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# On some applications of the space-time fractional derivative

Wasan Ajeel Ahmood<sup>1\*</sup> and Adem Kiliçman<sup>2</sup>

\*Correspondence:

nokiamsm84@yahoo.com

<sup>1</sup>Department of Mathematics,  
Universiti Putra Malaysia, UPM,  
Serdang, Selangor 43400, Malaysia  
Full list of author information is  
available at the end of the article

## Abstract

In this work, we study and extend the one-dimensional fractional derivative to the multidimensional space-time fractional derivative, determine the exact solution via the Laplace transform, and develop mathematical foundations of the respective operators. The multidimensional space-time fractional derivative is developed to augment the equations. The results show that this method is effective, convenient, and easily executable. The main advantage of the proposed approach is that it is an accurate analytical method (the Laplace transform method) that can be implemented for both space and time discretizations of the fractional derivatives and allows us to present new solutions to problems by certain applications for solving space-time fractional derivatives.

**Keywords:** fractional derivative; evolution equations; anomalous diffusion

## 1 Introduction

The integrals of integer order have clear physical interpretations in engineering and geometry and are features that help to solve applied problems in various scientific fields. Fractional calculus describes and is used to model physical processes via fractional differential equations. In the past, the investigation of nonlinear equations of traveling-wave solutions played an important role in the study of nonlinear physical phenomena (see [1–4]). Fractional differential equations are used to model various physical phenomena in porous, colloid materials, disordered, random, amorphous, geology, finance, and medicine [5, 6]. In [1, 3, 7, 8], many new mathematical models were developed to represent the limiting distribution of a specified stochastic, and the developed models successfully employ fractional derivatives in respective diffusion equations. Also, many physical situations were observed using anomalous diffusion.

Bhrawy *et al.* [9] proposed the implementation of a spectral method for time and space discretizations. They presented a new and efficient algorithm for solving a time fractional subdiffusion equation on a semiinfinite domain. The shifted Chebyshev-Gauss-Radau interpolation method is adapted for time discretization, along with the Laguerre-Gauss-Radau collocation scheme, which is used for space discretization on a semiinfinite domain.

Bhrawy [10] adapted an operational matrix formulation of the collocation method for one- and two-dimensional nonlinear fractional subdiffusion equations. They also used both double and triple shifted Jacobi polynomials as basis functions to elucidate approximate solutions of one- and two-dimensional cases. The space-time fractional derivatives

given in the underlined problems are expressed by Jacobi operational matrices. This helps to investigate spectral collocation schemes for both temporal and spatial discretizations. Bhrawy [11] also employed the shifted Legendre Gauss-Lobatto collocation scheme and Chebyshev Gauss-Radau collocation approximations for temporal and spatial discretizations. He then focused on implementing the new algorithm for two physical problems, time fractional modified anomalous subdiffusion, and fractional nonlinear superdiffusion equations.

The main type of spectral methods, collocation [12], shows that spectral methods not only have exponential rates of convergence, but also high levels of accuracy.

Bhrawy *et al.* [13] reported a new space-time spectral algorithm for obtaining an approximate solution for the space-time fractional Burgers equation based on spectral shifted Legendre collocation method in combination with the shifted Legendre operational matrix of fractional derivatives in the Caputo sense and proposed this method in both spatial and temporal discretizations for the space-time fractional Burgers equation to reduce such problems to that of solving a system of nonlinear algebraic equations.

Furthermore, Mainardi *et al.* [14] discussed the fundamental solutions of space-time diffusion equation of fractional order using the Fourier and Laplace integral transforms and Mittag-Leffler functions, in which the fundamental solution can only be expressed as a convolution form of a Green function and the initial value function, which is then computed.

In the case of the fundamental solutions of space and space-time Riesz fractional partial differential equations with regular conditions, in [15], the authors considered the fundamental solutions of the space-time fractional partial differential equations of the Riesz type with regular conditions; they are obtained from the partial differential equation by replacing the time derivative with fractional derivative of Caputo type and the space derivative with Riesz potential. Also, they derived explicit expressions of the fundamental solutions for the space Riesz fractional partial differential equation and the space-time Riesz fractional partial differential equation. Ebadian *et al.* [16] solved the fractional diffusion-wave equation using algorithms that are dependent on triangular function methods.

Wijker [17] discussed a random vibration of linear dynamic systems and considered spacecraft structures as behaving linearly and having an important role in the design and verification process for characteristic models. Similarly, Brooks *et al.* [18] constructed functions (spin-wave) of the inversion symmetry, obtained a relation between the basic functions of different members for the wave vector, introduced order parameters, and determined the properties of transformation under the operations of a paramagnetic crystal.

Ahmet [19] utilized the method of tanh-coth with the Riccati equation to handle nonlinear wave equations and to travel wave solutions of nonlinear evolution equations. Mark and Hans [20] developed limit theorems by triangular arrays for sequences of continuous-time random walks processes and solved fractional diffusion equations by transition densities of continuous-time random walks.

Wu *et al.* [21] proposed a Riesz Riemann-Liouville difference for modeling the diffusion equation on discrete media, proposed a lattice fractional diffusion equation, reduced Burgers equations to a system of differential equations with initial conditions, and adopted the Adomian decomposition method to obtain numerical solutions for the fractional partial difference equations.

Wu *et al.* [22] also proposed a fractional logistic map and Lorenz maps of Riemann-Liouville type and designed chaos synchronization of fractional chaotic maps according to the stability results. They extended the control method to discrete fractional equations and selected the control parameters according to the stability conditions.

In [23], Wu *et al.* proposed a Riesz-Caputo fractional difference within the discrete fractional calculus, suggested a fractional difference equation method for anomalous diffusion in discrete finite domains, and discussed the numerical simulation of a lattice fractional diffusion process for various difference orders.

Wu and Baleanu [24] studied master-slave synchronization for the fractional difference equation with a nonlinear coupling method. The designed synchronization method numerically can effectively synchronize the fractional logistic map. They adopted the Caputo-like delta derivative as the difference operator.

### 2 Space-time fractional derivative

Let  $\nu$  be a probability distribution on  $\mathfrak{R}_+ \times \mathfrak{R}^d = [0, \infty]$ , where  $\mathfrak{R}_+ = [0, \infty]$ . Define the Fourier-Laplace transform as

$$\hat{\nu} = \int_{\mathfrak{R}^d} \int_{\mathfrak{R}_+} e^{ix \cdot k} e^{-st} \nu(dt, dx),$$

where  $\nu^n = \nu * \dots * \nu$  is the  $n$ -fold convolution of  $\nu$ ,  $\nu$  is infinitely divisible for  $n = 1, 2, \dots$ , and  $\nu_n$  is a probability distribution such that  $\nu_n^n = \nu$ .

**Theorem 1** *Let  $\nu$  be finitely divisible on  $\mathfrak{R}_+ \times \mathfrak{R}^d$  and define*

$$T(x)f(y, t) = \int_0^t \int_{\mathfrak{R}^d} f(y - s_1, t - s_2) \nu^x(ds_1, ds_2)$$

for  $f \in L^1_w(\mathfrak{R}_+ \times \mathfrak{R}^d)$  and  $x \geq 0$ . Then  $M(x)_{x \geq 0}$  has the following properties for all  $f \in L^1_w(\mathfrak{R}_+ \times \mathfrak{R}^d)$ :

- (i)  $M(x + \nu)f = M(x)M(\nu)$  for  $\nu, x \geq 0$ .
- (ii)  $M(0)f = f$ .
- (iii)  $f > 0 \Rightarrow M(x)f > 0$  for  $x > 0$ .
- (iv)  $\|M(x)f\|_w \leq \|f\|_w$  for all  $x \geq 0$ .
- (v)  $\lim_{x \rightarrow 0} \|M(x)f - f\|_w = 0$ .

*Proof* Let  $f = I_P(y, t) = I((y, t) \in P)$ ,  $P = (y, t) \in \mathfrak{R}_+ \times \mathfrak{R}^d : a_0 \leq t \leq b_0$ , and  $a_i \leq y_i \leq b_i$ ,  $i = 1, 2, \dots, d$ .

Then

$$\begin{aligned} \int_0^\infty \int_{\mathfrak{R}^d} |M(x)f(y, t)| dy dt &= \int_0^\infty \int_{\mathfrak{R}^d} \left| \int_0^t \int_{\mathfrak{R}^d} f(y - s_1, t - s_2) \nu^x(ds_1, ds_2) \right| dy dt \\ &= \int_0^\infty \int_{\mathfrak{R}^d} \int \int_{s_1, s_2 + P} ds_1 ds_2 \nu^x(ds_1, ds_2), \end{aligned}$$

where  $\nu$  is infinitely divisible, and we have  $\nu^x \rightarrow \nu^0$  as  $x \downarrow 0$  and  $\nu^x(B) \rightarrow \nu(B)$  for all Borel subsets  $B \subseteq \mathfrak{R}_+ \times \mathfrak{R}^d$  such that  $\nu^0(\partial B) = 0$ , where  $\partial B = \overline{B} \setminus B^0$ ,  $\overline{B}$  is the closure of  $B$ , that is,

the intersection of all closed sets containing  $B$ , and  $B^0$  is the interior, that is, the union of all open sets contained in  $B$ .

If  $(0, 0) \in \partial B$ , then, for all  $(y, t) \in \mathfrak{R}_+ \times \mathfrak{R}^d$  with  $t \neq a_0, b_0$  and  $y_i \neq a_i, b_i$  for all  $i = 1, 2, \dots, d$ , we can get

$$\lim_{x \rightarrow 0^+} M(x)f(y, t) = \lim_{x \rightarrow 0^+} v^x((y, t) - P) = v^0((y, t) - P) = f(s_2, t).$$

By the dominated convergence theorem,

$$\begin{aligned} \int \int_P |M(x)f(y, t) - f(y, t)| \, dy \, dt &= \int \int_P |v^x((y, t) - P) - v^0((y, t) - P)| \, dy \, dt \\ &= \int_0^\infty \int_{\mathfrak{R}^d} \int \int_{s_1, s_2 + P} \, dy \, dt \, v^x(ds_1, ds_2) \end{aligned}$$

implies that

$$\int \int_{(y,t) \notin P} |M(x)f(y, t) - f(y, t)| \, dy \, dt = \int \int_{(y,t) \notin P} |M(x)f(y, t)| \, dy \, dt \rightarrow 0.$$

Hence,

$$\int \int_{(y,t) \notin P} |M(x)f(y, t) - f(y, t)| \, dy \, dt \rightarrow 0.$$

By the triangle inequality,

$$\|M(x)f - f\|_w \leq \|M(x)f - T(x)g_n^w\|_w + \|M(x)g_n^w - g_n^w\|_w + \|g_n^w - f\|_w.$$

By property (iv),

$$\|M(x)f - M(x)g_n^w\|_w \leq \|f - g_n^w\|_w \leq n^{-1}.$$

Thus, we have

$$\|M(x)f - f\|_w \leq n^{-1} + \|M(x)g_n^w - g_n^w\|_w + n^{-1}.$$

For a continuous semigroup  $M(x), x > 0$ , on  $Y$ , the generator is given by

$$Lf = \lim_{x \rightarrow 0} \frac{M(x)f - f}{x}.$$

Then  $\|x^{-1}(M(x)f - f) - Lf\| \rightarrow 0$  in  $\|Y\|$ . □

**Theorem 2** *If  $X = L_w^1 = \mathfrak{R}^+ \times \mathfrak{R}_c$ , then  $L$  is the generator of a continuous semigroup, and  $M(u)$  is defined as*

$$M(u)f = \int_0^t \int_{\mathfrak{R}_+} f(x - y, t - s)v^u(dy, ds).$$

Then,

$$\widehat{L}f(k, s) := \psi(k, s)\widehat{f}(k, s),$$

where  $\psi$  is given,  $H(L) = \{f \in X : \psi\widehat{f} = \widehat{h}(k, s) \exists h \in X\}$ , and  $\Sigma$  is a subset of  $L^1_w = \mathfrak{R}^+ \times \mathfrak{R}_c$ . Then  $\Sigma \subset H(L)$  for  $f \in \Sigma$ , and we can get

$$\begin{aligned} Lf(x, t) &= -a \cdot \nabla f(x, t) + b \cdot \frac{\partial f}{\partial t}(x, t) + \frac{1}{2} \nabla \cdot A \nabla f(x, t) \\ &+ \int_{\mathfrak{R}^d \times \mathfrak{R}_+(0,0)} \left( G(t-s)f(x-y, t-s) - f(x, t) + \frac{\nabla f(x, t) \cdot y}{1 - \|y\|^2} \right) \phi(dy, ds), \end{aligned}$$

where  $G(t)$  is the Heaviside step function.

*Proof* If  $f \in H(L) \subset L^1_w(\mathfrak{R}^+ \times \mathfrak{R}_c)$ , then its Fourier-Laplace transform exists. The Fourier-Laplace transform of a convolution is  $\widehat{T(u)f}(k, s) = \exp(u\psi(k, s))\widehat{f}(k, s)$ . Then

$$\begin{aligned} \widehat{L\{f\}}(k, s) &= \lim_{u \rightarrow 0^+} \frac{\exp(u\psi(k, s)) - 1}{u} \cdot \widehat{f}(k, s) \\ &= \psi(k, s)\widehat{f}(k, s) \end{aligned}$$

for  $f \in H(L)$ .

Let  $f \in X$ . Then  $g := \lambda f - h \in X$ . By semigroup theory,  $(\lambda I - L)^{-1}$  is a bounded linear for  $\lambda > 0$ .

Now, let  $q = (\lambda I - L)^{-1}g$ . Then

$$\widehat{q}(k, s) = \frac{\widehat{g}(k, s)}{\lambda - \psi(k, s)} = \frac{\lambda\widehat{f}(k, s) - \psi(k, s)\widehat{f}(k, s)}{\lambda - \psi(k, s)} = \widehat{f}(k, s).$$

Now,

$$\|f\|_\Sigma = \|f\|_w + \sum_{i=1}^c \left\| \frac{\partial f}{\partial x_i} \right\|_w + \sum_{i,j=1}^c \left\| \frac{\partial^2 f}{\partial x_i \partial x_j} \right\|_w + \left\| \frac{\partial f}{\partial t} \right\|_w,$$

and using a Taylor series expansion on  $x$ , we have

$$f(x-y, t) - f(x, t) + y \cdot \nabla f(x, t) = y \cdot \int_0^1 (1-r)M_{x-ry}y \, dr,$$

where  $P_x$  is the Hessian matrix of  $f$ , and

$$\int e^{-wt} \left| f(x-y, t) - f(x, t) + \frac{y \cdot \nabla f(x, t)}{1 + \|y\|^2} \right| dx dt \leq C\|f\|_\Sigma \frac{\|y\|^2}{1 + \|y\|^2}$$

for some  $C$ . Further, by Taylor series expansion we obtain

$$\int e^{-wt} |f(x, t-s) - f(x, t)| dx dt \leq H\|f\| \cdot \frac{s}{1+s}$$

for some constant  $H$ .

Then, by Fubini’s theorem we have

$$\begin{aligned} & \int e^{-wt} \left( \int \left| f(x-y, t-s) - f(x, t) + \frac{y \cdot \nabla f(x, t)}{1 + \|y\|^2} \right| \phi(dy, ds) \right) dx dt \\ &= \int \left( \int e^{-wt} \left| f(x-y, t-s) - f(x, t) + \frac{y \cdot \nabla f(x, t)}{1 + \|y\|^2} \right| dx dt \right) \phi(dy, ds) \\ &\leq \|f\|_{\Sigma} \left( C \int \frac{\|y\|^2}{1 + \|y\|^2} \phi(dy, ds) \right. \\ &\quad \left. + H \int \frac{s}{1+s} \phi(dy, ds) \right) \leq K \|f\|_{\Sigma}. \end{aligned}$$

We show that  $\Sigma \subset H(L)$ . If

$$L_1 f(x, t) = -a \cdot \nabla f(x, t) + b \cdot \frac{\partial f}{\partial t}(x, t) + \frac{1}{2} \nabla \cdot A \nabla \cdot f(x, t),$$

then

$$\|L_1 f\|_w \leq B \|f\|_{\Sigma}$$

for  $B$  independent of  $f$ ,

$$L_2 f(x, t) = \int_{\mathbb{R}^+ \times \mathbb{R}_c \setminus \{0\}} \left( f(x-y, t-s) - f(x, t) + \frac{\nabla f(x, t) \cdot y}{1 - \|y\|^2} \right) \phi(dy, ds).$$

Then,  $L = L_1 - L_2$  and

$$f \in \Sigma \cap C^\infty(\mathbb{R}^c \times \mathbb{R}_+), \quad \|L f\|_w \leq \|L_1 f\|_w + \|L_2 f\|_w \leq B \|f\|_{\Sigma} + K \|f\|_{\Sigma}.$$

If  $f$  is denoted by

$$\{f_n\}_{n=1}^\infty \subset \Sigma \cap C^\infty(\mathbb{R}^+ \times \mathbb{R}_c),$$

then  $f_n \rightarrow f$  in the  $\|L_w^1\|$ , and the above equation shows that  $\{f_n\}_{n=1}^\infty$  converges to  $g$  in the  $f_n \rightarrow f$ . Since  $\{L\}$  is closed,  $f \in D(L)$  and  $L\{f\} = g$ . □

### 3 Solution of the linear space-time fractional differential equations

Consider the space-time fractional order for a one-dimensional differential equation. The general form of derivative with respect to  $t$  is given by

$$\sum_{i=1}^m \frac{\partial^{v_i} u(x_i, t_i)}{\partial t_i^{v_i}} = \sum_{i=1}^m \frac{\partial^{n_i} u(x_i, t_i)}{\partial t_i^{n_i}} \left[ \sum_{i=1}^m \frac{\partial^{-n_i+v_i} u(x_i, t_i)}{\partial t_i^{-n_i+v_i}} \right],$$

where  $n_i$  is the smallest integer greater than  $v_i > 0$  and  $u_i = n_i - v_i$  for each  $i = 1, 2, \dots, m$ . We can write this equation as

$$\sum_{i=1}^m \frac{\partial^{v_i} u(x_i, t_i)}{\partial t_i^{v_i}} = \sum_{i=1}^m \frac{\partial^{n_i} u(x_i, t_i)}{\partial t_i^{n_i}} \left[ \sum_{i=1}^m \frac{\partial^{-(n_i-v_i)} u(x_i, t_i)}{\partial t_i^{-(n_i-v_i)}} \right]$$

together with the initial condition. By the Laplace transform we can get

$$\begin{aligned} L\left\{\sum_{i=1}^m \frac{\partial^{v_i} u(x_i, t_i)}{\partial t_i^{v_i}}\right\} &= L\left\{\sum_{i=1}^m \frac{\partial^{n_i} u(x_i, t_i)}{\partial t_i^{n_i}} \left[\sum_{i=1}^m \frac{\partial^{-(n_i-v_i)} u(x_i, t_i)}{\partial t_i^{-(n_i-v_i)}}\right]\right\} \\ &= \sum_{i=1}^m \left[ s_i^{n_i} L\left\{\sum_{i=1}^m \frac{\partial^{-(n_i-v_i)} u(x_i, t_i)}{\partial t_i^{-(n_i-v_i)}}\right\} - \sum_{i=1}^m s_i^{n_i-i} \frac{\partial^{i-1-n_i+v_i} u(x_i, 0)}{\partial t_i^{i-1-n_i+v_i}} \right] \\ &= \sum_{i=1}^m \left[ s_i^{n_i} (s_i^{-n_i+v_i} U(x_i, s_i)) - \sum_{i=1}^m s_i^{n_i-i} \frac{\partial^{i-1-n_i+v_i} u(x_i, 0)}{\partial t_i^{i-1-n_i+v_i}} \right] \\ &= \sum_{i=1}^m \left[ s_i^{v_i} U(x_i, s_i) - \sum_{i=1}^m s_i^{n_i-i} \frac{\partial^{i-1-n_i+v_i} u(x_i, 0)}{\partial t_i^{i-1-n_i+v_i}} \right]. \end{aligned}$$

Now, if  $n = 1$ , then the transform is given by

$$\sum_{i=1}^m \left[ s_i^{v_i} U(x_i, s_i) - \frac{\partial^{-1+v_i} u(x_i, 0)}{\partial t_i^{-1+v_i}} \right], \quad 0 < v_i \leq 1.$$

If  $n = 2$ , then

$$\Rightarrow \sum_{i=1}^m \left[ s_i^{v_i} U(x_i, s_i) - s_i \frac{\partial^{-2+v_i} u(x_i, 0)}{\partial t_i^{-2+v_i}} - \frac{\partial^{-1+v_i} u(x_i, 0)}{\partial t_i^{-1+v_i}} \right], \quad 1 < v_i \leq 2.$$

If  $n = 3$ , then

$$\sum_{i=1}^m \left[ s_i^{v_i} U(x_i, s_i) - s_i^2 \frac{\partial^{-3+v_i} u(x_i, 0)}{\partial t_i^{-3+v_i}} - s_i \frac{\partial^{-2+v_i} u(x_i, 0)}{\partial t_i^{-2+v_i}} - \frac{\partial^{-1+v_i} u(x_i, 0)}{\partial t_i^{-1+v_i}} \right],$$

where  $2 < v_i \leq 3$ .

Now, if  $n = 4$ , then we obtain

$$\begin{aligned} \sum_{i=1}^m \left[ s_i^{v_i} U(x_i, s_i) - s_i^3 \frac{\partial^{-4+v_i} u(x_i, 0)}{\partial t_i^{-4+v_i}} - s_i^2 \frac{\partial^{-3+v_i} u(x_i, 0)}{\partial t_i^{-3+v_i}} \right. \\ \left. - s_i \frac{\partial^{-2+v_i} u(x_i, 0)}{\partial t_i^{-2+v_i}} - \frac{\partial^{-1+v_i} u(x_i, 0)}{\partial t_i^{-1+v_i}} \right], \end{aligned}$$

where  $3 < v_i \leq 4$ .

Similarly, the general form of the derivative with respect to  $x$  is given by

$$L\left\{\sum_{i=1}^m \frac{\partial^{v_i} u(x_i, t_i)}{\partial x_i^{v_i}}\right\} = \sum_{i=1}^m \left[ s_i^{v_i} U(s_i, t_i) - \sum_{i=1}^m s_i^{n_i-i} \frac{\partial^{i-1-n_i+v_i} u(0, t_i)}{\partial x_i^{i-1-n_i+v_i}} \right].$$

Now, we provide some basic definitions and examples where the method is applied for solving linear space-time fractional differential equations. In fact, many different disciplines, such as biology, finance, semiconductor research and hydrology (see [25]), and space fractional differential equation (see [26, 27]) all utilize irregular transport in the context of the flow porous media, fractional space derivatives model, and huge motions

through largely conjunctive fractures. Now, we recall a method to solve the space-time fractional differential equations.

**Definition 1** (see [28]) The Laplace transform of  $f(x)$  for  $x > 0$  is denoted by  $L\{f(x)\}$  or  $F(s)$  and defined by

$$L\{f(x)\} = F(s) = \int_0^\infty e^{-sx} f(x) dx,$$

where  $s$  is a complex number.

**Example 1** Solve

$$\frac{\partial^{\frac{3}{2}}}{\partial x^{\frac{3}{2}}} y(x, t) - \frac{\partial^{\frac{1}{2}}}{\partial x^{\frac{1}{2}}} y(x, t) = e^{-\frac{3}{2}x} \left[ \cos ax + \left( \frac{\frac{1}{2} + \frac{3}{2}}{1} \right) \sin ax \right],$$

where  $a = 1$ ,  $1 < 3/2 \leq 2$ , and  $0 < 1/2 \leq 1$ .

*Solution:* By using the Laplace transform to the above equation we get:

$$\begin{aligned} &L\left\{ \frac{\partial^{\frac{3}{2}}}{\partial x^{\frac{3}{2}}} y(x, t) - \frac{\partial^{\frac{1}{2}}}{\partial x^{\frac{1}{2}}} y(x, t) \right\} \\ &= L\left\{ e^{-\frac{3}{2}x} \left[ \cos x + \left( \frac{\frac{1}{2} + \frac{3}{2}}{1} \right) \sin x \right] \right\}, \\ &s^{\frac{3}{2}} Y(s, t) + s^{\frac{1}{2}} Y(s, t) = \frac{s + \frac{1}{2}}{(s + \frac{3}{2})^2 + 1}, \\ &Y(s, t) = \frac{s + \frac{3}{2}}{s^{\frac{1}{2}}(s + 1)[(s + \frac{3}{2})^2 + 1]} \\ &= \frac{s + \frac{1}{2}}{s^{\frac{1}{2}}(s + 1)[(s + \frac{3}{2})^2 + 1]} + \frac{1}{s^{\frac{1}{2}}(s + 1)[(s + \frac{3}{2})^2 + 1]} \\ &= \frac{1}{s^{\frac{1}{2}}(s + 1)} \left[ \frac{s + \frac{1}{2}}{(s + \frac{3}{2})^2 + 1} + \frac{1}{(s + \frac{3}{2})^2 + 1} \right]. \end{aligned}$$

By using the inverse three-dimensional Laplace transform to the above equation we can get:

$$y(x, t) = x^{\frac{3}{2}} E_{1, \frac{3}{2}+1}(x) [e^{-\frac{3}{2}x} [\cos x + \sin x] + e^{-\frac{3}{2}x} \sin x].$$

**Example 2** (i) Letting  $\gamma = \frac{4}{3}$ , solve  $y''(x, t) - y^\gamma(x, t) = \delta'(x)$ .

*Solution:* By taking the Laplace transform of the above equation we can get:

$$L\{y''\} - L\{y^\gamma\} = L\{\delta'(x)\},$$

where  $\gamma = \frac{4}{3}$ ,

$$s^2 Y(s, t) - sy(0, t) - y'(0, t) - s^{\frac{4}{3}} Y(s, t) + sD^{-(2-4/3)}y(0, t) + D^{-(1-4/3)}y(0, t) = s$$

together with the initial conditions

$$y(0, t) = y'(0, t) = D^{-(2/3)}y(0, t) + D^{(1/3)}y(0, t) = 0.$$

We can get

$$s^2 Y(s, t) - s^{\frac{4}{3}} Y(s, t) = s.$$

Now, by solving for  $Y(s)$ , we obtain:

$$Y(s, t) = \frac{s}{s^2 - s^{\frac{4}{3}}}.$$

Finally,

$$\begin{aligned} y(x, t) &= L^{-1} \left\{ \frac{1}{s^{1/3}(s^{\frac{2}{3}} - 1)} \right\} \\ &= L^{-1} \left\{ \frac{1}{s^{1/3}} \right\} L^{-1} \left\{ \frac{1}{s^{\frac{2}{3}} - 1} \right\} \\ &= \left( \frac{x^{\frac{1}{3}-1}}{\Gamma(\frac{1}{3})} \right) (x^{(2/3)-1} E_{(2/3),(2/3)} x^{(2/3)}). \end{aligned}$$

(ii) Now let  $\gamma = \frac{8}{3}$  and  $1 < \alpha = \frac{5}{3} \leq 2$ . Then solve  $y^\gamma(x, t) + y'(x, t) = (\delta'(x) - \delta(x))(-x^\alpha \times \delta_{1, \frac{-2}{3}}(x))$ ,  $2 < \gamma \leq 3$ .

*Solution:* By taking the Laplace transform of the above equation we can get:-

$$L\{y^\gamma\} + L\{y'\} = L\{(\delta'(x) - \delta(x))(x^\alpha \delta_{1, \frac{-2}{3}}(x))\},$$

where  $\gamma = \frac{8}{3}$  and  $\alpha = \frac{5}{3}$ .

Together with the initial conditions, we can get:

$$s^{\frac{8}{3}} Y(s, t) + sY(s, t) = (s - 1) \left( \frac{s^{\frac{5}{3}}}{s - 1} \right).$$

Now, by solving for  $Y(s, t)$  we obtain:

$$Y(s, t) = \frac{s^{\frac{5}{3}}}{s(s^{\frac{5}{3}} + 1)}.$$

Finally,

$$y(x, t) = \delta_{\frac{5}{3}}(-x^{\frac{5}{3}}).$$

Note that, in particular, if  $\gamma = 1, 2$ , then the derivative is the classical ordinary derivative.

#### 4 Solution of space-time fractional linear partial differential equation

Consider the initial and boundary value problem of the space time fractional linear partial differential equation: The general form for multivariables with derivative with respect to one variable is given by

$$\begin{aligned} & \sum_{j=1}^{\infty} \frac{\partial^{\sum_{i=1}^n v_i} u(x_{j1}, t_{j1}, x_{j2}, t_{j2}, \dots, x_{jn}, t_{jn})}{\prod_{i=1}^n \partial x_i^{v_{ij}}} \\ &= \sum_{j=1}^{\infty} \frac{\partial^{\sum_{i=1}^n m_i} u(x_{j1}, t_{j1}, x_{j2}, t_{j2}, \dots, x_{jn}, t_{jn})}{\prod_{i=1}^n \partial x_i^{m_i}} \\ & \times \left[ \sum_{j=1}^{\infty} \frac{\partial^{\sum_{i=1}^n -(m_i-v_i)} u(x_{j1}, t_{j1}, x_{j2}, t_{j2}, \dots, x_{jn}, t_{jn})}{\prod_{i=1}^n \partial x_i^{-(m_i-v_i)}} \right]. \end{aligned}$$

Together with appropriate initial and boundary conditions, by taking the multidimensional Laplace transform, we can get:

$$\begin{aligned} & L_n \left\{ \sum_{j=1}^{\infty} \frac{\partial^{\sum_{i=1}^n v_i} u(x_{j1}, t_{j1}, x_{j2}, t_{j2}, \dots, x_{jn}, t_{jn})}{\prod_{i=1}^n \partial x_i^{v_{ij}}} \right\} \\ &= L_n \left\{ \sum_{j=1}^{\infty} \frac{\partial^{\sum_{i=1}^n m_i} u(x_{j1}, t_{j1}, x_{j2}, t_{j2}, \dots, x_{jn}, t_{jn})}{\prod_{i=1}^n \partial x_i^{m_i}} \right\} \\ & \times \left[ \sum_{j=1}^{\infty} \frac{\partial^{\sum_{i=1}^n -(m_i-v_i)} u(x_{j1}, t_{j1}, x_{j2}, t_{j2}, \dots, x_{jn}, t_{jn})}{\prod_{i=1}^n \partial x_i^{-(m_i-v_i)}} \right]. \end{aligned}$$

Thus,

$$\begin{aligned} & L_n \left\{ \frac{\partial^{\sum_{i=1}^n v_i} u(x_{j1}, t_{j1}, x_{j2}, t_{j2}, \dots, x_{jn}, t_{jn})}{\prod_{i=1}^n \partial x_i^{v_i}} \right\} \\ &= \prod_{i=1}^n s_i^{m_i} L_n \left\{ \frac{\partial^{\sum_{i=1}^n -(m_i-v_i)} u(x_{j1}, t_{j1}, x_{j2}, t_{j2}, \dots, x_{jn}, t_{jn})}{\prod_{i=1}^n \partial x_i^{-(m_i-v_i)}} \right\} \\ & + \sum_{k=1}^{n-1} (1)^{n-k} \sum_{\substack{i_1, i_2, \dots, i_k=1 \\ i_1, i_2, \dots, i_n}}^n \prod_{j=1}^k s_{ij}^{m_{ij}} \prod_{\substack{i=1 \\ i \neq l_j}}^n \sum_{l_i=1}^{m_i} s_i^{l_i-1} \\ & \quad \text{for each } j=1, 2, \dots, k \\ & \times L_k \left\{ \frac{\partial^{\sum_{i=1}^n (m_i-l_i+v_i)} u(0, t_{j1}, 0, t_{j2}, \dots, 0, t_{jn}, x_{i1}, t_{ij1}, 0, \dots, 0, x_{i2}, t_{ij2}, 0, \dots, 0, x_{ik}, t_{ijk}, 0, \dots, 0)}{\prod_{\substack{i=1 \\ i \neq l_j}}^n (m_i - l_i + v_i)} \right\} \\ & + (-1)^n \prod_{i=1}^n \sum_{l_i=1}^{m_i} s_i^{l_i-1} \frac{\partial^{-\sum_{i=1}^n (m_i-l_i+v_i)} u(0, t_{j1}, 0, t_{j2}, \dots, 0, t_{jn})}{\prod_{i=1}^n \partial x_i^{-(m_i-l_i+v_i)}}. \end{aligned}$$

**Example 3** Consider the initial and boundary value problem of the space time fractional order linear three-dimensional differential equation

$$\frac{\partial^{\frac{7}{3}} u(x_1, t_1, x_2, t_2, x_3, t_3)}{\partial x_1^{\frac{4}{3}} \partial x_2^{\frac{2}{3}} \partial x_3^{\frac{1}{3}}} + u(x_1, t_1, x_2, t_2, x_3, t_3) = 1$$

together with initial and boundary conditions

$$\begin{aligned} \frac{\partial^{\frac{4}{3}} u(x_1, t_1, 0, t_2, 0, t_3)}{\partial x_2^{\frac{-2}{3}} \partial x_3^{\frac{-1}{3}}} &= \frac{\partial^{\frac{-5}{3}} u(0, t_1, x_2, t_2, 0, t_3)}{\partial x_1^{\frac{-4}{3}} \partial x_3^{\frac{-1}{3}}} = \frac{\partial^{-2} u(0, t_1, 0, t_2, x_3, t_3)}{\partial x_1^{\frac{-4}{3}} \partial x_2^{\frac{-2}{3}}} \\ &= \frac{\partial^{\frac{-1}{3}} u(x_1, t_1, x_2, t_2, 0, t_3)}{\partial x_3^{\frac{-1}{3}}} = \frac{\partial^{\frac{-4}{3}} u(0, t_1, x_2, t_2, x_3, t_3)}{\partial x_1^{\frac{-4}{3}}} \\ &= \frac{\partial^{\frac{-2}{3}} u(x_1, t_1, 0, t_2, x_3, t_3)}{\partial x_2^{\frac{-2}{3}}} = \frac{\partial^{\frac{-7}{3}} u(0, t_1, 0, t_2, 0, t_3)}{\partial x_1^{\frac{4}{3}} \partial x_2^{\frac{2}{3}} \partial x_3^{\frac{1}{3}}} = 0. \end{aligned}$$

Then, by taking three-dimensional Laplace transforms we obtain

$$s_1^{\frac{4}{3}} s_2^{\frac{2}{3}} s_3^{\frac{1}{3}} U(s_1, t_1, s_2, t_2, s_3, t_3) + U(s_1, t_1, s_2, t_2, s_3, t_3) = \frac{1}{s_1 s_2 s_3}.$$

Then it follows that

$$U(s_1, t_1, s_2, t_2, s_3, t_3) = \frac{1}{(s_1 s_2 s_3)(s_1^{\frac{4}{3}} s_2^{\frac{2}{3}} s_3^{\frac{1}{3}} + 1)}.$$

Hence,

$$\begin{aligned} &L_3 \left\{ \frac{\partial^{\frac{7}{3}} u(x_1, t_1, x_2, t_2, x_3, t_3)}{\partial x_1^{\frac{4}{3}} \partial x_2^{\frac{2}{3}} \partial x_3^{\frac{1}{3}}} \right\} \\ &= s_1^{\frac{4}{3}} s_2^{\frac{2}{3}} s_3^{\frac{1}{3}} U(s_1, t_1, s_2, t_2, s_3, t_3) - s_1^{\frac{4}{3}} L_1 \left\{ \frac{\partial^{\frac{-3}{3}} u(x_1, t_1, 0, t_2, 0, t_3)}{\partial x_2^{\frac{-2}{3}} \partial x_3^{\frac{-1}{3}}} \right\} \\ &\quad - s_2^{\frac{2}{3}} L_1 \left\{ \frac{\partial^{\frac{-5}{3}} u(0, t_1, x_2, t_2, 0, t_3)}{\partial x_1^{\frac{-4}{3}} \partial x_3^{\frac{-1}{3}}} \right\} - s_3^{\frac{1}{3}} L_1 \left\{ \frac{\partial^{-2} u(0, t_1, 0, t_2, x_3, t_3)}{\partial x_1^{\frac{-4}{3}} \partial x_2^{\frac{-2}{3}}} \right\} \\ &\quad - s_1^{\frac{4}{3}} s_2^{\frac{2}{3}} L_2 \left\{ \frac{\partial^{\frac{-1}{3}} u(x_1, t_1, x_2, t_2, 0, t_3)}{\partial x_3^{\frac{-1}{3}}} \right\} - s_1^{\frac{4}{3}} s_3^{\frac{1}{3}} L_2 \left\{ \frac{\partial^{\frac{-2}{3}} u(x_1, t_1, 0, t_2, x_3, t_3)}{\partial x_2^{\frac{-2}{3}}} \right\} \\ &\quad - s_2^{\frac{2}{3}} s_3^{\frac{1}{3}} L_2 \left\{ \frac{\partial^{\frac{-4}{3}} u(0, t_1, x_2, t_2, x_3, t_3)}{\partial x_1^{\frac{-4}{3}}} \right\} - \left\{ \frac{\partial^{\frac{-7}{3}} u(0, t_1, 0, t_2, 0, t_3)}{\partial x_1^{\frac{4}{3}} \partial x_2^{\frac{2}{3}} \partial x_3^{\frac{1}{3}}} \right\}. \end{aligned}$$

Now, by taking the inverse three-dimensional Laplace transforms of the above equation we can get:

$$u(x_1, t_1, x_2, t_2, x_3, t_3) = (1 - E_{\frac{4}{3}}(-x_1^{\frac{4}{3}}))(1 - E_{\frac{2}{3}}(-x_2^{\frac{2}{3}}))(1 - E_{\frac{1}{3}}(-x_3^{\frac{1}{3}})).$$

**Example 4** As a more general example, consider

$$\begin{aligned} &\sum_{j=1}^{\infty} \frac{\partial^{\sum_{i=1}^n v_i} u(x_{j1}, t_{j1}, x_{j2}, t_{j2}, \dots, x_{jn}, t_{jn})}{\prod_{i=1}^n \partial x_{ji}^{v_i}} - \prod_{i=1}^n \lambda_i R_i u(x_{j1}, t_{j1}, x_{j2}, t_{j2}, \dots, x_{jn}, t_{jn}) \\ &= \sum_{j=1}^{\infty} \frac{\partial^{\sum_{i=1}^n m_i} u(x_{j1}, t_{j1}, x_{j2}, t_{j2}, \dots, x_{jn}, t_{jn})}{\prod_{i=1}^n \partial t_{ji}^{m_i}}. \end{aligned}$$

Then, by taking multidimensional Laplace transform to the above equation we get

$$\sum_{j=1}^{\infty} \frac{\partial^{\sum_{i=1}^n v_i} u(x_{j1}, t_{j1}, x_{j2}, t_{j2}, \dots, x_{jn}, t_{jn})}{\prod_{i=1}^n \partial x_{ji}^{v_i}} - \prod_{i=1}^n R_i(\lambda_i + s_i) u(x_{j1}, s_{j1}, x_{j2}, s_{j2}, \dots, x_{jn}, s_{jn}) = R_i u(x_{j1}, 0, x_{j2}, 0, \dots, x_{jn}, 0).$$

By transformation the above equation can be written as

$$\sum_{j=1}^{\infty} \frac{\partial^{\sum_{i=1}^n v_i} u(x_{j1}, t_{j1}, x_{j2}, t_{j2}, \dots, x_{jn}, t_{jn})}{\prod_{i=1}^n \partial x_{ji}^{v_i}} - \prod_{i=1}^n \tau_i u(x_{j1}, s_{j1}, x_{j2}, s_{j2}, \dots, x_{jn}, s_{jn}) = 0,$$

where  $t_{ji}$  is the variable of Laplace for the time-component. Let

$$u(x_{j1}, t_{j1}, x_{j2}, t_{j2}, \dots, x_{jn}, t_{jn}) = u(x_{j1}, x_{j2}, \dots, x_{jn}).$$

Then the equation becomes

$$\sum_{j=1}^{\infty} \frac{\partial^{\sum_{i=1}^n v_i} u(x_{j1}, t_{j1}, x_{j2}, t_{j2}, \dots, x_{jn}, t_{jn})}{\prod_{i=1}^n \partial x_{ji}^{v_i}} - \prod_{i=1}^n \tau_i u(x_{j1}, x_{j2}, \dots, x_{jn}) = 0.$$

By applying the Laplace operator we can get:

$$\begin{aligned} & L_n \{ u(p_{j1}, p_{j2}, \dots, p_{jn}) \} \\ &= \frac{\sum_{k=1}^{n-1} (1)^{n-k} \sum_{\substack{i_1, i_2, \dots, i_k=1 \\ i_1, i_2, \dots, i_n}} \prod_{j=1}^k s_{ij}^{m_{ij}} \prod_{\substack{i=1 \\ i \neq j}}^n \sum_{l_i=1}^{m_i} s_i^{l_i-1}}{\prod_{i=1}^n s_i^{m_i} L_n \left\{ \frac{\partial^{\sum_{i=1}^n -(m_i-v_i)} u(x_{j1}, t_{j1}, x_{j2}, t_{j2}, \dots, x_{jn}, t_{jn})}{\prod_{i=1}^n \partial x_i^{-(m_i-v_i)}} \right\} - 1} \\ & \times L_k \left\{ \frac{\partial^{\sum_{i=1}^n (m_i-l_i+v_i)} u(0, t_{j1}, 0, t_{j2}, \dots, 0, t_{jn}, x_{i1}, t_{ij1}, 0, \dots, 0, x_{i2}, t_{ij2}, 0, \dots, 0, x_{ik}, t_{ijk}, 0, \dots, 0)}{\prod_{i=1}^n (m_i - l_i + v_i)} \right\} \\ & + \frac{(-1)^n \prod_{i=1}^n \sum_{l_i=1}^{m_i} s_i^{l_i-1} \partial^{-\sum_{i=1}^n (m_i-l_i+v_i)} u(0, t_{j1}, 0, t_{j2}, \dots, 0, t_{jn})}{\prod_{i=1}^n \partial x_i^{-(m_i-l_i+v_i)}}, \\ & \prod_{i=1}^n s_i^{m_i} L_n \left\{ \frac{\partial^{\sum_{i=1}^n -(m_i-v_i)} u(x_{j1}, t_{j1}, x_{j2}, t_{j2}, \dots, x_{jn}, t_{jn})}{\prod_{i=1}^n \partial x_i^{-(m_i-v_i)}} \right\} - 1 \end{aligned}$$

and

$$= \frac{1}{n!} L_n \left\{ \prod_{i=1}^n \left( x_i^{\alpha_i n + \alpha_i - j} \left( \sum_{j=1}^{\infty} \frac{\partial^{\sum_{i=1}^n v_i}}{\prod_{i=1}^n \partial x_{ji}^{v_i}} \right) E_{\alpha_i - \beta_i, \alpha_i n \beta_i + 1 - j} (\mu_i x_i^{\alpha_i - \beta_i}) \right) \right\},$$

where

$$\left( \sum_{j=1}^{\infty} \frac{\partial^{\sum_{i=1}^n v_i}}{\prod_{i=1}^n \partial x_{ji}^{v_i}} \right) E_{\alpha_i, \beta_i(xji)} = \prod_{i=1}^n \left( \sum_{j=0}^{\infty} \frac{\Gamma(n+j+1)}{\Gamma(n\alpha_i + \beta_i + \alpha_j i)} \frac{x_i^j}{j!} \right).$$

Hence,

$$= \sum_{n=0}^{\infty} \frac{\prod_{i=1}^n \tau_i^{n_i}}{n!} \left\{ \prod_{i=1}^n \left( x_i^{\alpha_i n + \alpha_i - j} \left( \sum_{j=1}^{\infty} \frac{\partial^{\sum_{i=1}^n v_i}}{\prod_{i=1}^n \partial x_i^{v_i}} \right) E_{\alpha_i - \beta_i, \alpha_i n \beta_i + 1 - j}(\mu_i x_i^{\alpha_i - \beta_i}) \right) \right\}.$$

Thus

$$y(x) = \sum_{i=1}^2 h_i y_i(x).$$

## 5 Conclusion

This work involved an implementation of the multidimensional Laplace transform to solve the multidimensional space-time fractional differential equation. Space-time derivatives of fractional order can be defined as the originators of semigroups to calculate the Fourier-Laplace signs. Hence, by the fractional calculus we established the space-time fractional diffusion equation, which is also applicable in this field. It was shown that the present method is a powerful and effective scheme for solving other fractional problems.

### Competing interests

The authors declare that they have no competing interests.

### Authors' contributions

Both authors jointly worked on deriving the results and approved the final manuscript.

### Author details

<sup>1</sup>Department of Mathematics, Universiti Putra Malaysia, UPM, Serdang, Selangor 43400, Malaysia. <sup>2</sup>Institute for Mathematical Research, Department of Mathematics, Universiti Putra Malaysia, UPM, Serdang, Selangor 43400, Malaysia.

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