{\tau,\tau',\upsilon,\upsilon',\sigma} t^{\mu+n-1}\bigr)(x).
\end{aligned} \tag{2.3}$$

Applying (1.1) and (1.3) with ρ replaced by $\eta + n$ yields

$$\begin{aligned}
\mathcal{I} = {}&\sum_{n=0}^{\infty} \frac{\mathcal{B}^{(\{\kappa_l\}_{l\in\mathbb{N}_0})}(\lambda+n, 1-\lambda;p)}{B(\lambda,1-\lambda)} \frac{1}{\Gamma(\xi n+\zeta)} \\
&\times \frac{\Gamma(\mu+n)\Gamma(\mu+n+\tau-\upsilon)\Gamma(\mu+n+\tau+\tau'+\upsilon'-\sigma)}{\Gamma(\mu+n-\upsilon)\Gamma(\mu+n-\sigma+\tau+\tau')\Gamma(\mu+n-\sigma+\tau+\upsilon')} x^{\mu+n+\tau+\tau'-\sigma-1},
\end{aligned} \tag{2.4}$$

which, in view of (1.13), is equal to the right-hand side of (2.2). This completes the proof. □

Theorem 2 *Let* $x > 0$, $\tau, \tau', \upsilon, \upsilon', \sigma, \mu, \xi, \zeta, \lambda, t, p \in \mathbb{C}$ ($\Re(\xi) > 0, \Re(\zeta) > 0, \Re(\lambda) > 1, \Re(p) \geq 0$) *be such that*

$$\Re(\varrho) > \max\bigl\{\Re\bigl(-\upsilon'\bigr), \Re\bigl(\tau + \tau' - \sigma\bigr), \Re\bigl(\tau' + \upsilon - \sigma\bigr) + \Re(\sigma)\bigr\}. \tag{2.5}$$

Then the following formula holds:

$$
\begin{aligned}
&\left(D_{-}^{\tau,\tau',\upsilon,\upsilon',\sigma} t^{-\mu} E_{\xi,\zeta}^{(\{\kappa_l\}_{l\in\mathbb{N}_0};\lambda)}(1/t;p)\right)(x)\\
&\quad=\frac{x^{-\mu+\tau+\tau'-\sigma}}{\Gamma(\lambda)}\\
&\qquad\times {}_4\psi_4^{(\{\kappa_l\}_{l\in\mathbb{N}_0})}\left[\begin{matrix}(\mu+\upsilon',1), & (\mu-\upsilon-\tau'+\sigma,1), & (\mu-\tau-\tau'+\sigma,1), & (\gamma,1)\\ (\mu,1), & (\mu-\tau-\tau'-\upsilon+\sigma,1), & (\mu-\tau'+\upsilon,1), & (\zeta,\xi)\end{matrix}\middle|\left(x^{-1};p\right)\right].
\end{aligned}
\tag{2.6}
$$

Proof We establish the result by a similar argument as in the proof of Theorem 1 using (1.4) instead of (1.3). Therefore, we omit the details. □

3 Special cases

The results in Theorems 1 and 2 can be easily specialized to yield the corresponding formulas involving simpler functions like Mittag-Leffler-type functions and extended confluent hypergeometric functions given by (1.8)-(1.12) after appropriate selection of the sequence $\{\kappa_l\}_{l\in\mathbb{N}_0}$.

Setting $\kappa_l=\frac{(\phi)_l}{(\varphi)_l}$ ($l\in\mathbb{N}_0$), we obtain the following results from Theorems 1 and 2, respectively.

Corollary 1 *Let* $x>0$, $\tau,\tau',\upsilon,\upsilon',\sigma,\mu,\xi,\zeta,\lambda,\phi,\varphi,t,p\in\mathbb{C}$ ($\Re(\xi)>0,\Re(\zeta)>0,\Re(p)\geq 0$) *be such that*

$$
\Re(\mu)>\max\left\{0,\Re(-\tau+\upsilon),\Re\left(-\tau-\tau'-\upsilon'-\sigma\right)\right\}.
\tag{3.1}
$$

Then the following formula holds:

$$
\begin{aligned}
&\left(D_{0+}^{\tau,\tau',\upsilon,\upsilon',\sigma} t^{\mu-1} E_{\xi,\zeta}^{(\phi,\varphi);\lambda}(t;p)\right)(x)\\
&\quad=\frac{x^{\mu+\tau+\tau'-\sigma-1}\Gamma(\varphi)}{\Gamma(\lambda)\Gamma(\phi)}\\
&\qquad\times {}_5\psi_5\left[\begin{matrix}(\mu,1), & (\mu-\upsilon+\tau,1), & (\mu+\tau+\tau'+\upsilon'-\sigma,1), & (\gamma,1), & (\phi,1)\\ (\mu-\upsilon,1), & (\mu+\tau+\tau'-\sigma,1), & (\mu-\sigma+\tau+\upsilon',1), & (\zeta,\xi), & (\varphi,1)\end{matrix}\middle|(x;p)\right].
\end{aligned}
\tag{3.2}
$$

Corollary 2 *Let* $x>0$, $\tau,\tau',\upsilon,\upsilon',\sigma,\mu,\xi,\zeta,\lambda,\phi,\varphi t,p\in\mathbb{C}$ ($\Re(\xi)>0,\Re(\zeta)>0,\Re(\lambda)>1,\Re(p)\geq 0$) *be such that*

$$
\Re(\varrho)>\max\left\{\Re\left(-\upsilon'\right),\Re\left(\tau+\tau'-\sigma\right),\Re\left(\tau'+\upsilon-\sigma\right)+\Re(\sigma)\right\}.
\tag{3.3}
$$

Then the following formula holds:

$$
\begin{aligned}
&\left(D_{-}^{\tau,\tau',\upsilon,\upsilon',\sigma} t^{-\mu} E_{\xi,\zeta}^{(\phi,\varphi);\lambda}(1/t;p)\right)(x)\\
&\quad=\frac{x^{-\mu+\tau+\tau'-\sigma}\Gamma(\varphi)}{\Gamma(\lambda)\Gamma(\phi)}\\
&\qquad\times {}_5\psi_5\left[\begin{matrix}(\mu+\upsilon',1), & (\mu-\upsilon-\tau'+\sigma,1), & (\mu-\tau-\tau'+\sigma,1), & (\gamma,1), & (\phi,1)\\ (\mu,1), & (\mu-\tau-\tau'-\upsilon+\sigma,1), & (\mu-\tau'+\upsilon,1), & (\zeta,\xi), & (\varphi,1)\end{matrix}\middle|\left(x^{-1};p\right)\right].
\end{aligned}
\tag{3.4}
$$

For $\kappa_l=1$, Theorems 1 and 2 become as follows.

Corollary 3 *Let* $x > 0$, $\tau, \tau', \upsilon, \upsilon', \sigma, \mu, \xi, \zeta, \lambda, t, p \in \mathbb{C}$ $(\Re(\xi) > 0, \Re(\zeta) > 0, \Re(p) \geq 0)$ *be such that*

$$\Re(\mu) > \max\{0, \Re(-\tau + \upsilon), \Re(-\tau - \tau' - \upsilon' - \sigma)\}. \tag{3.5}$$

Then the following formula holds:

$$\begin{aligned}&\left(D_{0+}^{\tau,\tau',\upsilon,\upsilon',\sigma} t^{\mu-1} E_{\xi,\zeta}^{\lambda}(t;p)\right)(x)\\ &\quad= \frac{x^{\mu+\tau+\tau'-\sigma-1}}{\Gamma(\lambda)}\\ &\qquad\times {}_4\psi_4\left[\begin{matrix}(\mu,1), & (\mu-\upsilon+\tau,1), & (\mu+\tau+\tau'+\upsilon'-\sigma,1), & (\gamma,1)\\ (\mu-\upsilon,1), & (\mu+\tau+\tau'-\sigma,1), & (\mu-\sigma+\tau+\upsilon',1), & (\zeta,\xi)\end{matrix}\middle|(x;p)\right].\end{aligned} \tag{3.6}$$

Corollary 4 *Let* $x > 0$, $\tau, \tau', \upsilon, \upsilon', \sigma, \mu, \xi, \zeta, \lambda, t, p \in \mathbb{C}$ $(\Re(\xi) > 0, \Re(\zeta) > 0, \Re(\lambda) > 1, \Re(p) \geq 0)$ *be such that*

$$\Re(\varrho) > \max\{\Re(-\upsilon'), \Re(\tau + \tau' - \sigma), \Re(\tau' + \upsilon - \sigma) + \Re(\sigma)\}. \tag{3.7}$$

Then the following formula holds:

$$\begin{aligned}&\left(D_{-}^{\tau,\tau',\upsilon,\upsilon',\sigma} t^{-\mu} E_{\xi,\zeta}^{\lambda}(1/t;p)\right)(x)\\ &\quad= \frac{x^{-\mu+\tau+\tau'-\sigma}\Gamma(\varphi)}{\Gamma(\lambda)\Gamma(\phi)}\\ &\qquad\times {}_4\psi_4\left[\begin{matrix}(\mu+\upsilon',1), & (\mu-\upsilon-\tau'+\sigma,1), & (\mu-\tau-\tau'+\sigma,1), & (\gamma,1)\\ (\mu,1), & (\mu-\tau-\tau'-\upsilon+\sigma,1), & (\mu-\tau'+\upsilon,1), & (\zeta,\xi)\end{matrix}\middle|(x^{-1};p)\right].\end{aligned} \tag{3.8}$$

Similarly, putting $p = 0$ in Corollaries 3 and 4, we get the fractional differential formulas involving the Prabhakar-type [28] Mittag-Leffler function. We omit the details.

Following the same way, setting $\xi = \zeta = 1$, from Theorems 1 and 2 and Corollaries 1 and 2 we obtain the following interesting results.

Corollary 5 *Let* $x > 0$, $\tau, \tau', \upsilon, \upsilon', \sigma, \mu, \lambda, \phi, \varphi, t, p \in \mathbb{C}$ $(\Re(\xi) > 0, \Re(\zeta) > 0, \Re(p) \geq 0)$ *be such that*

$$\Re(\mu) > \max\{0, \Re(-\tau + \upsilon), \Re(-\tau - \tau' - \upsilon' - \sigma)\}. \tag{3.9}$$

Then the following formula holds:

$$\begin{aligned}&\left(D_{0+}^{\tau,\tau',\upsilon,\upsilon',\sigma} t^{\mu-1} \Phi_p^{(\{\kappa_l\}_{l\in\mathbb{N}_0})}(\lambda;1;t)\right)(x)\\ &\quad= \frac{x^{\mu+\tau+\tau'-\sigma-1}}{\Gamma(\lambda)}\\ &\qquad\times {}_4\psi_4^{(\{\kappa_l\}_{l\in\mathbb{N}_0})}\left[\begin{matrix}(\mu,1), & (\mu-\upsilon+\tau,1), & (\mu+\tau+\tau'+\upsilon'-\sigma,1), & (\gamma,1)\\ (\mu-\upsilon,1), & (\mu+\tau+\tau'-\sigma,1), & (\mu-\sigma+\tau+\upsilon',1), & (1,1)\end{matrix}\middle|(x;p)\right].\end{aligned} \tag{3.10}$$

Corollary 6 *Let* $x > 0$, $\tau, \tau', \upsilon, \upsilon', \sigma, \mu, \lambda, t, p \in \mathbb{C}$ $(\Re(\xi) > 0, \Re(\zeta) > 0, \Re(\lambda) > 1, \Re(p) \geq 0)$ *be such that*

$$\Re(\varrho) > \max\{\Re(-\upsilon'), \Re(\tau + \tau' - \sigma), \Re(\tau' + \upsilon - \sigma) + \Re(\sigma)\}. \tag{3.11}$$

Then the following formula holds:

$$\begin{aligned}&\left(D_{-}^{\tau,\tau',\upsilon,\upsilon',\sigma} t^{-\mu} \Phi_p^{(\{\kappa_l\}_{l\in\mathbb{N}_0})}\left(\lambda;1;\frac{1}{t}\right)\right)(x)\\&\quad=\frac{x^{-\mu+\tau+\tau'-\sigma}\Gamma(\varphi)}{\Gamma(\lambda)\Gamma(\phi)}\\&\qquad\times{}_4\psi_4\left[\begin{matrix}(\mu+\upsilon',1), & (\mu-\upsilon-\tau'+\sigma,1), & (\mu-\tau-\tau'+\sigma,1), & (\gamma,1)\\ (\mu,1), & (\mu-\tau-\tau'-\upsilon+\sigma,1), & (\mu-\tau'+\upsilon,1), & (1,1)\end{matrix}\middle|\left(x^{-1};p\right)\right].\end{aligned}\tag{3.12}$$

Corollary 7 *Let* $x>0$, $\tau,\tau',\upsilon,\upsilon',\sigma,\mu,\lambda,\phi,\varphi,t,p\in\mathbb{C}$ $(\Re(\xi)>0,\Re(\zeta)>0,\Re(p)\geq 0)$ *be such that*

$$\Re(\mu)>\max\left\{0,\Re(-\tau+\upsilon),\Re\left(-\tau-\tau'-\upsilon'-\sigma\right)\right\}.\tag{3.13}$$

Then the following formula holds:

$$\begin{aligned}&\left(D_{0+}^{\tau,\tau',\upsilon,\upsilon',\sigma} t^{\mu-1} \Phi_p^{(\phi,\varphi)}(\lambda;1;t)\right)(x)\\&\quad=\frac{x^{\mu+\tau+\tau'-\sigma-1}\Gamma(\varphi)}{\Gamma(\lambda)\Gamma(\phi)}\\&\qquad\times{}_5\psi_5\left[\begin{matrix}(\mu,1), & (\mu-\upsilon+\tau,1), & (\mu+\tau+\tau'+\upsilon'-\sigma,1), & (\gamma,1), & (\phi,1)\\ (\mu-\upsilon,1), & (\mu+\tau+\tau'-\sigma,1), & (\mu-\sigma+\tau+\upsilon',1), & (1,1), & (\varphi,1)\end{matrix}\middle|(x;p)\right].\end{aligned}\tag{3.14}$$

Corollary 8 *Let* $x>0$, $\tau,\tau',\upsilon,\upsilon',\sigma,\mu,\lambda,\phi,\varphi,t,p\in\mathbb{C}$ $(\Re(\xi)>0,\Re(\zeta)>0,\Re(\lambda)>1,\Re(p)\geq 0)$ *be such that*

$$\Re(\varrho)>\max\left\{\Re\left(-\upsilon'\right),\Re\left(\tau+\tau'-\sigma\right),\Re\left(\tau'+\upsilon-\sigma\right)+\Re(\sigma)\right\}.\tag{3.15}$$

Then the following formula holds:

$$\begin{aligned}&\left(D_{-}^{\tau,\tau',\upsilon,\upsilon',\sigma} t^{-\mu} \Phi_p^{(\{\kappa_l\}_{l\in\mathbb{N}_0})}(\lambda;1;t)\right)(x)\\&\quad=\frac{x^{-\mu+\tau+\tau'-\sigma}\Gamma(\varphi)}{\Gamma(\lambda)\Gamma(\phi)}\\&\qquad\times{}_5\psi_5\left[\begin{matrix}(\mu+\upsilon',1), & (\mu-\upsilon-\tau'+\sigma,1), & (\mu-\tau-\tau'+\sigma,1), & (\gamma,1), & (\phi,1)\\ (\mu,1), & (\mu-\tau-\tau'-\upsilon+\sigma,1), & (\mu-\tau'+\upsilon,1), & (1,1), & (\varphi,1)\end{matrix}\middle|\left(x^{-1};p\right)\right].\end{aligned}\tag{3.16}$$

4 Concluding remarks

In this study, we established some fractional differential formulas involving a family of Mittag-Leffler functions. Due to practical importance of the Mittag-Leffler functions, our results are of general character and hence encompass several cases of interest.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors read and approved the final manuscript.

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