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The wave equation with internal control in non-cylindrical domains

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Abstract

In this paper, we shall be concerned with interior controllability for a one-dimensional wave equation in a domain with moving boundary. When the speed of the moving endpoint is less than a certain constant which is less than the characteristic speed, we obtain exact controllability for this equation.

Keywords: interior controllability; wave equation; non-cylindrical domains

1 Introduction and main results

Given $T > 0$. For any $0 < k < 1$, set

$$\alpha_k(t) = 1 + kt, \quad \text{for } t \in [0, T]. \tag{1.1}$$

Also, define the following non-cylindrical domains:

$$\widehat{Q}_T^k = \left\{ (y, t) \in \mathbb{R}^2; 0 < y < \alpha_k(t), t \in [0, T] \right\},$$

and for any $0 < m < m' < n' < n < 1$,

$$\widehat{Q_1} = \left\{ (y, t) \in \mathbb{R}^2; m\alpha_k(t) < y < n\alpha_k(t), t \in [0, T] \right\},$$
$$\widehat{Q_2} = \left\{ (y, t) \in \mathbb{R}^2; m'\alpha_k(t) < y < n'\alpha_k(t), t \in [0, T] \right\}.$$

Consider the following controlled wave equation:

$$\begin{cases} u_{tt} - u_{yy} = \widehat{B}v, & (y, t) \in \widehat{Q}_T^k, \\ u(0, t) = 0, \quad u(\alpha_k(t), t) = 0, & t \in (0, T), \\ u(y, 0) = u^0(y), \quad u_t(y, 0) = u^1(y), & y \in (0, 1), \end{cases} \tag{1.2}$$

where $v \in [H^1(\widehat{Q_1})]'$ is control variable, u is state variable, $(u^0, u^1) \in L^2(0, 1) \times H^{-1}(0, 1)$ is any given initial value and $\widehat{B} \in C_0^\infty(\widehat{Q}_T^k)$,

$$\widehat{B}(y, t) \begin{cases} = 0, & (y, t) \in \widehat{Q}_T^k \setminus \widehat{Q_1}, \\ = 1, & (y, t) \in \widehat{Q_2}, \\ \in (0, 1), & (y, t) \in \widehat{Q_1} \setminus \widehat{Q_2}. \end{cases}$$

By [1], it is easy to check that (1.2) has a unique weak solution u:

$$u \in C\big([0,T];L^2\big(0,\alpha_k(t)\big)\big) \cap C^1\big([0,T];H^{-1}\big(0,\alpha_k(t)\big)\big).$$

The main purpose of this paper is to study exact controllability of (1.2) in the following sense.

Definition 1.1 Equation (1.2) is called exactly controllable at the time T, if for any initial value $(u^0,u^1) \in L^2(0,1) \times H^{-1}(0,1)$ and any target $(u_d^0,u_d^1) \in L^2(0,\alpha_k(T)) \times H^{-1}(0,\alpha_k(T))$, one can always find a control $v \in [H^1(\widehat{Q_1})]'$ such that the corresponding weak solution u of (1.2) satisfies

$$u(T) = u_d^0, \qquad u_t(T) = u_d^1.$$

The main result of this paper is stated as follows.

Theorem 1.1 *Suppose that* $0 < \tilde{k} < 1, 0 < k < \tilde{k}$. *For any given* $T > T_k^*$, (1.2) *is exactly controllable at time* T *in the sense of Definition* 1.1.

Remark 1.1 $\tilde{k}$ and T_k^* will be defined during proof of this theorem.

There is a variety of literature on interior and boundary controllability problems of wave equations in a cylindrical domain (see *e.g.* [2–6]). However, there is only little work of wave equations defined in non-cylindrical domains. We refer to [7–14] for some known results in this respect. In practical situations, many processes evolve in domains whose boundary has moving parts. A simple model, *e.g.*, is the interface of ice water mixture when temperature increases. To study the controllability problem of wave equations with moving boundary or free boundary is very meaningful. In [7–13], boundary controllability for wave equations with a moving boundary has been obtained. In [7], some controllability results for wave equations with Dirichlet boundary conditions in suitable non-cylindrical domains were investigated. In [7], in the one-dimensional case, the following condition seems necessary:

$$\int_0^\infty \big|\alpha_k'(t)\big|\,dt < \infty.$$

In [8–13], the above condition is removed. In [14], exact controllability of a multi-dimensional wave equation with constant coefficients in a non-cylindrical domain was established, while a control entered the system through the whole non-cylindrical domain. Now we consider interior controllability for a one-dimensional wave equation with moving boundary when the moving endpoint moves along a line. Meanwhile, in our paper, we consider locally distributed control of a one-dimensional wave equation in a certain non-cylindrical domain. In order to overcome this difficulty, we transform (1.2) into an equivalent wave equation with variable coefficients in the cylindrical domain and establish exact interior controllability of this equation. In [6], a one-dimensional wave equation with variable coefficients with locally distributed control in cylindrical domains was proved. The variable coefficients are dependent of the space variable x, not dependent of

the time variable t. Moreover, in the published work, variable coefficients are only dependent of space variable x in most cases. Meanwhile, in our paper, the variable coefficients are dependent of space variable x and time variable t. To solve this, motivated by [6], the key point is to construct a suitable multiplier different from that in [6].

The rest of this paper is organized as follows. In Section 2, we reduce the controllability problem of (1.2) to that of a wave equation with variable coefficients in a cylindrical domain. Section 3 is devoted to proving an observability inequality of a wave equation with variable coefficients in a cylindrical domain.

2 Reduction to controllability problems in a cylindrical domain

When $0 < k < 1$, in order to prove Theorem 1.1, we first transform (1.2) into a wave equation with variable coefficients in a cylindrical domain in this section. Set

$$Q = (0,1) \times (0,T), \qquad Q_1 = (m,n) \times (0,T), \qquad Q_2 = (m',n') \times (0,T).$$

To this aim, set

$$x = \frac{y}{\alpha_k(t)} \quad \text{and} \quad w(x,t) = u(y,t) = u(\alpha_k(t)x,t), \quad \text{for } (y,t) \in \widehat{Q}_T^k.$$

Then it is easy to check that (x,t) varies in Q. Also, (1.2) is transformed into the following equivalent wave equation in Q:

$$\begin{cases} w_{tt} - \left[\frac{\beta_k(x,t)}{\alpha_k(t)} w_x\right]_x + \frac{\gamma_k(x)}{\alpha_k(t)} w_{tx} = B\overline{v}(x,t), & (x,t) \in Q, \\ w(0,t) = 0, \qquad w(1,t) = 0, & t \in (0,T), \\ w(x,0) = w^0, \qquad w_t(x,0) = w^1, & x \in (0,1), \end{cases} \tag{2.1}$$

where

$$\overline{v}(x,t) = v(\alpha_k(t)x,t) = v(y,t), \qquad \beta_k(x,t) = \frac{1 - k^2 x^2}{\alpha_k(t)},$$

$$\gamma_k(x) = -2kx, \qquad w^0 = u^0, \qquad w^1 = u^1 + kxu_x^0,$$

$$B(x,t) \begin{cases} = 0, & (x,t) \in Q \setminus Q_1, \\ = 1, & (x,t) \in Q_2, \\ \in (0,1), & (x,t) \in Q_1 \setminus Q_2. \end{cases} \tag{2.2}$$

For any given initial value $(w^0, w^1) \in L^2(0,1) \times H^{-1}(0,1)$ and any control $\overline{v} \in [H^1(Q_1)]'$, (2.1) admits a unique weak solution

$$w \in C\big([0,T]; L^2(0,1)\big) \cap C^1\big([0,T]; H^{-1}(0,1)\big).$$

Therefore, exact controllability of (1.2) (Theorem 1.1) is reduced to the above interior controllability result for (2.1).

To prove this, we first solve the following system.

$$\begin{cases} \xi_{tt} - [\frac{\beta_k(x,t)}{\alpha_k(t)}\xi_x]_x + \frac{\gamma_k(x)}{\alpha_k(t)}\xi_{tx} = 0, & (x,t) \in Q, \\ \xi(0,t) = 0, \quad \xi(1,t) = 0, & t \in (0,T), \\ \xi(x,0) = w^0, \quad \xi_t(x,0) = w^1, & x \in (0,1). \end{cases} \tag{2.3}$$

For any given $(w^0, w^1) \in L^2(0,1) \times H^{-1}(0,1)$, this system has a unique weak solution

$$\xi \in C\big([0,T]; L^2(0,1)\big) \cap C^1\big([0,T]; H^{-1}(0,1)\big).$$

Then in order to obtain interior controllability of (2.1), we only prove interior controllability result for the following wave equation:

$$\begin{cases} \eta_{tt} - [\frac{\beta_k(x,t)}{\alpha_k(t)}\eta_x]_x + \frac{\gamma_k(x)}{\alpha_k(t)}\eta_{tx} = B\overline{v}(x,t), & (x,t) \in Q, \\ \eta(0,t) = \eta(1,t) = 0, & t \in (0,T), \\ \eta(x,0) = \eta_t(x,0) = 0, & x \in (0,1). \end{cases} \tag{2.4}$$

Theorem 2.1 *Let $T > T_k^*$. Then, for any target $(w_d^0, w_d^1) \in L^2(0,1) \times H^{-1}(0,1)$, there exists a control $\overline{v} \in [H^1(Q_1)]'$ such that corresponding weak solution η of (2.4) satisfies*

$$\eta(T) = w_d^0, \qquad \eta_t(T) = w_d^1.$$

Write $H = [H^1(Q_1)]'$, $F = L^2(0,1) \times H^{-1}(0,1)$ and $F' = H_0^1(0,1) \times L^2(0,1)$. Define a linear operator A:

$$A : H \to F,$$

$$Ag = \big(\alpha_k(T)\eta_t(x,T) - k\eta(x,T) + \gamma_k(x)\eta_x(x,T), \alpha_k(T)\eta(x,T)\big), \quad \forall g \in H.$$

Then A is surjective is equivalent to interior controllability of the wave equation (2.4). And A is surjective is derived from an observability inequality of the form

$$\big|A'\big(z^0, z^1\big)\big|_{H'}^2 \geq C\big|\big(z^0, z^1\big)\big|_{F'}^2, \quad \forall\big(z^0, z^1\big) \in F'(C > 0), \tag{2.5}$$

for the dual operator $A' : F' \to H'$ for $T > T_k^*$.

3 Observability inequality of wave equations with variable coefficients

First, we define A'. A' is described by the following homogeneous wave equation:

$$\begin{cases} \alpha_k(t)z_{tt} - [\beta_k(x,t)z_x]_x + \gamma_k(x)z_{tx} = 0, & \text{in } Q, \\ z(0,t) = 0, \quad z(1,t) = 0, & \text{on } (0,T), \\ z(0) = z^0, \quad z_t(0) = z^1, & \text{in } (0,1), \end{cases} \tag{3.1}$$

where $k \in (0,1)$, $(z^0, z^1) \in H_0^1(0,1) \times L^2(0,1)$ is any given initial value, and α_k, β_k and γ_k are given in (2.2). Equation (3.1) has a unique weak solution,

$$z \in C\big([0,T]; H_0^1(0,1)\big) \cap C^1\big([0,T]; L^2(0,1)\big).$$

Set B' the adjoint of the extension operator B in (1.2), and if $\bar{v} \in [H^1(Q_1)]'$, then B' : $H^1(Q) \to H^1(Q_1)$. Hence A' is defined as follows:

$$A'\left(z^0, z^1\right) = B'\left(\alpha_k(t)z\right) = \alpha_k(t)z(x,t), \quad (x,t) \in Q_1, \forall\left(z^0, z^1\right) \in F',$$

where z is the solution of (3.1). Therefore, (2.5) is equivalent to the following inequality:

$$\left|\alpha_k(t)z\right|_{H^1(Q_1)} \geq C\left|\left(z^0, z^1\right)\right|_{F'}, \quad \forall\left(z^0, z^1\right) \in F'. \tag{3.2}$$

In the following, we shall give a proof of (3.2) by the multiplier method.

Define the following weighted energy for (3.1):

$$E(t) = \frac{1}{2} \int_0^1 \left[\alpha_k(t)\left|z_t(x,t)\right|^2 + \beta_k(x,t)\left|z_x(x,t)\right|^2\right] dx \quad \text{for } t \geq 0,$$

It follows that

$$E_0 \triangleq E(0) = \frac{1}{2} \int_0^1 \left[\left|z^1(x)\right|^2 + \beta_k(x,0)\left|z_x^0(x)\right|^2\right] dx.$$

We obtain the following lemma (see the detailed proof in [8]).

Lemma 3.1 *For any $(z^0, z^1) \in H_0^1(0,1) \times L^2(0,1)$ and $t \in [0,T]$, any solution z of (3.1) satisfies the following estimate*:

$$E(t) = \frac{1}{\alpha_k(t)} E_0. \tag{3.3}$$

Equation (3.2) is derived with a special multiplier. Set

$$F(x,k) = \frac{\beta_{k,x}(x,t)}{\beta_k(x,t)} = \frac{-2k^2 x}{1 - k^2 x^2} \leq 0, \quad (x,k) \in [0,1] \times (0,1).$$

It is easy to check $F_x(x,k) < 0, (x,k) \in [0,1] \times (0,1)$. We have

$$F(1,k) = \frac{-2k^2}{1 - k^2} \leq F(x,k), \quad (x,k) \in [0,1] \times (0,1).$$

We see $F_k(1,k) = \frac{-4k}{(1-k^2)^2} < 0, k \in (0,1)$. Therefore, we obtain, for any $\eta > 0$,

$$F_0 \triangleq F(1, 1 - \eta) = \frac{-2(1-\eta)^2}{1 - (1-\eta)^2} \leq F(1,k), \quad k \in (0, 1-\eta].$$

Hence, we derive

$$F_0 \leq F(1,k) \leq F(x,k) \leq 0, \quad (x,k) \in [0,1] \times (0, 1-\eta). \tag{3.4}$$

Assume that $\lambda \in (0,1)$ and a point $x_0 \in (m,n)$ to be unknown for now. We require the multiplier to satisfy the following lemma.

Lemma 3.2 *Assume that $p(x)$ be a solution of first-order linear differential equation*

$$
p'(x) = \begin{cases} \lambda - 1, & x \in (0, m), \\ \lambda, & x \in (m, x_0), \\ \lambda + F_0 p(x), & x \in (x_0, n), \\ \lambda - 1 + F_0 p(x), & x \in (n, 1), \end{cases}
\tag{3.5}
$$

then there exist a unique $\lambda \in (0,1)$ and a unique $x_0 \in (m, n)$ such that $p(x)$ belongs to $C[0,1]$ and satisfies

$$
p(0) = p(1) = p(x_0) = 0.
\tag{3.6}
$$

Proof By (3.5), (3.6) and the constant variation method, it follows that

$$
p(x) = \begin{cases} (\lambda - 1)x \le 0, & x \in (0, m), \\ (x - x_0)\lambda \le 0, & x \in (m, x_0), \\ \frac{\lambda}{F_0}[e^{F_0(x - x_0)} - 1] \ge 0, & x \in (x_0, n), \\ \frac{(\lambda - 1)}{F_0}[e^{F_0(x - 1)} - 1] \ge 0, & x \in (n, 1). \end{cases}
$$

Note that $p(x) \in C[0,1]$, we have

$$
p(m - 0) = p(m + 0), \qquad p(n - 0) = p(n + 0).
$$

From this, we obtain

$$
\lambda \triangleq \lambda_- = \frac{m}{x_0}, \quad x_0 \in (m, n),
\tag{3.7}
$$

$$
\lambda \triangleq \lambda_+ = \frac{e^{F_0(n-1)} - 1}{e^{F_0(n-1)} - e^{F_0(n - x_0)}}, \quad x_0 \in (m, n).
\tag{3.8}
$$

By (3.7) and (3.8), we have λ_- and λ_+ are monotone decreasing and increasing with respect to x_0 and satisfy

$$
\lambda_-, \lambda_+ \in (0, 1); \qquad \lambda_-(m) > \lambda_+(m); \qquad \lambda_-(n) < \lambda_+(n).
$$

Hence there exists a unique $x_0 \in (m, n)$ of the equation $\lambda_- = \lambda_+$, the corresponding value

$$
\lambda = \lambda_- = \lambda_+ \in (0, 1). \qquad \square
$$

Remark 3.1 It is easy to verify that

$$
M \triangleq \max_{0 \le x \le 1} |p(x)| = \max\{|p(m)|, p(n)\}.
$$

In the following, we prove (3.2) by the above multiplier $p(x)$. Multiplying the first equation of (3.1) by qz_x and integrating on Q, we have

$$
\begin{aligned}
0 &= \int_0^T \int_0^1 \alpha_k(t) z_{tt}(x,t) p(x) z_x(x,t)\, dx\, dt - \int_0^T \int_0^1 \big[\beta_k(x,t) z_x(x,t)\big]_x p(x) z_x(x,t)\, dx\, dt \\
&\quad + \int_0^T \int_0^1 \gamma_k(x) z_{tx}(x,t) p(x) z_x(x,t)\, dx\, dt \\
&\triangleq D_1 + D_2 + D_3.
\end{aligned}
$$

In the following, we calculate the above three integrals D_i ($i = 1, 2, 3$), respectively. It is easy to check that

$$
\begin{aligned}
D_1 &= \int_0^1 \alpha_k(t) p(x) z_t(x,t) z_x(x,t)\, dx \Big|_0^T \\
&\quad - \int_0^T \int_0^1 \big[\alpha_{k,t}(t) p(x) z_t(x,t) z_x(x,t) + \alpha_k(t) p(x) z_t(x,t) z_{tx}(x,t)\big]\, dx\, dt \\
&= \int_0^1 \alpha_k(t) p(x) z_t(x,t) z_x(x,t)\, dx \Big|_0^T \\
&\quad - \int_0^T \int_0^1 \alpha_{k,t}(t) p(x) z_t(x,t) z_x(x,t)\, dx\, dt \\
&\quad + \frac{1}{2} \int_0^T \int_0^1 \alpha_k(t) p_x(x) \big|z_t(x,t)\big|^2\, dx\, dt, \hspace{3cm} (3.9)
\end{aligned}
$$

$$
\begin{aligned}
D_2 &= - \int_0^T \beta_k(x,t) p(x) \big|z_x(x,t)\big|^2\, dt \Big|_0^1 \\
&\quad + \int_0^T \int_0^1 \big[\beta_k(x,t) p_x(x) \big|z_x(x,t)\big|^2 + \beta_k(x,t) p(x) z_x(x,t) z_{xx}(x,t)\big]\, dx\, dt \\
&= - \int_0^T \beta_k(x,t) p(x) \big|z_x(x,t)\big|^2\, dt \Big|_0^1 \\
&\quad + \int_0^T \int_0^1 \beta_k(x,t) p_x(x) \big|z_x(x,t)\big|^2\, dx\, dt \\
&\quad + \frac{1}{2} \int_0^T \beta_k(x,t) p(x) \big|z_x(x,t)\big|^2\, dt \Big|_0^1 \\
&\quad - \frac{1}{2} \int_0^T \int_0^1 \big[\beta_k(x,t) p(x)\big]_x \big|z_x(x,t)\big|^2\, dx\, dt \\
&= \frac{1}{2} \int_0^T \int_0^1 \big[\beta_k(x,t) p_x(x) \big|z_x(x,t)\big|^2 - \beta_{k,x}(x,t) p(x) \big|z_x(x,t)\big|^2\big]\, dx\, dt \\
&= \frac{1}{2} \int_0^T \int_0^1 \left[\frac{p(x)}{\beta_k(x,t)}\right]_x \big[\beta_k(x,t)\big]^2 \big|z_x(x,t)\big|^2\, dx\, dt \hspace{2cm} (3.10)
\end{aligned}
$$

and

$$
D_3 = \frac{1}{2} \int_0^1 \gamma_k(x) p(x) \big|z_x(x,t)\big|^2\, dx \Big|_0^T. \hspace{4cm} (3.11)
$$

Write

$$i(t) \triangleq \int_0^1 \left[\alpha_k(t)p(x)z_t(x,t)z_x(x,t) + \frac{1}{2}\gamma_k(x)p(x)|z_x(x,t)|^2 \right] dx. \tag{3.12}$$

By (3.9)-(3.12), we obtain

$$\frac{1}{2}\int_0^T \int_0^1 \left[\alpha_k(t)p_x(x)|z_t(x,t)|^2 + \left[\frac{p(x)}{\beta_k(x,t)} \right]_x [\beta_k(x,t)]^2 |z_x(x,t)|^2 \right] dx\,dt$$
$$= \int_0^T \int_0^1 \alpha_{k,t}(t)p(x)z_t(x,t)z_x(x,t)\,dx\,dt + i(0) - i(T). \tag{3.13}$$

By Lemma 3.2, it follows that

$$p_x(x) = \lambda - 1, \quad x \in [0,m]$$

and

$$p(x) \le 0, \quad x \in [0,m].$$

Then we have

$$F(x,k)p(x) = \frac{\beta_{k,x}(x,t)}{\beta_k(x,t)}p(x) \ge 0, \quad (x,t) \in [0,m] \times [0,T]$$

and

$$p_x(x) \le \frac{\beta_{k,x}(x,t)}{\beta_k(x,t)}p(x) + \lambda - 1, \quad (x,t) \in [0,m] \times [0,T].$$

From this, we have

$$p_x(x)\beta_k(x,t) - \beta_{k,x}(x,t)p(x) \le (\lambda - 1)\beta_k(x,t), \quad (x,t) \in [0,m] \times [0,T]. \tag{3.14}$$

Similarly, we obtain

$$p_x(x)\beta_k(x,t) - \beta_{k,x}(x,t)p(x) \le \lambda\beta_k(x,t), \quad (x,t) \in [m,x_0] \times [0,T]; \tag{3.15}$$

for any $\eta > 0$, $k \in (0, 1-\eta]$,

$$p_x(x)\beta_k(x,t) - \beta_{k,x}(x,t)p(x) \le \lambda\beta_k(x,t), \quad (x,t) \in [x_0,n] \times [0,T], \tag{3.16}$$

and

$$p_x(x)\beta_k(x,t) - \beta_{k,x}(x,t)p(x) \le (\lambda - 1)\beta_k(x,t), \quad (x,t) \in [n,1] \times [0,T]. \tag{3.17}$$

By (3.13)-(3.17), it follows that for any $\eta > 0$, $k \in (0, 1-\eta]$,

$$\frac{\lambda}{2}\int_0^T \int_m^n \left[\alpha_k(t)|z_t(x,t)|^2 + \beta_k(x,t)|z_x(x,t)|^2 \right] dx\,dt$$

$$+ \frac{\lambda - 1}{2} \int_0^T \int_{(0,1)\backslash(m,n)} \left[\alpha_k(t) \left| z_t(x,t) \right|^2 + \beta_k(x,t) \left| z_x(x,t) \right|^2 \right] dx \, dt$$

$$\geq \int_0^T \int_0^1 \alpha_{k,t}(t) p(x) z_t(x,t) z_x(x,t) \, dx \, dt + i(0) - i(T).$$

Therefore, we have, for any $\eta > 0$, $k \in (0, 1 - \eta]$,

$$\frac{1}{2} \int_0^T \int_m^n \left[\alpha_k(t) \left| z_t(x,t) \right|^2 + \beta_k(x,t) \left| z_x(x,t) \right|^2 \right] dx \, dt$$

$$\geq \frac{1 - \lambda}{2} \int_0^T \int_0^1 \left[\alpha_k(t) \left| z_t(x,t) \right|^2 + \beta_k(x,t) \left| z_x(x,t) \right|^2 \right] dx \, dt$$

$$+ \int_0^T \int_0^1 k p(x) z_t(x,t) z_x(x,t) \, dx \, dt + i(0) - i(T). \tag{3.18}$$

For each $t \in [0, T]$ and $\varepsilon > 0$, we have

$$\left| i(t) \right| = \left| \int_0^1 \left[\alpha_k(t) q(x) z_t(x,t) z_x(x,t) + \frac{1}{2} \gamma_k(x) q(x) \left| z_x(x,t) \right|^2 \right] dx \right|$$

$$\leq \left| \int_0^1 \alpha_k(t) q(x) z_t(x,t) z_x(x,t) \, dx \right| + k \left| \int_0^1 x q(x) \left| z_x(x,t) \right|^2 dx \right|$$

$$\leq \sqrt{1 + kt} \left[\frac{1}{2\varepsilon} \int_0^1 \alpha_k(t) \left| z_t(x,t) \right|^2 dx + \frac{\varepsilon}{2} \int_0^1 q^2(x) \left| z_x(x,t) \right|^2 dx \right]$$

$$+ kM \int_0^1 \left| z_x(x,t) \right|^2 dx$$

$$\leq \frac{\sqrt{1 + kt}}{2\varepsilon} \int_0^1 \alpha_k(t) \left| z_t(x,t) \right|^2 dx + \frac{\sqrt{1 + kt}\,\varepsilon M^2 + 2kM}{2} \int_0^1 \left| z_x(x,t) \right|^2 dx$$

$$\leq \frac{\sqrt{1 + kt}}{\varepsilon} \frac{1}{2} \int_0^1 \alpha_k(t) \left| z_t(x,t) \right|^2 dx$$

$$+ \frac{(\sqrt{1 + kt}\,\varepsilon M^2 + 2kM)(1 + kt)}{1 - k^2} \frac{1}{2} \int_0^1 \beta_k(x,t) \left| z_x(x,t) \right|^2 dx.$$

Take $\varepsilon = \frac{1-k}{\sqrt{1+ktM}}$, then it is easy to check that

$$\varepsilon > 0 \quad \text{and} \quad \frac{\sqrt{1 + kt}}{\varepsilon} = \frac{(\sqrt{1 + kt}\,\varepsilon M^2 + 2kM)(1 + kt)}{1 - k^2} = \frac{M(1 + kt)}{1 - k}.$$

This implies that, for any $t \in [0, T]$,

$$\left| i(t) \right| \leq \frac{M(1 + kt)}{1 - k} E(t) = \frac{M}{1 - k} E_0.$$

It follows that

$$\left| \int_0^1 \left[\alpha_k(t) q(x) z_t(x,t) z_x(x,t) + \frac{1}{2} \gamma_k(x) q(x) \left| z_x(x,t) \right|^2 \right] dx \Big|_0^T \right| \leq \frac{2M}{1 - k} E_0. \tag{3.19}$$

For each $\varepsilon \in (0, 1 - \lambda)$, we have

$$\left| \int_0^T \int_0^1 kp(x)z_t(x,t)z_x(x,t)\,dx\,dt \right|$$

$$= \left| \int_0^T \int_0^1 \sqrt{1+kt}z_t(x,t)\frac{\sqrt{1-k^2x^2}}{\sqrt{1+kt}}z_x(x,t)\frac{kp(x)}{\sqrt{1-k^2x^2}}\,dx\,dt \right|$$

$$\leq \frac{\varepsilon}{2}\int_0^T \int_0^1 \alpha_k(t)|z_t(x,t)|^2\,dx\,dt$$

$$+ \frac{1}{2\varepsilon}\int_0^T \int_0^1 \beta_k(x,t)|z_x(x,t)|^2\frac{k^2p^2(x)}{1-k^2x^2}\,dx\,dt. \tag{3.20}$$

Define

$$G(x) = \frac{k^2p^2(x)}{1-k^2x^2}, \quad x \in [0,1].$$

We see

$$M_1 = \max_{x\in[0,1]} G(x) = \max\{G(m), G(n)\} = \max\left\{ \frac{k^2p^2(m)}{1-k^2m^2}, \frac{k^2p^2(n)}{1-k^2n^2} \right\}.$$

By (3.20), we obtain, for each $\varepsilon \in (0, 1 - \lambda)$,

$$\left| \int_0^T \int_0^1 kp(x)z_t(x,t)z_x(x,t)\,dx\,dt \right|$$

$$\leq \frac{\varepsilon}{2}\int_0^T \int_0^1 \alpha_k(t)|z_t(x,t)|^2\,dx\,dt + \frac{M_1}{2\varepsilon}\int_0^T \int_0^1 \beta_k(x,t)|z_x(x,t)|^2\,dx\,dt. \tag{3.21}$$

Take $\varepsilon = \sqrt{M_1} < 1 - \lambda$, then it is easy to check that

$$\varepsilon = \frac{M_1}{\varepsilon} = \sqrt{M_1} < 1 - \lambda,$$

i.e.,

$$\max\left\{ \frac{k^2p^2(m)}{1-k^2m^2}, \frac{k^2p^2(n)}{1-k^2n^2} \right\} < (1-\lambda)^2.$$

From the above inequality, it follows that

$$k \in \left(0, \min\left\{ \frac{1-\lambda}{\sqrt{p^2(m)+(1-\lambda)^2m^2}}, \frac{1-\lambda}{\sqrt{p^2(n)+(1-\lambda)^2n^2}} \right\} \right).$$

From (3.21), we get

$$k \in \left(0, \min\left\{ \frac{1-\lambda}{\sqrt{p^2(m)+(1-\lambda)^2m^2}}, \frac{1-\lambda}{\sqrt{p^2(n)+(1-\lambda)^2n^2}} \right\} \right),$$

$$\left| \int_0^T \int_0^1 kp(x)z_t(x,t)z_x(x,t)\,dx\,dt \right| \leq \sqrt{M_1}\int_0^T E(t)\,dt. \tag{3.22}$$

Write

$$\tilde{k} = \min\left\{ \frac{1-\lambda}{\sqrt{p^2(m) + (1-\lambda)^2 m^2}}, \frac{1-\lambda}{\sqrt{p^2(n) + (1-\lambda)^2 n^2}}, 1-\eta \right\}.$$

By (3.3), (3.18), (3.20) and (3.22), we derive, for each $k \in (0, \tilde{k})$,

$$
\begin{aligned}
\frac{1}{2} \int_0^T \int_m^n &\left[\alpha_k(t) |z_t(x,t)|^2 + \beta_k(x,t) |z_x(x,t)|^2 \right] \\
&\geq (1 - \lambda - \sqrt{M_1}) \int_0^T E(t)\,dt - \frac{2M}{1-k} E_0 \\
&= \left[\frac{1 - \lambda - \sqrt{M_1}}{k} \ln(1 + kT) - \frac{2M}{1-k} \right] E_0.
\end{aligned}
\tag{3.23}
$$

Set

$$T_k^* = \frac{e^{\frac{2kM}{(1-\lambda-\sqrt{M_1})(1-k)}} - 1}{k}.$$

If $T > T_k^*$, we have $\frac{1-\lambda-\sqrt{M_1}}{k} \ln(1 + kT) - \frac{2M}{1-k} > 0$. Also,

$$
\begin{aligned}
\frac{1}{2} \int_0^T \int_m^n &\left[\alpha_k(t) |z_t(x,t)|^2 + \beta_k(x,t) |z_x(x,t)|^2 \right] \\
&\geq \left[\frac{1 - \lambda - \sqrt{M_1}}{k} \ln(1 + kT) - \frac{2M}{1-k} \right] E_0 \\
&\geq C \left[\frac{1 - \lambda - \sqrt{M_1}}{k} \ln(1 + kT) - \frac{2M}{1-k} \right] \left(|z^0|_{H_0^1(0,1)}^2 + |z^1|_{L^2(0,1)}^2 \right).
\end{aligned}
\tag{3.24}
$$

Equation (3.2) is deduced by (3.24).

Remark 3.2 We can finally check that

$$T^0 \triangleq \lim_{k \to 0} T_k^* = 2\max\{m, 1-n\}.$$

It is well known that (1.2) in the cylindrical domain is interiorly controllable at any time $T > T^0$. However, we do not know whether T_k^* is sharp.

4 Conclusions

In this paper, we consider interior controllability for a one-dimensional wave equation in a domain with moving boundary. When the speed of the moving endpoint is less than a certain constant which is less than the characteristic speed, we obtain exact controllability for this equation. In the future, we hope that we may consider controllability problem of wave equations with free boundary.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors read and approved the final manuscript.

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