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Asymptotic behavior and numerical simulations of a Lotka-Volterra mutualism system with white noises

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Abstract

In the present paper, a stochastic mutualism model subject to white noises is established. We first investigate the existence and uniqueness of globally positive solution of the stochastic model. Then we study its asymptotic behavior, such as stochastic permanence and extinction, and estimate the limit of the average in time of the sample paths of every component. We also show that the stochastic system is globally attractive under some appropriate conditions. Finally, numerical simulations are presented to justify the analytical results.

Keywords: mutualism system; white noise; stochastic permanence; extinction; global attractivity

1 Introduction

The relationship between organisms and their living environments is very close in the nature, a small change of ecological environment usually could lead to a great influence on the organisms. On the other hand, there exist many complicated interactions among various species in the biological communities, and these phenomena, such as competition, predation and mutualism, are extensive. Through the years, more and more mathematical models have been used to describe the relationship between species and their living environments in ecology. The models mainly study the population dynamic behaviors, for example, permanence, extinction, global attractivity, etc. They have long been one of the popular themes in mathematical biology due to their universal existence and importance. As far as we know, many relevant papers and monographs on population dynamics have been reported, and many important results can be found in Refs. [1–9]. In this paper, we are concerned with a Lotka-Volterra mutualism system expressed by

$$\begin{cases} x_1'(t) = x_1(t)[r_1 - a_1x_1(t) + \frac{c_2x_2(t)}{b_2+x_2(t)}], \\ x_2'(t) = x_2(t)[r_2 - a_2x_2(t) + \frac{c_1x_1(t)}{b_1+x_1(t)}], \end{cases} \tag{1.1}$$

where $x_1(t)$, $x_2(t)$ denote the population size, the positive coefficients r_1, r_2 and a_1, a_2 are the intrinsic growth rate and self-inhibition rate, respectively, positive coefficients c_j measure the interspecific mutualism effects of species x_j on species x_i ($i,j = 1,2$, $i \neq j$), and b_1, b_2 are positive control constants.

It is well known that many factors may affect the population dynamic behaviors of biology, for instance, the fluctuating environment, delays, victuals and population density. Especially, the ecological systems in practical world are often perturbed by various types of environmental noises. Just as May and Allen show that the birth rate, the death rate and other parameters usually show random fluctuation to a certain extent due to environmental fluctuation (see Ref. [5]). In fact, species undergoing environmental noises is also one of the most prevalent phenomena in the nature. Mao points out that a reasonable mathematical interpretation for the noise is the so-called white noise $\dot{B}(t)$, which is formally regarded as the derivative of the Brownian motion $B(t)$, i.e., $\dot{B}(t) = dB(t)/dt$ (see Ref. [6]). Hence it might happen that $r_i(t)$ are not completely known but subject to some environmental noises. In other words, $r_i(t)$ could be estimated by an average value plus an error term

$$r_i(t) \to r_i(t) + \sigma_i(t)\dot{B}_i(t), \quad i = 1, 2,$$

where $\sigma_i^2(t)$ represent the intensities of the noises, $B_i(t)$ are the independent standard Brownian motions defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, P)$ with a filtration $\{\mathcal{F}\}_{t\geq 0}$ satisfying the usual conditions. In this sense, such systems subject to environmental white noises tend to be more suitably modeled by stochastic differential equations. Therefore, what is of most interest for our present purposes is the modification of considering the possible effects of environmental white noises for system (1.1) and establishing the stochastic mutualism model

$$\begin{cases} dx_1(t) = x_1(t)[r_1 - a_1x_1(t) + \frac{c_2x_2(t)}{b_2+x_2(t)}]\,dt + \sigma_1x_1(t)\,dB_1(t), \\ dx_2(t) = x_2(t)[r_2 - a_2x_2(t) + \frac{c_1x_1(t)}{b_1+x_1(t)}]\,dt + \sigma_2x_2(t)\,dB_2(t), \end{cases} \tag{1.2}$$

with the initial value $x(0) \in R_+^2$. In recent years, stochastic differential equations have been widely used in population biology because they could accurately characterize some realistic processes by means of stochastic models. Many interesting and valuable results including extinction, persistence and stability can be found in Refs. [10–14]. But to our best knowledge, there exist few published papers concerning system (1.2).

Motivated by the existing results, our contribution is as follows:

- We introduce the white noise to model the evolution of a mutualism system.
- We investigate the existence and uniqueness of globally positive solution, which shows that the positive solution of system (1.2) could not explode to infinity at any finite time.
- We estimate the limit of the average in time of the sample paths of every component of the positive solution.
- We derive sufficient conditions for stochastic permanence, extinction and global attractivity, and the corresponding numerical simulations are provided.

2 Preliminaries

In this section, we shall state some definitions and lemmas which will be useful for establishing our main results.

Definition 2.1 System (1.2) is said to be extinct exponentially with probability one if for any initial condition $x(0) \in R_+^2$, the solution $x(t) = (x_1(t), x_2(t))$ satisfies $\limsup_{t\to+\infty} \ln x_i(t)/t < 0$, $i = 1, 2$ a.s.

Definition 2.2 System (1.2) is said to be stochastically permanent if for every $\varepsilon \in (0,1)$, there exists a pair of positive constants α, β such that for any initial condition $x(0) \in R_+^2$, the solution $x(t)$ satisfies

$$\liminf_{t\to+\infty} P\{|x(t)| \geq \alpha\} \geq 1-\varepsilon, \qquad \liminf_{t\to+\infty} P\{|x(t)| \leq \beta\} \geq 1-\varepsilon.$$

Definition 2.3 Let $x(t) = (x_1(t), x_2(t))$, $u(t) = (u_1(t), u_2(t))$ be any two solutions of system (1.2) with initial conditions $x(0) \in R_+^2$, $u(0) \in R_+^2$, respectively. If $\lim_{t\to+\infty} |x_i(t) - u_i(t)| = 0$ a.s., $i = 1, 2$, then system (1.2) is said to be globally attractive.

Lemma 2.1 *Suppose that* $a_1, a_2, \ldots, a_n$ *are real numbers, then the following inequality*

$$|a_1 + a_2 + \cdots + a_n|^p \leq C_p\big(|a_1|^p + |a_2|^p + \cdots + |a_n|^p\big)$$

holds, where $p > 0$ *and*

$$C_p = \begin{cases} 1, & 0 < p \leq 1, \\ n^{p-1}, & p > 1. \end{cases}$$

Lemma 2.2 ([15]) *Assume that an n-dimensional stochastic process* $X(t)$ *on* $t \geq 0$ *satisfies the condition*

$$E|X(t) - X(s)|^{\eta} \leq \mu|t-s|^{1+\varsigma}, \quad 0 \leq s, t < \infty,$$

for positive constants η, ς, μ. *Then there exists a continuous version* $\tilde{X}(t)$ *of* $X(t)$ *which has the property that for every* $\vartheta \in (0, \varsigma/\eta)$, *there is a positive random variable* $\psi(\omega)$ *such that*

$$\mathbb{P}\left\{\omega : \sup_{0<|t-s|<\psi(\omega), 0\leq s,t<\infty} \frac{|\tilde{X}(t,\omega) - X(t,\omega)|}{|t-s|^{\vartheta}} \leq \frac{2}{1-2^{-\vartheta}}\right\} = 1.$$

In other words, almost every sample path of $\tilde{X}(t)$ *is locally but uniformly Hölder continuous with exponent* ϑ.

Lemma 2.3 ([16]) *Let* $f(t)$ *be a nonnegative function on* $t \geq 0$ *such that* $f(t)$ *is integrable on* $t \geq 0$ *and is uniformly continuous on* $t \geq 0$. *Then* $\lim_{t\to+\infty} f(t) = 0$.

3 Existence and uniqueness of global solution

To begin with, we first show the existence and uniqueness of global solution on system (1.2) which is fundamental in the present paper.

Theorem 3.1 *For initial value* $x(0) = (x_1(0), x_2(0)) \in R_+^2$, *there is a unique solution* $x(t)$ *to system* (1.2) *for all* $t \geq 0$, *and* $x(t)$ *will remain in* R_+^2 *with probability one.*

Proof The proof of this theorem is standard. It is obvious that the coefficients of system (1.2) satisfy the local Lipschitz condition, then for any given initial value $x(0) = (x_1(0), x_2(0)) \in R_+^2$, there exists a unique local solution $(x_1(t), x_2(t))$ on $[0, \tau_e)$, where τ_e is the explosion time. To show that the positive solution is global, we only need to show that $\tau_e = +\infty$ a.s.

Let n_0 be sufficiently large such that every component of $x(0)$ remains in the interval $[\frac{1}{n_0}, n_0]$. For each integer $n \geq n_0$, we define the stopping time

$$\tau_n = \inf\left\{ t \in [0, \tau_e) : x_1(t) \notin \left(\frac{1}{n}, n\right) \text{ or } x_1(t) \notin \left(\frac{1}{n}, n\right) \right\}.$$

Here we set $\inf \emptyset = +\infty$ ($\emptyset$ denotes the empty set). Obviously, τ_n is increasing as $n \to +\infty$. Assign $\tau_{+\infty} = \lim_{n\to+\infty} \tau_n$, whence $\tau_{+\infty} \leq \tau_e$ a.s., if we can show that $\tau_{+\infty} = +\infty$ a.s., then $\tau_e = +\infty$ a.s. and $x(0) = (x_1(0), x_2(0)) \in R_+^2$ a.s. for all $t \geq 0$.

To complete the proof, we only need to show that $\tau_{+\infty} = +\infty$ a.s. By reduction to absurdity, we assume that there exists a pair of constants $T > 0$ and $\varepsilon \in (0, 1)$ such that

$$P\{\tau_{+\infty} < T\} > \varepsilon.$$

As a result, there is an integer $n_1 \geq n_0$ such that for $n \geq n_1$,

$$P\{\tau_n \leq T\} \geq \varepsilon.$$

Define a C^2-function $\widetilde{V}(x) : R_+^2 \to R_+$ as

$$\widetilde{V}(x_1, x_2) = x_1 - 1 - \ln x_1 + x_2 - 1 - \ln x_2.$$

The nonnegativity of this function can be seen from $y - 1 - \ln y \geq 0$ for $y > 0$. Using Itô's formula one can show that

$$\begin{aligned}
d\widetilde{V}(x_1, x_2) &= \left(1 - \frac{1}{x_1}\right) dx_1 + 0.5\frac{1}{x_1^2}(dx_1)^2 + \left(1 - \frac{1}{x_2}\right) dx_1 + 0.5\frac{1}{x_2^2}(dx_2)^2 \\
&= (x_1 - 1)\left(r_1 - a_1 x_1 + \frac{c_2 x_2}{b_2 + x_2}\right) dt + (x_1 - 1)\sigma_1\, dB_1(t) + \frac{1}{2}\sigma_1^2\, dt \\
&\quad + (x_2 - 1)\left(r_2 - a_2 x_2 + \frac{c_1 x_1}{b_1 + x_1}\right) dt + (x_2 - 1)\sigma_2\, dB_2(t) + \frac{1}{2}\sigma_2^2\, dt \\
&= \left\{ -a_1 x_1^2 + (r_1 + a_1)x_1 + \frac{c_2 x_1 x_2}{b_2 + x_2} - \frac{c_2 x_2}{b_2 + x_2} + \frac{1}{2}\sigma_1^2 - r_1 \right. \\
&\quad \left. - a_2 x_2^2 + (r_2 + a_2)x_2 + \frac{c_1 x_1 x_2}{b_1 + x_1} - \frac{c_1 x_1}{b_1 + x_1} + \frac{1}{2}\sigma_2^2 - r_2 \right\} dt \\
&\quad + (x_1 - 1)\sigma_1\, dB_1(t) + (x_2 - 1)\sigma_2\, dB_2(t) \\
&\leq \left\{ -a_1 x_1^2 + (r_1 + a_1 + c_2)x_1 + \frac{1}{2}\sigma_1^2 - r_1 - a_2 x_2^2 + (r_2 + a_2 + c_1)x_2 \right. \\
&\quad \left. + \frac{1}{2}\sigma_2^2 - r_2 \right\} dt + (x_1 - 1)\sigma_1\, dB_1(t) + (x_2 - 1)\sigma_2\, dB_2(t) \\
&= G(x_1, x_2)\, dt + (x_1 - 1)\sigma_1\, dB_1(t) + (x_2 - 1)\sigma_2\, dB_2(t),
\end{aligned} \tag{3.1}$$

where

$$G(x_1, x_2) = -a_1 x_1^2 + (r_1 + a_1 + c_2)x_1 + \frac{1}{2}\sigma_1^2 - r_1 - a_2 x_2^2 + (r_2 + a_2 + c_1)x_2 + \frac{1}{2}\sigma_2^2 - r_2.$$

Clearly, $G(x_1,x_2)$ is upper bounded, denoted by K. So we have

$$d\widetilde{V}(x_1,x_2) \le K\,dt + (x_1-1)\sigma_1\,dB_1(t) + (x_2-1)\sigma_2\,dB_2(t).$$

Integrating both sides from 0 to $\tau_k \wedge T$ yields

$$\int_0^{\tau_k\wedge T} d\widetilde{V}(x_1,x_2) \le \int_0^{\tau_k\wedge T} K\,dt + \int_0^{\tau_k\wedge T} (x_1-1)\sigma_1\,dB_1(t) + \int_0^{\tau_k\wedge T} (x_2-1)\sigma_2\,dB_2(t).$$

Taking expectations leads to

$$\begin{aligned} E\bigl[\widetilde{V}\bigl(x_1(\tau_k\wedge T),x_2(\tau_k\wedge T)\bigr)\bigr] &\le \widetilde{V}\bigl(x_1(0),x_2(0)\bigr) + KE(\tau_k\wedge T) \\ &\le \widetilde{V}\bigl(x_1(0),x_2(0)\bigr) + KT. \end{aligned} \tag{3.2}$$

Let $\Omega_n = \{\tau_n \le T\}$ for $n \ge n_1$, then $P(\Omega_n) \ge \varepsilon$. Note that for arbitrary $\omega \in \Omega_n$, there exist some i such that $x_i(\tau_n,\omega)$ equals either n or $\frac{1}{n}$, and thus $\widetilde{V}(x(\tau_n,\omega))$ is no less than either

$$(n-1-\ln n) \wedge \left(\frac{1}{n} - 1 + \ln n\right).$$

It then follows from (3.2) that

$$\begin{aligned} \widetilde{V}\bigl(x_1(0),x_2(0)\bigr) + KT &\ge E\bigl[1_{\Omega_n}(\omega)\widetilde{V}\bigl(x_1(\tau_n,\omega),x_2(\tau_n,\omega)\bigr)\bigr] \\ &\ge \varepsilon\left\{(n-1-\ln n) \wedge \left(\frac{1}{n} - 1 + \ln n\right)\right\}, \end{aligned}$$

where 1_{Ω_n} is the indicator function of Ω_n. Letting $n \to +\infty$ leads to the contradiction

$$+\infty > \widetilde{V}\bigl(x_1(0),x_2(0)\bigr) + KT = +\infty.$$

So we must have $\tau_{+\infty} = +\infty$ a.s.

This completes the proof of Theorem 3.1. □

4 Stochastic permanence and extinction

In this section, we prove that the pth moment of the solution of system (1.2) is upper bounded, and then discuss the stochastic permanence and extinction.

Lemma 4.1 *Assign $p > 1$, then there exists a positive constant $K(p)$ such that the solution $x(t)$ of system* (1.2) *has the following property*:

$$\limsup_{t\to+\infty} E\bigl|x(t)\bigr|^p \le K(p).$$

Proof Assign $p > 1$, we define $V(x_1,x_2) = x_1^p + x_2^p$. By Itô's formula, we have

$$\begin{aligned} dV(x_1,x_2) &= px_1^p\,dx_1 + 0.5p(p-1)x_1^{p-2}(dx_1)^2 + px_2^{p-1}\,dx_2 + 0.5p(p-1)x_2^{p-2}(dx_2)^2 \\ &= px_1^p\left(r_1 - a_1x_1 + \frac{c_2x_2}{b_2+x_2}\right)dt + p\sigma_1x_1^p\,dB_1(t) + 0.5p(p-1)\sigma_1^2x_1^p\,dt \end{aligned}$$

$$
\begin{aligned}
&+ px_2^p\left(r_2 - a_2x_2 + \frac{c_1x_1}{b_1 + x_1}\right)dt + p\sigma_2x_2^p\,dB_2(t) + 0.5p(p-1)\sigma_2^2x_2^p\,dt \\
&= px_1^p\left(r_1 - a_1x_1 + \frac{c_2x_2}{b_2 + x_2} + 0.5(p-1)\sigma_1^2\right)dt + p\sigma_1x_1^p\,dB_1(t) + x_1^p\,dt \\
&\quad + px_2^p\left(r_2 - a_2x_2 + \frac{c_1x_1}{b_1 + x_1} + 0.5(p-1)\sigma_2^2\right)dt + p\sigma_2x_2^p\,dB_2(t) + x_2^p\,dt \\
&\quad - V(x_1,x_2)\,dt \\
&\le x_1^p\big(1 + r_1p + c_2p + 0.5p(p-1)\sigma_1^2 - a_1px_1\big)\,dt + p\sigma_1x_1^p\,dB_1(t) \\
&\quad + x_2^p\big(1 + r_2p + c_1p + 0.5p(p-1)\sigma_2^2 - a_2px_2\big)\,dt + p\sigma_2x_2^p\,dB_2(t) - V(x_1,x_2) \\
&= \big(L(x_1,x_2) - V(x_1,x_2)\big)\,dt + p\sigma_1x_1^p\,dB_1(t) + p\sigma_2x_2^p\,dB_2(t), \qquad (4.1)
\end{aligned}
$$

where

$$
\begin{aligned}
L(x_1,x_2) = {}& x_1^p\big(1 + r_1p + c_2p + 0.5p(p-1)\sigma_1^2 - a_1px_1\big) \\
&+ x_2^p\big(1 + r_2p + c_1p + 0.5p(p-1)\sigma_2^2 - a_2px_2\big).
\end{aligned}
$$

Since $x_i^p(1 + r_ip + c_jp + 0.5p(p-1)\sigma_i^2 - a_ipx_i)$ has a max value $\frac{p(1+r_ip+c_jp+0.5p(p-1)\sigma_i^2)}{(p+1)a_ip}$, let

$$
K^*(p) = \sum_{i=1}^{2}\frac{p(1 + r_ip + c_jp + 0.5p(p-1)\sigma_i^2)}{(p+1)a_ip} > 0,
$$

then $K^*(p)$ is the upper bound of $L(x_1,x_2)$. Recalling (4.1) and integrating on both sides yield

$$
\begin{aligned}
\int_0^t dV\big(x_1(s),x_2(s)\big) \le {}& \int_0^t \big(K^*(p) - V\big(x_1(s),x_2(s)\big)\big)\,ds + \int_0^t p\sigma_1x_1^p(s)\,dB_1(s) \\
&+ \int_0^t p\sigma_2x_2^p(s)\,dB_2(s). \qquad (4.2)
\end{aligned}
$$

Taking expectation on both sides yields

$$
E\big[V\big(x_1(t),x_2(t)\big)\big] \le V\big(x_1(0),x_2(0)\big) + \int_0^t K^*(p) - E\big[V\big(x_1(s),x_2(s)\big)\big]\,ds. \qquad (4.3)
$$

It then follows from Gronwall's inequality that

$$
E\big[V\big(x_1(t),x_2(t)\big)\big] \le \exp(-t)\big[V\big(x_1(0),x_2(0)\big) + K^*(p)\big(\exp(t) - 1\big)\big]. \qquad (4.4)
$$

This implies that

$$
\limsup_{t\to+\infty} E\big|x(t)\big|^p \le 2^{p-1}\limsup_{t\to+\infty} E\big[V\big(x_1(t),x_2(t)\big)\big] \le 2^{p-1}K^*(p) := K(p). \qquad (4.5)
$$

This completes the proof of Lemma 4.1. □

Next we investigate the stochastic permanence.

Theorem 4.1 *If $r_1 > 0.5\sigma_1^2$ and $r_2 > 0.5\sigma_2^2$ hold, then system* (1.2) *with initial value $(x_1(0), x_2(0)) \in R_+^2$ is stochastically permanent.*

Proof Assign $0 < \varepsilon < 1$ arbitrarily, we first prove that there is a constant $\alpha > 0$ such that

$$\liminf_{t\to+\infty} P\{|x(t)| \geq \alpha\} \geq 1 - \varepsilon.$$

It follows from $r_i > 0.5\sigma_i^2$ that we can choose a constant $m > 0$ such that

$$r_i - 0.5\sigma_i^2 - 0.5m\sigma_i^2 > 0, \quad i = 1, 2.$$

Define

$$V_1(x_1, x_2) = m^{-1} \sum_1^2 \left(1 + x_i^{-1}\right)^m, \quad (x_1, x_2) \in R_+^2.$$

By Itô's formula, one derives that

$$\begin{aligned}
dV_1(x_1,x_2) &= \left(1+x_1^{-1}\right)^{m-1} dx_1^{-1} + 0.5(m-1)\left(1+x_1^{-1}\right)^{m-2}\left(dx_1^{-1}\right)^2 \\
&\quad + \left(1+x_2^{-1}\right)^{m-1} dx_2^{-1} + 0.5(m-1)\left(1+x_2^{-1}\right)^{m-2}\left(dx_2^{-1}\right)^2 \\
&= \left(1+x_1^{-1}\right)^{m-1}\left[-x_1^{-2}\,dx_1 + x_1^{-3}(dx_1)^2\right] + 0.5(m-1)\left(1+x_1^{-1}\right)^{m-2}\sigma_1^2 x_1^{-2}\,dt \\
&\quad + \left(1+x_2^{-1}\right)^{m-1}\left[-x_2^{-2}\,dx_2 + x_2^{-3}(dx_2)^2\right] + 0.5(m-1)\left(1+x_2^{-1}\right)^{m-2}\sigma_1^2 x_2^{-2}\,dt \\
&= \left(1+x_1^{-1}\right)^{m-2}\left\{\left(1+x_1^{-1}\right)\left[-x_1^{-1}\left(r_1 - a_1x_1 + \frac{c_2x_2}{b_2+x_2}\right)dt - \sigma_1 x_1^{-1}\,dB_1(t)\right.\right. \\
&\quad \left.\left. + \sigma_1^2 x_1^{-1}\,dt\right] + 0.5(m-1)\sigma_1^2 x_1^{-2}\,dt\right\} + \left(1+x_2^{-1}\right)^{m-2}\left\{\left(1+x_2^{-1}\right)\left[-x_2^{-1}\left(r_2\right.\right.\right. \\
&\quad \left.\left.\left. - a_2x_2 + \frac{c_1x_1}{b_1+x_1}\right)dt - \sigma_2 x_2^{-1}\,dB_2(t) + \sigma_2^2 x_2^{-1}\,dt\right] + 0.5(m-1)\sigma_2^2 x_2^{-2}\,dt\right\} \\
&= \left(1+x_1^{-1}\right)^{m-2}\left\{-x_1^{-2}\left(r_1 + \frac{c_2x_2}{b_2+x_2} - 0.5\sigma_1^2 - 0.5m\sigma_1^2\right) + x_1^{-1}\left(-r_1 - \frac{c_2x_2}{b_2+x_2}\right.\right. \\
&\quad \left.\left. + \sigma_1^2 + a_1\right) + a_1\right\}dt - \left(1+x_1^{-1}\right)^{m-1}\sigma_1 x_1^{-1}\,dB_1(t) + \left(1+x_2^{-1}\right)^{m-2}\left\{-x_2^{-2}\left(r_2\right.\right. \\
&\quad \left.\left. + \frac{c_1x_1}{b_1+x_1} - 0.5\sigma_2^2 - 0.5m\sigma_2^2\right) + x_2^{-1}\left(-r_2 - \frac{c_1x_1}{b_1+x_1} + \sigma_2^2 + a_2\right) + a_2\right\}dt \\
&\quad - \left(1+x_2^{-1}\right)^{m-1}\sigma_2 x_2^{-1}\,dB_2(t) \\
&\leq \left(1+x_1^{-1}\right)^{m-2}\left\{-x_1^{-2}\left(r_1 - 0.5\sigma_1^2 - 0.5m\sigma_1^2\right) + x_1^{-1}\left(-r_1 + \sigma_1^2 + a_1\right) + a_1\right\}dt \\
&\quad - \left(1+x_1^{-1}\right)^{m-1}\sigma_1 x_1^{-1}\,dB_1(t) + \left(1+x_2^{-1}\right)^{m-2}\left\{-x_2^{-2}\left(r_2 - 0.5\sigma_2^2 - 0.5m\sigma_2^2\right)\right. \\
&\quad \left. + x_2^{-1}\left(-r_2 + \sigma_2^2 + a_2\right) + a_2\right\}dt - \left(1+x_2^{-1}\right)^{m-1}\sigma_2 x_2^{-1}\,dB_2(t).
\end{aligned} \tag{4.6}$$

Let k be sufficiently small to satisfy

$$0 < \frac{k}{m} < r_i - 0.5\sigma_i^2 - 0.5m\sigma_i^2, \quad i = 1, 2.$$

We continue to define

$$V_2(x_1,x_2) = e^{kt} V_1(x_1,x_2).$$

By virtue of Itô's formula again, we have

$$\begin{aligned}
dV_2(x_1,x_2) &= ke^{kt} V_1\,dt + e^{kt}\,dV_1 \\
&\le ke^{kt} V_1\,dt + e^{kt}\big\{\big(1+x_1^{-1}\big)^{m-2}\big\{-x_1^{-2}\big(r_1 - 0.5\sigma_1^2 - 0.5m\sigma_1^2\big) \\
&\quad + x_1^{-1}\big(-r_1+\sigma_1^2+a_1\big)+a_1\big\}\,dt - \big(1+x_1^{-1}\big)^{m-1}\sigma_1 x_1^{-1}\,dB_1(t) \\
&\quad + \big(1+x_2^{-1}\big)^{m-2}\big\{-x_2^{-2}\big(r_2 - 0.5\sigma_2^2 - 0.5m\sigma_2^2\big) \\
&\quad + x_2^{-1}\big(-r_2+\sigma_2^2+a_2\big)+a_2\big\}\,dt - \big(1+x_2^{-1}\big)^{m-1}\sigma_2 x_2^{-1}\,dB_2(t)\big\} \\
&= e^{kt}\big(1+x_1^{-1}\big)^{m-2}\big\{km^{-1}\big(1+x_1^{-1}\big)^{-2} - x_1^{-2}\big(r_1 - 0.5\sigma_1^2 - 0.5m\sigma_1^2\big) \\
&\quad + x_1^{-1}\big(-r_1+\sigma_1^2+a_1\big)+a_1\big\}\,dt + e^{kt}\big(1+x_2^{-1}\big)^{m-2}\big\{km^{-1}\big(1+x_2^{-1}\big)^{-2} \\
&\quad - x_2^{-2}\big(r_2 - 0.5\sigma_2^2 - 0.5m\sigma_2^2\big) + x_2^{-1}\big(-r_2+\sigma_2^2+a_2\big)+a_2\big\}\,dt \\
&\quad - \big(1+x_1^{-1}\big)^{m-1}\sigma_1 x_1^{-1}e^{kt}\,dB_1(t) - \big(1+x_2^{-1}\big)^{m-1}\sigma_2 x_2^{-1}e^{kt}\,dB_2(t) \\
&= e^{kt}\big(1+x_1^{-1}\big)^{m-2}\left\{-x_1^{-2}\left(r_1 - 0.5\sigma_1^2 - 0.5m\sigma_1^2 - \frac{k}{m}\right)\right. \\
&\quad \left. + x_1^{-1}\left(-r_1+\sigma_1^2+a_1+\frac{2k}{m}\right)+a_1+\frac{k}{m}\right\}dt + e^{kt}\big(1+x_2^{-1}\big)^{m-2}\left\{-x_2^{-2}\left(r_2\right.\right. \\
&\quad \left.\left. - 0.5\sigma_2^2 - 0.5m\sigma_2^2 - \frac{k}{m}\right) + x_2^{-1}\left(-r_2+\sigma_2^2+a_2+\frac{2k}{m}\right)+a_2+\frac{k}{m}\right\}dt \\
&\quad - \big(1+x_1^{-1}\big)^{m-1}\sigma_1 x_1^{-1}e^{kt}\,dB_1(t) - \big(1+x_2^{-1}\big)^{m-1}\sigma_2 x_2^{-1}e^{kt}\,dB_2(t) \\
&= e^{kt}\mathcal{L}(x_1,x_2)\,dt - \big(1+x_1^{-1}\big)^{m-1}\sigma_1 x_1^{-1}e^{kt}\,dB_1(t) \\
&\quad - \big(1+x_2^{-1}\big)^{m-1}\sigma_2 x_2^{-1}e^{kt}\,dB_2(t),
\end{aligned} \tag{4.7}$$

where

$$\begin{aligned}
\mathcal{L}(x_1,x_2) &= \big(1+x_1^{-1}\big)^{m-2}\left\{-x_1^{-2}\left(r_1 - 0.5\sigma_1^2 - 0.5m\sigma_1^2 - \frac{k}{m}\right) + x_1^{-1}\left(-r_1+\sigma_1^2+a_1+\frac{2k}{m}\right)\right. \\
&\quad \left. + a_1 + \frac{k}{m}\right\} + \big(1+x_2^{-1}\big)^{m-2}\left\{-x_2^{-2}\left(r_2 - 0.5\sigma_2^2 - 0.5m\sigma_2^2 - \frac{k}{m}\right)\right. \\
&\quad \left. + x_2^{-1}\left(-r_2+\sigma_2^2+a_2+\frac{2k}{m}\right)+a_2+\frac{k}{m}\right\}.
\end{aligned}$$

Now, we verify that $\mathcal{L}(x_1,x_2)$ is upper bounded for $(x_1,x_2)\in R_+^2$. Denote

$$\mathcal{A}_i = r_i - 0.5\sigma_i^2 - 0.5m\sigma_i^2 - \frac{k}{m}, \qquad \mathcal{B}_i = -r_i + \sigma_i^2 + a_i + \frac{2k}{m}, \qquad \mathcal{C}_i = a_i + \frac{k}{m}, \quad i = 1,2.$$

Obviously, $\mathcal{A}_i > 0$, $\mathcal{B}_i > 0$, $\mathcal{C}_i > 0$.

Then we only need to prove

$$\mathcal{G}(x_i) := \big(1+x_i^{-1}\big)^{m-2}\big(-\mathcal{A}_i x_i^{-2} + \mathcal{B}_i x_i^{-1} + \mathcal{C}_i\big) = \big(1+x_i^{-1}\big)^{m-2}\overline{\mathcal{L}}(x_i)$$

is upper bounded. In the following, we consider cases (I) and (II). Let

$$\mathcal{X} = \frac{\mathcal{B}_i + \sqrt{\mathcal{B}_i^2 + 4\mathcal{A}_i\mathcal{C}_i}}{2\mathcal{A}_i}, \qquad \mathcal{U} = \frac{\mathcal{B}_i^2 + 4\mathcal{A}_i\mathcal{C}_i}{4\mathcal{A}_i}.$$

Case (I). If $\frac{1}{x_i} \geq \mathcal{X}$, then $\overline{\mathcal{L}}(x_i) \leq 0$ and so $\mathcal{G}(x_i) \leq 0$.

Case (II). If $0 < \frac{1}{x_i} \leq \mathcal{X}$, then $\overline{\mathcal{L}}(x_i) \leq \mathcal{U}$. As $(1 + x_i^{-1})^{m-2} \leq (1 + \mathcal{X})^{m-2}$ for $m \geq 2$, while $(1 + x_i^{-1})^{m-2} \leq 1$ for $0 < m < 2$. Cases (I) and (II) show that $\mathcal{G}(x_i)$ is upper bounded and so $\mathcal{L}(x_1, x_2)$ is, we denote it by $\mathcal{M}$.

Now let us return to (4.7), which leads to

$$dV_2(x_1, x_2) \leq \mathcal{M}e^{kt}\,dt - \left(1 + x_1^{-1}\right)^{m-1}\sigma_1 x_1^{-1}e^{kt}\,dB_1(t) - \left(1 + x_2^{-1}\right)^{m-1}\sigma_2 x_2^{-1}e^{kt}\,dB_2(t). \quad (4.8)$$

Integrating on both sides and taking expectations, we get

$$E\left[V_2(x_1, x_2)\right] = e^{kt}E\left[V_1(x_1, x_2)\right] \leq V_1\left(x_1(0), x_2(0)\right) + \frac{\mathcal{M}}{k}\left(e^{kt} - 1\right). \quad (4.9)$$

This implies that

$$\begin{aligned}
\limsup_{t\to+\infty} E\left[\left|x(t)\right|^{-m}\right] &\leq \limsup_{t\to+\infty}(2\sqrt{2})^{-m}E\left[\left(\frac{1}{x_1} + \frac{1}{x_2}\right)^m\right] \\
&\leq \limsup_{t\to+\infty}(2\sqrt{2})^{-m}E\left[\left(1 + x_1^{-1} + 1 + x_2^{-1}\right)^m\right] \\
&\leq \limsup_{t\to+\infty}(2\sqrt{2})^{-m}C_m E\left[\sum_{i=1}^{2}\left(1 + x_i^{-1}\right)^m\right] \\
&= m(2\sqrt{2})^{-m}C_m \limsup_{t\to+\infty} E\left[V_1(x_1, x_2)\right] \\
&\leq \frac{m(2\sqrt{2})^{-m}C_m\mathcal{M}}{k} := \delta. \qquad (4.10)
\end{aligned}$$

For arbitrary $\varepsilon \in (0, 1)$, let $\alpha = (\frac{\varepsilon}{\delta})^{\frac{1}{m}}$. By Chebyshev's inequality, we have

$$P\left\{\left|x(t)\right| < \alpha\right\} = P\left\{\left|x(t)\right|^{-m} > \alpha^{-m}\right\} \leq \frac{E[|x(t)|^{-m}]}{\alpha^{-m}} = \alpha^m E\left[\left|x(t)\right|^{-m}\right].$$

This gives that

$$\limsup_{t\to+\infty} P\left\{\left|x(t)\right| < \alpha\right\} \leq \alpha^m\delta = \varepsilon.$$

Furthermore,

$$\liminf_{t\to+\infty} P\left\{\left|x(t)\right| \geq \alpha\right\} \geq 1 - \varepsilon.$$

In the following, we turn to proving that for arbitrary fixed $\varepsilon \in (0, 1)$, there is a constant $\beta > 0$ such that

$$\liminf_{t\to+\infty} P\left\{\left|x(t)\right| \leq \beta\right\} \geq 1 - \varepsilon.$$

Let $\beta = [\frac{K(p)}{\varepsilon}]^{\frac{1}{p}}$, then by Chebyshev's inequality and Lemma 4.1, we have

$$P\{|x(t)| > \beta\} = P\{|x(t)|^p > \beta^p\} \leq \frac{E[|x(t)|^p]}{\beta^p},$$

which implies that

$$\limsup_{t\to+\infty} P\{|x(t)| > \beta\} \leq \frac{K(p)}{\beta^p} = \varepsilon.$$

Consequently,

$$\liminf_{t\to+\infty} P\{|x(t)| \leq \beta\} \geq 1 - \varepsilon.$$

This completes the proof of Theorem 4.1. □

Remark 4.1 The definition of stochastic permanence here is not a very appropriate one for stochastic population models. Many authors have introduced some more appropriate definitions of permanence for stochastic population model, for example, stochastic persistence in probability (see Refs. [17, 18]) or a new definition of stochastic permanence (see Ref. [19]). We would like to study them in our future work.

To end with, we show that a large noise may lead to exponential extinction.

Theorem 4.2 *If* $r_1 + c_2 < 0.5\sigma_1^2$ *and* $r_2 + c_1 < 0.5\sigma_2^2$ *hold, then system* (1.2) *with initial value* $(x_1(0), x_2(0)) \in R_+^2$ *is extinct exponentially with probability one.*

Proof An application of Itô's formula yields

$$\begin{aligned} d\ln x_i &= \frac{1}{x_i}\,dx_i - \frac{(dx_i)^2}{2x_i^2} \\ &= \left(r_i - a_i x_i + \frac{c_j x_j}{b_j + x_j} - 0.5\sigma_i^2\right) dt + \sigma_i\,dB_i(t) \\ &\leq \left(r_i + c_j - 0.5\sigma_i^2\right) dt + \sigma_i\,dB_i(t), \quad i = 1, 2; i \neq j. \end{aligned} \tag{4.11}$$

Integrating on both sides yields

$$\ln x_i(t) \leq \ln x_i(0) + \left(r_i + c_j - 0.5\sigma_i^2\right)t + \int_0^t \sigma_i\,dB_i(t), \quad i = 1, 2. \tag{4.12}$$

Let $t \to +\infty$ and applying the strong law of large numbers for local martingales, we obtain that

$$\limsup_{t\to+\infty} \frac{\ln x_i(t)}{t} \leq r_i + c_j - 0.5\sigma_i^2 < 0, \quad i = 1, 2; i \neq j.$$

This completes the proof of Theorem 4.2. □

5 The limit of the average in time of sample paths

In this section, we estimate the limit of the average in time of the sample paths of every component of the positive solution. To begin with, we need to introduce the following important lemma which Liu and Wang obtained and proved in Ref. [13].

Lemma 5.1 *Suppose $z(t) \in C[\Omega \times R_+, R_+]$, where $R_+ := \{a | a > 0, a \in R\}$.*

(1) *If there are positive constants λ_0, T and $\lambda \geq 0$ such that*

$$\ln z(t) \leq \lambda t - \lambda_0 \int_0^t z(s)\,ds + \sum_{i=1}^n \rho_i B_i(t)$$

for $t \geq T$, where ρ_i is a constant, then

$$\limsup_{t \to +\infty} \frac{\int_0^t z(s)\,ds}{t} \leq \frac{\lambda}{\lambda_0}, \quad a.s.$$

(2) *If there are positive constants λ_0, T and $\lambda \geq 0$ such that*

$$\ln z(t) \geq \lambda t - \lambda_0 \int_0^t z(s)\,ds + \sum_{i=1}^n \rho_i B_i(t)$$

for $t \geq T$, where ρ_i is a constant, then

$$\liminf_{t \to +\infty} \frac{\int_0^t z(s)\,ds}{t} \geq \frac{\lambda}{\lambda_0}, \quad a.s.$$

Now we are in the position to establish our threshold theorems.

Theorem 5.1 *Assume that $x(t) = (x_1(t), x_2(t))$ is a solution of system* (1.2) *with initial value $(x_1(0), x_2(0)) \in R_+^2$, if $r_i > 0.5\sigma_i^2$, then the component $x_i(t)$ has the property*

$$\limsup_{t \to +\infty} \frac{\int_0^t x_i(s)\,ds}{t} \leq \frac{r_i + c_j - 0.5\sigma_i^2}{a_i}, \quad a.s.$$

Moreover,

$$\liminf_{t \to +\infty} \frac{\int_0^t x_i(s)\,ds}{t} \geq \frac{r_i - 0.5\sigma_i^2}{a_i}, \quad a.s., i = 1, 2.$$

Proof For arbitrarily fixed $\epsilon > 0$, one has $-\epsilon \leq \frac{\ln x_i(0)}{t} \leq \epsilon$. Recalling (4.11), we have

$$\begin{aligned} d \ln x_i &= \frac{1}{x_i}\,dx_i - \frac{(dx_i)^2}{2x_i^2} \\ &= \left(r_i - a_i x_i + \frac{c_j x_j}{b_j + x_j} - 0.5\sigma_i^2\right) dt + \sigma_i\,dB_i(t) \\ &\leq \left(r_i + c_j - 0.5\sigma_i^2 - a_i x_i\right) dt + \sigma_i\,dB_i(t), \quad i = 1, 2; i \neq j. \end{aligned} \tag{5.1}$$

Integrating on both sides, one has

$$\begin{aligned}\ln x_i(t) &\le \ln x_i(0) + \left(r_i + c_j - 0.5\sigma_i^2\right)t - a_i \int_0^t x_i(s)\,ds + \int_0^t \sigma_i\,dB_i(s)\\ &\le \left(r_i + c_j + \epsilon - 0.5\sigma_i^2\right)t - a_i \int_0^t x_i(s)\,ds + \int_0^t \sigma_i\,dB_i(s), \quad i = 1,2; i \neq j.\end{aligned} \tag{5.2}$$

It then follows from (1) of Lemma 5.1 that

$$\limsup_{t\to+\infty} \frac{\int_0^t x_i(s)\,ds}{t} \le \frac{r_i + c_j + \epsilon - 0.5\sigma_i^2}{a_i}, \quad \text{a.s.}$$

Recalling (4.11) again, we have

$$\begin{aligned}d\ln x_i &= \frac{1}{x_i}\,dx_i - \frac{(dx_i)^2}{2x_i^2}\\ &= \left(r_i - a_i x_i + \frac{c_j x_j}{b_j + x_j} - 0.5\sigma_i^2\right)dt + \sigma_i\,dB_i(t)\\ &\ge \left(r_i - 0.5\sigma_i^2 - a_i x_i\right)dt + \sigma_i\,dB_i(t), \quad i = 1,2; i \neq j.\end{aligned} \tag{5.3}$$

Integrating on both sides, we obtain

$$\begin{aligned}\ln x_i(t) &\ge \ln x_i(0) + \left(r_i - 0.5\sigma_i^2\right)t - a_i \int_0^t x_i(s)\,ds + \int_0^t \sigma_i\,dB_i(s)\\ &\ge \left(r_i - \epsilon - 0.5\sigma_i^2\right)t - a_i \int_0^t x_i(s)\,ds + \int_0^t \sigma_i\,dB_i(s), \quad i = 1,2; i \neq j.\end{aligned} \tag{5.4}$$

In view of (2) of Lemma 5.1, we have

$$\liminf_{t\to+\infty} \frac{\int_0^t x_i(s)\,ds}{t} \ge \frac{r_i - \epsilon - 0.5\sigma_i^2}{a_i}, \quad \text{a.s.}$$

Then the desired assertion follows from the arbitrariness of ϵ.

This completes the proof of Theorem 5.1. □

6 Global attractivity

In this section, we will establish the sufficient criteria for global attractivity of system (1.2).

Lemma 6.1 *Let $x(t) = (x_1(t), x_2(t))$ be a solution of system* (1.2) *with initial value $(x_1(0), x_2(0)) \in R_+^2$, then almost every sample path of $(x_1(t), x_2(t))$ is uniformly continuous for $t \ge 0$.*

Proof We first prove $x_1(t)$. Let us consider the following integral equation:

$$x_1(t) = x_1(0) + \int_0^t f_1(s)\,ds + \int_0^t g_1(s)\,dB_1(s),$$

where

$$f_1(s) = x_1(s)\left[r_1 - a_1 x_1(s) + \frac{c_2 x_2(s)}{b_2 + x_2(s)}\right], \qquad g_1(s) = \sigma_1 x_1(s).$$

Then from Lemmas 2.1 and 4.1, one derives that

$$\begin{aligned}
E\left|f_1(t)\right|^p &= E\left|x_1(t)\left[r_1 - a_1x_1(t) + \frac{c_2x_2}{b_2+x_2}\right]\right|^p \\
&= E\left[\left|x_1(t)\right|^p\left|r_1 - a_1x_1(t) + \frac{c_2x_2}{b_2+x_2}\right|^p\right] \\
&\leq 0.5E\left|x_1(t)\right|^{2p} + 0.5E\left|r_1 - a_1x_1(t) + \frac{c_2x_2}{b_2+x_2}\right|^{2p} \\
&\leq 0.5E\left|x_1(t)\right|^{2p} + 0.5\cdot 3^{2p-1}r_1^{2p} + 0.5\cdot 3^{2p-1}a_1^{2p}E\left|x_1(t)\right|^{2p} + 0.5\cdot 3^{2p-1}c_2^{2p} \\
&\leq 0.5K(2p) + 0.5\cdot 3^{2p-1}r_1^{2p} + 0.5\cdot 3^{2p-1}a_1^{2p}K(2p) + 0.5\cdot 3^{2p-1}c_2^{2p} := \mathcal{R}(p)
\end{aligned}$$

and

$$E\left|g_1(t)\right|^p = E\left|\sigma_1x_1(t)\right|^p = \sigma_1^pE\left|x_1(t)\right|^p \leq \sigma_1^pK(p) := \Gamma(p).$$

Furthermore, in view of the moment inequality for stochastic integral, we can obtain that for $0 \leq t_1 < t_2 < +\infty$ and $p > 2$,

$$\begin{aligned}
E\left|\int_{t_1}^{t_2} g_1(s)\,dB_1(s)\right|^p &\leq \left[\frac{p(p-1)}{2}\right]^{p/2}(t_2-t_1)^{(p-2)/2}\int_{t_1}^{t_2} E\left|g_1(s)\right|^p\,ds \\
&\leq \left[\frac{p(p-1)}{2}\right]^{p/2}(t_2-t_1)^{p/2}\Gamma(p).
\end{aligned}$$

Let

$$t_2 - t_1 \leq 1, \qquad \frac{1}{p} + \frac{1}{q} = 1,$$

then we have

$$\begin{aligned}
E\left|x_1(t_2) - x_1(t_1)\right|^p &= E\left|\int_{t_1}^{t_2} f_1(s)\,ds + \int_{t_1}^{t_2} g_1(s)\,dB_1(s)\right|^p \\
&= 2^{p-1}E\left[\int_{t_1}^{t_2}\left|f_1(s)\right|\,ds\right]^p + 2^{p-1}E\left|\int_{t_1}^{t_2} g_1(s)\,dB_1(s)\right|^p \\
&\leq 2^{p-1}E\left\{\left[\int_{t_1}^{t_2} 1^q\,ds\right]^{\frac{1}{q}}\left[\int_{t_1}^{t_2}\left|f_1(s)\right|^p\,ds\right]^{\frac{1}{p}}\right\}^p \\
&\quad + 2^{p-1}\left[\frac{p(p-1)}{2}\right]^{p/2}(t_2-t_1)^{p/2}\Gamma(p) \\
&= 2^{p-1}(t_2-t_1)^{\frac{p}{q}}\int_{t_1}^{t_2} E\left|f_1(s)\right|^p\,ds + 2^{p-1}\left[\frac{p(p-1)}{2}\right]^{p/2}(t_2-t_1)^{p/2}\Gamma(p) \\
&\leq 2^{p-1}(t_2-t_1)^{\frac{p}{q}+1}\mathcal{R}(p) + 2^{p-1}\left[\frac{p(p-1)}{2}\right]^{\frac{p}{2}}(t_2-t_1)^{\frac{p}{2}}\Gamma(p) \\
&= 2^{p-1}(t_2-t_1)^{\frac{p}{2}}\left\{(t_2-t_1)^{\frac{p}{2}}\mathcal{R}(p) + \left[\frac{p(p-1)}{2}\right]^{\frac{p}{2}}\Gamma(p)\right\} \\
&\leq 2^{p-1}(t_2-t_1)^{\frac{p}{2}}\left\{\mathcal{R}(p) + \left[\frac{p(p-1)}{2}\right]^{\frac{p}{2}}\Gamma(p)\right\}.
\end{aligned}$$

Thus it follows from Lemma 2.2 that almost every sample path of $x_1(t)$ is locally but uniformly Hölder continuous with exponent ϑ for $\vartheta \in (0,(p-2)/2p)$, and therefore almost every sample path of $x_1(t)$ is uniformly continuous on $t \geq 0$. By a similar procedure as above, we can demonstrate that almost every sample path of $x_2(t)$ is uniformly continuous on $t \geq 0$. The proof of Lemma 6.1 is completed. □

Assign

$$a_1 \cdot a_2 > \frac{c_1}{b_1} \cdot \frac{c_2}{b_2}, \tag{6.1}$$

then it is easy to verify that there exists a pair of positive constants φ_1 and φ_2 such that

$$\frac{\frac{c_2}{b_2}}{a_2} < \frac{\varphi_2}{\varphi_1} < \frac{a_1}{\frac{c_1}{b_1}},$$

thus $\varphi_2 a_2 - \varphi_1 \frac{c_2}{b_2} > 0$, $\varphi_1 a_1 - \varphi_2 \frac{c_1}{b_1} > 0$.

Theorem 6.1 *If* (6.1) *holds, then system* (1.2) *with initial value* $(x_1(0),x_2(0)) \in R_+^2$ *is globally attractive.*

Proof Let $x(t) = (x_1(t),x_2(t))$ and $u(t) = (u_1(t),u_2(t))$ be two arbitrary solutions of system (1.2) with initial values $(x_1(0),x_2(0)) \in R_+^2$, respectively. By Itô's formula, we obtain that

$$\begin{aligned} d\ln x_i(t) &= \left[r_i - \frac{1}{2}\sigma_i^2 - a_i x_i(t) + \frac{c_j x_j(t)}{b_j + x_j(t)}\right)\right] dt + \sigma_i\, dB_i(t), \quad i,j = 1,2; i \neq j, \\ d\ln u_i(t) &= \left[r_i - \frac{1}{2}\sigma_i^2 - a_i u_i(t) + \frac{c_j u_j(t)}{b_j + u_j(t)}\right)\right] dt + \sigma_i\, dB_i(t), \quad i,j = 1,2; i \neq j. \end{aligned}$$

Then

$$\begin{aligned} &d\bigl[\ln x_i(t) - \ln u_i(t)\bigr] \\ &\quad = \left\{-a_i\bigl[x_i(t) - u_i(t)\bigr] + c_j\left[\frac{x_j(t)}{b_j + x_j(t)} - \frac{u_j(t)}{b_j + u_j(t)}\right]\right\} dt, \quad i,j = 1,2; i \neq j. \end{aligned}$$

Define a Lyapunov function by

$$\widehat{V}(t) = \varphi_1\bigl|\ln x_1(t) - u_1(t)\bigr| + \varphi_2\bigl|\ln x_2(t) - u_2(t)\bigr|.$$

A direct calculation of the right differential $d^+\widehat{V}(t)$ of $\widehat{V}(t)$ along the solutions yields

$$\begin{aligned} d^+\widehat{V}(t) &= \varphi_1 \operatorname{sgn}\bigl(x_1(t) - u_1(t)\bigr)\, d\bigl[\ln x_1(t) - \ln u_1(t)\bigr] \\ &\quad + \varphi_2 \operatorname{sgn}\bigl(x_2(t) - u_2(t)\bigr)\, d\bigl[\ln x_2(t) - \ln u_2(t)\bigr] \\ &= \varphi_1 \operatorname{sgn}\bigl(x_1(t) - u_1(t)\bigr)\left\{-a_1\bigl[x_1(t) - u_1(t)\bigr] + c_2\left[\frac{x_2(t)}{b_2 + x_2(t)} - \frac{u_2(t)}{b_2 + u_2(t)}\right]\right\} dt \\ &\quad + \varphi_2 \operatorname{sgn}\bigl(x_2(t) - u_2(t)\bigr)\left\{-a_2\bigl[x_2(t) - u_2(t)\bigr] + c_1\left[\frac{x_1(t)}{b_1 + x_1(t)} - \frac{u_1(t)}{b_1 + u_1(t)}\right]\right\} dt \end{aligned}$$

$$\begin{aligned}
&\le \left\{-\varphi_1 a_1\left|x_1(t)-u_1(t)\right| + \frac{\varphi_1 c_2 b_2 |x_2(t)-u_2(t)|}{[b_2+x_2(t)]+[b_2+u_2(t)]}\right.\\
&\quad \left. -\varphi_2 a_2\left|x_2(t)-u_2(t)\right| + \frac{\varphi_2 c_1 b_1 |x_1(t)-u_1(t)|}{[b_1+x_1(t)]+[b_1+u_1(t)]}\right\} dt\\
&\le \left\{-\varphi_1 a_1\left|x_1(t)-u_1(t)\right| + \varphi_1\frac{c_2}{b_2}\left|x_2(t)-u_2(t)\right| - \varphi_2 a_2\left|x_2(t)-u_2(t)\right|\right.\\
&\quad \left. +\varphi_2\frac{c_1}{b_1}\left|x_1(t)-u_1(t)\right|\right\} dt\\
&= -\left(\varphi_1 a_1-\varphi_2\frac{c_1}{b_1}\right)\left|x_1(t)-u_1(t)\right| dt - \left(\varphi_2 a_2-\varphi_1\frac{c_2}{b_2}\right)\left|x_2(t)-u_2(t)\right| dt.
\end{aligned}$$

Integrating on both sides from 0 to t, we have

$$\begin{aligned}
&\widehat{V}(t)-\widehat{V}(0)\\
&\quad\le -\left(\varphi_1 a_1-\varphi_2\frac{c_1}{b_1}\right)\int_0^t\left|x_1(s)-u_1(s)\right| ds - \left(\varphi_2 a_2-\varphi_1\frac{c_2}{b_2}\right)\int_0^t\left|x_2(s)-u_2(s)\right| ds.
\end{aligned}$$

That is,

$$\begin{aligned}
&\widehat{V}(t)+\left(\varphi_1 a_1-\varphi_2\frac{c_1}{b_1}\right)\int_0^t\left|x_1(s)-u_1(s)\right| ds + \left(\varphi_2 a_2-\varphi_1\frac{c_2}{b_2}\right)\int_0^t\left|x_2(s)-u_2(s)\right| ds\\
&\quad\le \widehat{V}(0)<+\infty.
\end{aligned}$$

As a result, $|x_1(t)-u_1(t)|\in L^1[0,+\infty)$, $|x_2(t)-u_2(t)|\in L^1[0,+\infty)$. It then follows from Lemmas 2.3 and 6.1 that

$$\lim_{t\to+\infty}\left|x_1(t)-u_1(t)\right|=0,\qquad \lim_{t\to+\infty}\left|x_2(t)-u_2(t)\right|=0\quad\text{a.s.}$$

The proof of Theorem 6.1 is completed. □

7 Numerical simulations

In this paper, we derived the sufficient conditions for stochastic permanence, extinction and global attractivity of system (1.2). In order to illustrate the above analytical results, we will give several specific examples in the following. Motivated by the Milsten method mentioned by Higham (see Ref. [20]), we obtain the following discrete version of system (1.2):

$$\begin{cases}
x_1(k+1)=x_1(k)+x_1(k)[r_1-a_1x_1(k)+\frac{c_2x_2(k)}{b_2+x_2(k)}]\Delta t\\
\qquad\qquad +\sigma_1x_1(k)\sqrt{\Delta t}\xi_1(k)+0.5\sigma_1^2x_1(k)[\xi_1^2(k)-1]\Delta t,\\
x_2(k+1)=x_2(k)+x_2(k)[r_2-a_2x_2(k)+\frac{c_1x_1(k)}{b_1+x_1(k)}]\Delta t\\
\qquad\qquad +\sigma_2x_2(k)\sqrt{\Delta t}\xi_2(k)+0.5\sigma_2^2x_2(k)[\xi_2^2(k)-1]\Delta t,
\end{cases}\tag{7.1}$$

where $\xi_1(k)$ and $\xi_2(k)$ are Gaussian random variables that follow $N(0,1)$. Let us assign

$$\begin{aligned}
&r_1=0.4,\qquad a_1=0.2,\qquad c_2=0.03,\qquad b_2=0.2;\\
&r_2=0.5,\qquad a_2=0.3,\qquad c_1=0.012,\qquad b_1=0.1,
\end{aligned}$$

Figure 1 **Computer simulations of the permanence for system (7.1) without white noises.**

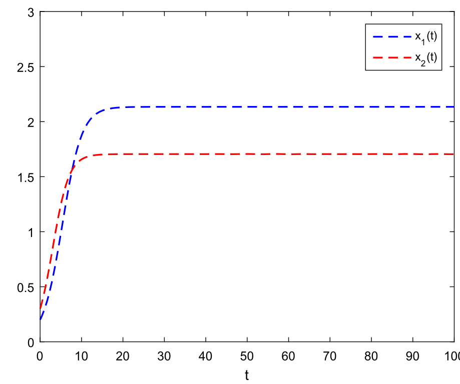
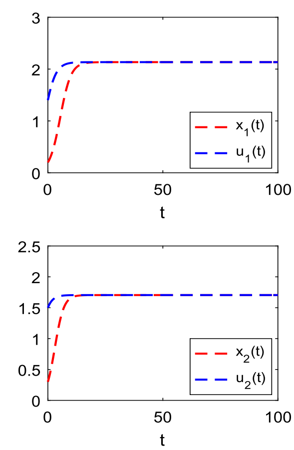


Figure 2 **Computer simulations of the global attractivity for system (7.1) without white noises.**



and the initial value $(x_1(0), x_2(0)) = (0.2, 0.3)$. Theorem 3.1 shows that the positive solution of system (7.1) will not explode to infinity at any finite time, which is fundamental to our analytical results. From Figures 1 and 2, we know that system (7.1) without white noises is permanent and globally attractive.

Example 7.1 (Stochastic permanence) Theorem 4.1 shows that system (7.1) is stochastically permanent under a small noise. Let $\sigma_1 = 0.6, \sigma_2 = 0.8, \Delta t = 0.01$, then $r_1 = 0.4 > 0.5\sigma_1^2 = 0.5 \times 0.36 = 0.18$, $r_2 = 0.5 > 0.5\sigma_2^2 = 0.5 \times 0.64 = 0.32$. It then follows from Theorem 4.1 that system (7.1) is stochastically permanent, see Figure 3.

Example 7.2 (Extinction) Theorem 4.2 shows that a large noise may make the system extinct exponentially with probability one. Let $\sigma_1 = \sqrt{1.4}$, $\sigma_2 = \sqrt{1.4}$, $\Delta t = 0.01$, then $r_1 + c_2 = 0.4 + 0.03 = 0.43 < 0.5\sigma_1^2 = 0.5 \times 1.4 = 0.7$, $r_2 + c_1 = 0.5 + 0.012 = 0.512 < 0.5\sigma_2^2 = 0.5 \times 1.4 = 0.7$. In view of Theorem 4.2, we know that system (7.1) is extinct exponentially with probability one, see Figure 4.

Example 7.3 (Global attractivity) Let $(u_1(0), u_2(0)) = (1.4, 1.5)$, $\sigma_1 = 0.3, \sigma_2 = 0.3, \Delta t = 0.01$. Since $a_1 \cdot a_2 = 0.2 \times 0.3 = 0.06 > \frac{c_1}{b_1} \cdot \frac{c_2}{b_2} = \frac{0.012}{0.1} \times \frac{0.03}{0.2} = 0.018$, then by Theorem 6.1, we conclude that system (7.1) is globally attractive, see Figure 5.

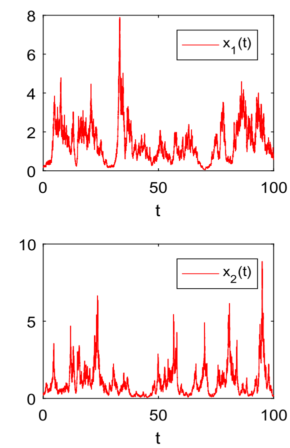


Figure 3 Computer simulations of the stochastic permanence for system (7.1).

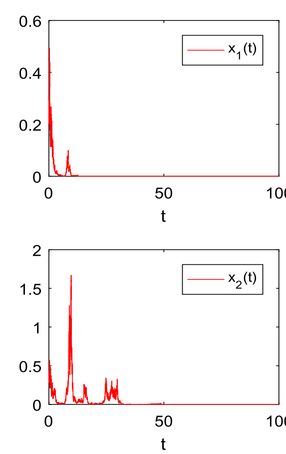


Figure 4 Computer simulations of the extinction for system (7.1).

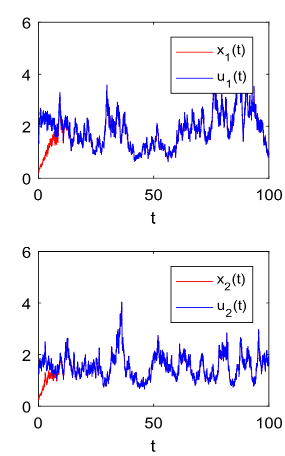


Figure 5 Computer simulations of the global attractivity for system (7.1).

8 Conclusion

In this paper, we have considered a two-species mutualism model perturbed by white noises. We first show that the model has a unique globally positive solution, and then study the stochastic permanence and extinction of the model. The limit of the average in time of the sample paths is also estimated. Furthermore, we establish the sufficient conditions for the global attractivity of the solutions. Finally, we provide several specific examples to illustrate the analytical results. According to the numerical simulations, we find that a small noise could not disrupt the original permanence (see Figures 1 and 3), while a large noise could make species in an equilibrium state tend to be extinct (see Figures 1 and 4), and that the environment noises have no influence on the global attractivity (see Figures 2 and 5).

The stability of the positive equilibrium state is one of the most interesting topics in the study of population models. For models with environmental noises, however, they could not keep the positive equilibrium state of the corresponding deterministic systems. In recent years, many authors have investigated the stability in distribution of stochastic population models (see Refs. [21–23]). We would like to mention that the stability in distribution could be an interesting problem associated with the study of system (1.2), and we leave this for future investigation.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors have read and approved the final manuscript.

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