Hu *Advances in Difference Equations* (2016) 2016:200
DOI 10.1186/s13662-016-0924-1

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Existence of solutions to a coupled system of fractional differential equations with infinite-point boundary value conditions at resonance

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Abstract

By means of coincidence degree theory, we present an existence result for the solution of a higher-order coupled system of nonlinear fractional differential equations with infinite-point boundary conditions at resonance.

MSC: 26A33; 34B15

Keywords: fractional differential equation; infinite-point boundary value conditions; coincidence degree; resonance

1 Introduction

In this paper, we investigate the existence of solutions for the following higher-order coupled fractional differential equation with infinite-point boundary value conditions:

$$\begin{cases} D_{0^+}^{\alpha}u(t) = f(t, v(t), v'(t), \ldots, v^{(n-1)}(t)), & 0 < t < 1, \\ D_{0^+}^{\beta}v(t) = g(t, u(t), u'(t), \ldots, u^{(n-1)}(t)), & 0 < t < 1, \\ u'(0) = \cdots = u^{(n-1)}(0) = 0, & u(1) = \sum_{i=1}^{\infty} a_i u(\xi_i), \\ v'(0) = \cdots = v^{(n-1)}(0) = 0, & v(1) = \sum_{i=1}^{\infty} b_i v(\eta_i), \end{cases} \tag{1.1}$$

where $n-1 < \alpha, \beta < n$, $n \geq 2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_i < \xi_{i+1} < \cdots < 1$, $0 < \eta_1 < \eta_2 < \cdots < \eta_i < \eta_{i+1} < \cdots < 1$, $0 < a_i, b_i < 1$; $D_{0^+}^{\alpha}$ and $D_{0^+}^{\beta}$ denote the Caputo fractional derivatives, f, g are given continuous functions, and

$$\sum_{i=1}^{\infty} a_i = 1, \qquad \sum_{i=1}^{\infty} b_i = 1,$$

which implies that BVP (1.1) is at resonance.

During the past decades, fractional differential equations have attracted considerable interest because of their wide applications in various sciences, such as physics, mechanics, chemistry, engineering, electromagnetic, control, *etc.* (see [1–4]). In recent years, boundary value problems of fractional differential equations or systems of fractional differential equations at resonance have been discussed in some papers, such as [5–10]. Most of the results on the existence of solutions for fractional boundary value problems at resonance are concerned with finite points. For example, Wang *et al.* [5] discussed the following coupled system of fractional $2m$-point boundary value problem at resonance:

$$\begin{cases} D_{0^+}^{\alpha}u(t) = f(t, v(t), D_{0^+}^{\beta-1}v(t), D_{0^+}^{\beta-2}v(t)), \quad 0 < t < 1, \\ D_{0^+}^{\beta}v(t) = g(t, u(t), D_{0^+}^{\alpha-1}u(t), D_{0^+}^{\alpha-2}u(t)), \quad 0 < t < 1, \\ I_{0^+}^{3-\alpha}u(t)|_{t=0} = 0, \qquad D_{0^+}^{\alpha-2}u(1) = \sum_{i=1}^{m} a_i D_{0^+}^{\alpha-2}u(\xi_i), \\ I_{0^+}^{3-\beta}v(t)|_{t=0} = 0, \qquad D_{0^+}^{\beta-2}v(1) = \sum_{i=1}^{m} c_i D_{0^+}^{\beta-2}v(\gamma_i), \\ u(1) = \sum_{i=1}^{m} b_i u(\eta_i), \qquad v(1) = \sum_{i=1}^{m} d_i v(\delta_i), \end{cases}$$

where $2 < \alpha, \beta \leq 3$, $0 < \xi_1 < \cdots < \xi_m < 1$, $0 < \eta_1 < \cdots < \eta_m < 1$, $0 < \gamma_1 < \cdots < \gamma_m < 1$, $0 < \delta_1 < \cdots < \delta_m < 1$, $a_i, b_i, c_i, d_i \in \mathbb{R}$, $f, g : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$, f, g satisfy the Carathéodory conditions, $D_{0^+}^{\alpha}$, $I_{0^+}^{\alpha}$ are standard Riemann-Liouville fractional operators.

In [6], Liu *et al.* discussed the following boundary value problem for a coupled system of fractional differential equations at resonance:

$$\begin{cases} D_{0^+}^{\alpha}u(t) = f(t, v(t), D_{0^+}^{p}v(t)), \quad 0 < t < 1, \\ D_{0^+}^{\beta}v(t) = g(t, u(t), D_{0^+}^{q}u(t)), \quad 0 < t < 1, \\ u(0) = 0, \qquad D_{0^+}^{p}u(1) = \sum_{i=1}^{m-2} a_i D_{0^+}^{p}u(\xi_i), \\ v(0) = 0, \qquad D_{0^+}^{q}v(1) = \sum_{i=1}^{m-2} b_i D_{0^+}^{q}v(\eta_i), \end{cases}$$

where $1 < \alpha, \beta \leq 2$, $0 < p, q < 1$, $\alpha - p - 1, \beta - q - 1 \geq 0$, $a_i, b_i \geq 0$, $0 < \xi_i, \eta_i < 1$ ($i = 1, 2, \ldots, m-2$), $\sum_{i=1}^{m-2} a_i \xi_i^{\alpha-p-1} = \sum_{i=1}^{m-2} b_i \eta_i^{\beta-q-1} = 1$; $D_{0^+}^{\alpha}$, $D_{0^+}^{\beta}$ are standard Riemann-Liouville fractional derivatives.

Very recently, the infinite-point boundary value problems of fractional differential equations have been discussed by researchers, whose excellent results extend many previous results; see [11–14].

In 2015, Zhang [11] considered the existence of positive solutions of the following nonlinear fractional differential equation with infinite-point boundary value conditions:

$$\begin{cases} D_{0^+}^{\alpha}u(t) + q(t)f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ v^{(i)}(1) = \sum_{j=1}^{\infty} \alpha_j u(\xi_j), \end{cases}$$

where $\alpha > 2$, $n - 1 < \alpha \leq n$, $a_j \geq 0$, $0 < \xi_1 < \xi_2 < \cdots < \xi_j < \cdots < 1$ ($j = 1, 2, \ldots$), $\Delta - \sum_{j=1}^{\infty} \alpha_j \xi_j^{\alpha-1} > 0$, $\Delta = (\alpha - 1)(\alpha - 2)\cdots(\alpha - i)$, $i \in [1, n-2]$ is a fixed integer, $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative.

In [14], Ge *et al.* considered the existence of solutions of the following nonlinear fractional differential equation with infinitely many points boundary value problems at resonance:

$$\begin{cases} D_{0^+}^{\alpha}x_1(t) = f_1(t, x_1(t), D_{0^+}^{\beta-1}x_2(t)), \\ D_{0^+}^{\beta}x_2(t) = f_2(t, x_2(t), D_{0^+}^{\alpha-1}x_1(t)), \\ x_1(0) = 0, \qquad \lim_{t\to\infty} D_{0^+}^{\alpha-1}x_1(t) = \sum_{i=1}^{+\infty} \gamma_i x_1(\eta_i), \\ x_2(0) = 0, \qquad \lim_{t\to\infty} D_{0^+}^{\beta-1}x_2(t) = \sum_{i=1}^{+\infty} \sigma_i x_2(\xi_i), \end{cases}$$

where $1 < \alpha, \beta \leq 2$, $0 < \eta_1 < \eta_2 < \cdots < \eta_i < \cdots$, $0 < \xi_1 < \xi_2 < \cdots < \xi_i < \cdots$, $\lim_{i\to\infty} \eta_i = \infty$, $\lim_{i\to\infty} \xi_i = \infty$, and $\sum_{i=1}^{\infty} |\gamma_i|\eta_i^{\alpha} < \infty$, $\sum_{i=1}^{\infty} |\sigma_i|\xi_i^{\beta} < \infty$. Here, $f_1, f_2 : [0,+\infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfy the Carathéodory conditions, $D_{0^+}^{\alpha}$, $D_{0^+}^{\beta}$ are the standard Riemann-Liouville fractional derivatives.

From the above work, we note that it is meaningful and interesting to study the existence of solutions for fractional boundary value problems with infinite-point boundary conditions. Although fractional boundary value problems at resonance have been studied by some authors, to the best of our knowledge, fractional differential equations subject to infinite points at resonance have not been studied till now. Motivated by the work above, we considered the existence of solutions for BVP (1.1).

The rest of this paper is organized as follows. In Section 2, we give some necessary notations, definitions, and lemmas. In Section 3, we study the existence of solutions of (1.1) by the coincidence degree theory due to Mawhin [14]. Finally, an example is given to illustrate our results in Section 4.

2 Preliminaries

We present the necessary definitions and lemmas from fractional calculus theory that will be used to prove our main theorems.

Definition 2.1 ([1]) The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0,\infty) \to \mathbb{R}$ is given by

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds,$$

provided that the right-hand side is pointwise defined on $(0,\infty)$.

Definition 2.2 ([1]) The Caputo fractional derivative of order $\alpha > 0$ of a continuous function $f : (0,\infty) \to \mathbb{R}$ is given by

$$D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}}\, ds,$$

where $n-1 < \alpha \leq n$, provided that the right-hand side is pointwise defined on $(0,\infty)$.

Lemma 2.1 ([1]) *Let $n-1 < \alpha \leq n$, $u \in C(0,1) \cap L^1(0,1)$, then*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + c_0 + c_1 t + \cdots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0,1,\ldots,n-1$.

Lemma 2.2 ([1]) *If $\beta > 0$, $\alpha + \beta > 0$, then the equation*

$$I_{0^+}^{\alpha} I_{0^+}^{\beta} f(x) = I_{0^+}^{\alpha+\beta} f(x),$$

is satisfied for a continuous function f.

First of all, we briefly recall some definitions on the coincidence degree theory. For more details, see [14].

Let Y, Z be real Banach spaces, $L : \operatorname{dom} L \subset Y \to Z$ be a Fredholm map of index zero and $P : Y \to Y$, $Q : Z \to Z$ be continuous projectors such that

$$\operatorname{Ker} L = \operatorname{Im} P, \qquad \operatorname{Im} L = \operatorname{Ker} Q, \qquad Y = \operatorname{Ker} L \oplus \operatorname{Ker} P, \qquad Z = \operatorname{Im} L \oplus \operatorname{Im} Q.$$

It follows that

$$L|_{\operatorname{dom} L \cap \operatorname{Ker} P} : \operatorname{dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$$

is invertible. We denote the inverse of this map by K_P.

If Ω is an open bounded subset of Y, the map N will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_{P,Q}N = K_P(I - Q)N : \overline{\Omega} \to Y$ is compact.

Theorem 2.1 *Let L be a Fredholm operator of index zero and N be L-compact on $\overline{\Omega}$. Suppose that the following conditions are satisfied*:

(1) $Lx \neq \lambda Nx$ *for each* $(x, \lambda) \in [(\operatorname{dom} L \setminus \operatorname{Ker} L) \cap \partial\Omega] \times (0, 1)$;

(2) $Nx \notin \operatorname{Im} L$ *for each* $x \in \operatorname{Ker} L \cap \partial\Omega$;

(3) $\deg(JQN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) \neq 0$, *where* $Q : Z \to Z$ *is a continuous projection as above with* $\operatorname{Im} L = \operatorname{Ker} Q$ *and* $J : \operatorname{Im} Q \to \operatorname{Ker} L$ *is any isomorphism.*

Then the equation $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$.

3 Main results

In this paper, we will always suppose the following condition holds:

(H1) $\sum_{i=1}^{\infty} a_i \xi_i^{\alpha} \neq 1$, $\sum_{i=1}^{\infty} b_i \eta_i^{\beta} \neq 1$.

Denote by E the Banach space $E = C[0, 1]$ with the norm $\|u\|_\infty = \max_{0 \le t \le 1} |u(t)|$. We denote a Banach space $X = \{u(t) | u^{(i)}(t) \in E, i = 1, 2, \ldots, n - 1\}$ with the norm $\|u\|_X = \max\{\|u\|_\infty, \|u'\|_\infty, \ldots, \|u^{(n-1)}\|_\infty\}$. Let $Y = X \times X$ be endowed with the norm $\|(u, v)\|_Y = \max\{\|u\|_X, \|v\|_X\}$, and $Z = E \times E$ is a Banach space with the norm defined by $\|(x, y)\|_Z = \max\{\|x\|_\infty, \|y\|_\infty\}$.

Define the linear operator $L_1 : \operatorname{dom} L_1 \to E$ by setting

$$\operatorname{dom} L_1 = \left\{ u \in X \,\middle|\, u'(0) = \cdots = u^{(n-1)}(0) = 0, u(1) = \sum_{i=1}^{\infty} a_i u(\xi_i) \right\}$$

and

$$L_1 u = D_{0^+}^{\alpha} u, \quad u \in \operatorname{dom} L_1.$$

Define the linear operator L_2 from $\operatorname{dom} L_2 \to E$ by setting

$$\operatorname{dom} L_2 = \left\{ v \in X \,\middle|\, v'(0) = \cdots = v^{(n-1)}(0) = 0, v(1) = \sum_{i=1}^{\infty} b_i v(\eta_i) \right\}$$

and

$$L_2 v = D_{0^+}^{\beta} v, \quad v \in \operatorname{dom} L_2.$$

Define the operator $L : \operatorname{dom} L \to Z$ with

$$\operatorname{dom} L = \{(u,v) \in Y | u \in \operatorname{dom} L_1, v \in \operatorname{dom} L_2\}$$

and

$$L(u,v) = (L_1 u, L_2 v).$$

Let $N : Y \to Z$ be the Nemytski operator

$$N(u,v) = (N_1 v, N_2 u),$$

where $N_1 : X \to E$ is defined by

$$N_1 v(t) = f\bigl(t, v(t), v'(t), v''(t), \dots, v^{(n-1)}(t)\bigr),$$

and $N_2 : X \to E$ is defined by

$$N_2 u(t) = g\bigl(t, u(t), u'(t), u''(t), \dots, u^{(n-1)}(t)\bigr).$$

Then BVP (1.1) can be written as $L(u,v) = N(u,v)$.

Lemma 3.1 *L is defined as above, then*

$$\operatorname{Ker} L = \{(u,v) \in X | (u,v) = (c_0, d_0), c_0, d_0 \in \mathbb{R}\}, \tag{3.1}$$

$$\operatorname{Im} L = \left\{(x,y) \in Y \middle| I_{0^+}^{\alpha} x(1) - \sum_{i=1}^{\infty} a_i I_{0^+}^{\alpha} x(\xi_i) = 0, I_{0^+}^{\beta} y(1) - \sum_{i=1}^{\infty} b_i I_{0^+}^{\beta} y(\eta_i) = 0\right\}. \tag{3.2}$$

Proof By Lemma 2.1, the equation $D_{0^+}^{\alpha} u(t) = 0$ has the solution

$$u(t) = c_0 + c_1 t + \cdots + c_{n-1} t^{n-1}.$$

Combining with $u^{(i)}(0) = 0$, $i = 1, 2, \dots, n-1$, one has $c_i = 0$, $i = 1, 2, \dots, n-1$. Then $u(t) = c_0$. Similarly, for $v \in \operatorname{Ker} L_2$, we have $v(t) = d_0$. Thus, we obtain (3.1).

Next we prove (3.2) holds. Let $(x,y) \in \operatorname{Im} L$, so there exists $(u,v) \in \operatorname{dom} L$ such that $x(t) = D_{0^+}^{\alpha} u(t)$, $y(t) = D_{0^+}^{\beta} v(t)$. By Lemma 2.1, we have

$$u(t) = I_{0^+}^{\alpha} x(t) + \sum_{i=0}^{n-1} c_i t^i, \qquad v(t) = I_{0^+}^{\beta} y(t) + \sum_{i=0}^{n-1} d_i t^i, \quad c_i, d_i \in \mathbb{R}, i = 0, 1, \dots, n-1.$$

In view of $u^{(i)}(0) = v^{(i)}(0) = 0$, $i = 1, 2, \dots, n-1$, we get $c_i = d_i = 0$, $i = 1, 2, \dots, n-1$. Hence, we have

$$u(t) = I_{0^+}^{\alpha} x(t) + c_0, \qquad v(t) = I_{0^+}^{\beta} y(t) + d_0.$$

According to $u(1) = \sum_{i=1}^{\infty} a_i u(\xi_i)$ and $v(1) = \sum_{i=1}^{\infty} b_i v(\eta_i)$, we have

$$I_{0^+}^{\alpha} x(1) + c_0 = \sum_{i=1}^{\infty} a_i u(\xi_i) = \sum_{i=1}^{\infty} a_i \bigl(I_{0^+}^{\alpha} x(\xi_i) + c_0\bigr) = \sum_{i=1}^{\infty} a_i I_{0^+}^{\alpha} x(\xi_i) + c_0,$$
$$I_{0^+}^{\beta} y(1) + d_0 = \sum_{i=1}^{\infty} b_i v(\xi_i) = \sum_{i=1}^{\infty} b_i \bigl(I_{0^+}^{\beta} y(\eta_i) + c_0\bigr) = \sum_{i=1}^{\infty} b_i I_{0^+}^{\beta} y(\eta_i) + d_0,$$

that is,

$$I_{0^+}^{\alpha} x(1) = \sum_{i=1}^{\infty} a_i I_{0^+}^{\alpha} x(\xi_i), \qquad I_{0^+}^{\beta} y(1) = \sum_{i=1}^{\infty} b_i I_{0^+}^{\beta} y(\eta_i).$$

On the other hand, suppose (x, y) satisfies the above equations. Let $u(t) = I_{0^+}^{\alpha} x(t)$ and $v(t) = I_{0^+}^{\beta} y(t)$, we can prove $(u(t), v(t)) \in \operatorname{dom} L$ and $L(u(t), v(t)) = (x, y)$. Then (3.2) holds. □

Lemma 3.2 *The mapping $L : \operatorname{dom} L \subset Y \to Z$ is a Fredholm operator of index zero.*

Proof The linear continuous projector operator $P(u, v) = (P_1 u, P_2 v)$ can be defined as

$$P_1 u = u(0), \qquad P_2 v = v(0).$$

Obviously, $P_1^2 = P_1$ and $P_2^2 = P_2$.

It is clear that

$$\operatorname{Ker} P = \bigl\{(u, v) | u(0) = 0, v(0) = 0\bigr\}.$$

It follows from $(u, v) = (u, v) - P(u, v) + P(u, v)$ that $Y = \operatorname{Ker} P + \operatorname{Ker} L$. For $(u, u) \in \operatorname{Ker} L \cap \operatorname{Ker} P$, then $u = c_0$, $v = d_0$, $c_0, d_0 \in \mathbb{R}$. Furthermore, by the definition of $\operatorname{Ker} P$, we have $c_0 = d_0 = 0$. Thus, we get

$$Y = \operatorname{Ker} L \oplus \operatorname{Ker} P.$$

The linear operator $Q(x, y) = (Q_1 x, Q_2 y)$ can be defined as

$$Q_1 x(t) = \frac{\Gamma(1+\alpha)}{1 - \sum_{i=1}^{\infty} a_i \xi_i^{\alpha}} \Biggl[I_{0^+}^{\alpha} x(1) - \sum_{i=1}^{\infty} a_i I_{0^+}^{\alpha} x(\xi_i) \Biggr],$$
$$Q_2 y(t) = \frac{\Gamma(1+\beta)}{1 - \sum_{i=1}^{\infty} b_i \eta_i^{\beta}} \Biggl[I_{0^+}^{\beta} y(1) - \sum_{i=1}^{\infty} b_i I_{0^+}^{\beta} y(\eta_i) \Biggr].$$

Obviously, $Q(x, y) = (Q_1 x(t), Q_2 y(t)) \cong \mathbb{R}^2$.

For $x(t) \in E$, we have

$$Q_1\bigl(Q_1 x(t)\bigr) = Q_1 x(t) \cdot \frac{\Gamma(1+\alpha)}{1 - \sum_{i=1}^{\infty} a_i \xi_i^{\alpha}} \Biggl[\bigl(I_{0^+}^{\alpha} 1\bigr)|_{t=1} - \sum_{i=1}^{\infty} a_i \bigl(I_{0^+}^{\alpha} 1\bigr)|_{t=\xi_i} \Biggr] = Q_1 x(t).$$

Similarly, $Q_2^2 = Q_2$, that is to say, the operator Q is idempotent. It follows from $(x,y) = (x,y) - Q(x,y) + Q(x,y)$ that $Z = \operatorname{Im} L + \operatorname{Im} Q$. Moreover, by $\operatorname{Ker} Q = \operatorname{Im} L$ and $Q_2^2 = Q_2$, we get $\operatorname{Im} L \cap \operatorname{Im} Q = \{(0,0)\}$. Hence,

$$Z = \operatorname{Im} L \oplus \operatorname{Im} Q.$$

Now, $\operatorname{Ind} L = \dim \operatorname{Ker} L - \operatorname{codim} \operatorname{Im} L = 0$, and so L is a Fredholm mapping of index zero. □

For every $(u,v) \in Y$,

$$\left\|P(u,v)\right\|_Y = \max\left\{\|P_1 u\|_X; \|P_2 v\|_X\right\} = \max\left\{\left|u(0)\right|; \left|v(0)\right|\right\}. \tag{3.3}$$

Furthermore, the operator $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ can be defined by

$$K_P(x,y) = \left(I_{0^+}^{\alpha} x, I_{0^+}^{\beta} y\right). \tag{3.4}$$

For $(x,y) \in \operatorname{Im} L$, we have

$$LK_P(x,y) = L\left(I_{0^+}^{\alpha} x, I_{0^+}^{\beta} y\right) = \left(D_{0^+}^{\alpha} I_{0^+}^{\alpha} x, D_{0^+}^{\beta} I_{0^+}^{\beta} y\right) = (x,y).$$

On the other hand, for $(u,v) \in \operatorname{dom} L \cap \operatorname{Ker} P$, according to Lemma 2.1, we have

$$\begin{aligned} I_{0^+}^{\alpha} L_1 u(t) &= I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + c_0 + c_1 t + \cdots + c_{n-1} t^{n-1}, \\ I_{0^+}^{\beta} L_2 v(t) &= I_{0^+}^{\beta} D_{0^+}^{\beta} v(t) = v(t) + d_0 + d_1 t + \cdots + d_{n-1} t^{n-1}. \end{aligned}$$

By the definitions of $\operatorname{dom} L$ and $\operatorname{Ker} P$, one has $u^{(i)}(0) = v^{(i)}(0)$, $i = 0,1,\ldots,n-1$, which implies that $c_i = d_i$, $i = 0,1,\ldots,n-1$. Thus, we obtain

$$K_p L(x,y) = \left(I_{0^+}^{\alpha} D_{0^+}^{\alpha} x, I_{0^+}^{\beta} D_{0^+}^{\beta} y\right) = (x,y). \tag{3.5}$$

Combining (3.4) and (3.5), we get $K_P = (L_{\operatorname{dom} L \cap \operatorname{Ker} P})^{-1}$.

For simplicity of notation, we set $a = \frac{1}{\Gamma(\alpha-n+2)}$, $b = \frac{1}{\Gamma(\beta-n+2)}$.

For $(x,y) \in \operatorname{Im} L$, we have

$$\begin{aligned} \left\|K_P(x,y)\right\|_Y &= \left\|\left(I_{0^+}^{\alpha} x, I_{0^+}^{\beta} y\right)\right\|_Y = \max\left\{\left\|I_{0^+}^{\alpha} x\right\|_X; \left\|I_{0^+}^{\beta} y\right\|_X\right\} \\ &\le \max\left\{\frac{1}{\Gamma(\alpha-n+2)}\|x\|_\infty; \frac{1}{\Gamma(\beta-n+2)}\|y\|_\infty\right\} \\ &= \max\left\{a\|x\|_\infty; b\|y\|_\infty\right\}. \end{aligned} \tag{3.6}$$

Again for $(u,v) \in \Omega_1$, $(u,v) \in \operatorname{dom}(L) \setminus \operatorname{Ker}(L)$, then $(I-P)(u,v) \in \operatorname{dom} L \cap \operatorname{Ker} P$ and $LP(u,v) = (0,0)$, thus from (3.6), we have

$$\begin{aligned} \left\|(I-P)(u,v)\right\|_Y &= \left\|K_P L(I-P)(u,v)\right\|_Y = \left\|K_P(L_1 u, L_2 v)\right\|_Y \\ &\le \max\left\{a\|N_1 v\|_\infty; b\|N_2 u\|_\infty\right\}. \end{aligned} \tag{3.7}$$

With a similar proof to [15], we have the following lemma.

Lemma 3.3 *$K_P(I-Q)N : Y \to Y$ is completely continuous.*

Theorem 3.1 *Assume* (H1) *and the following conditions hold*:

(H2) *There exist nonnegative functions $\varphi_i(t), \psi_i(t) \in E$, $i = 0, 1, \ldots, n$, such that, for all $t \in [0,1]$, $(u_1, u_2, \ldots, u_n), (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$, one has*

$$\begin{aligned} \bigl|f(t, u_1, u_2, \ldots, u_n)\bigr| &\le \varphi_0(t) + \varphi_1(t)|u_1| + \varphi_2(t)|u_2| + \cdots + \varphi_n(t)|u_n|, \\ \bigl|g(t, v_1, v_2, \ldots, v_n)\bigr| &\le \psi_0(t) + \psi_1(t)|v_1| + \psi_2(t)|v_2| + \cdots + \psi_n(t)|v_n|. \end{aligned}$$

(H3) *There exists $A > 0$ such that, for $(u, u', \ldots, u^{(n-1)})$, $(v, v', \ldots, v^{(n-1)})$, if $|u| > A$ or $|v| > A$, $\forall t \in [0,1]$, one has*

$$\begin{aligned} &u \cdot \Biggl[I_{0^+}^{\alpha} f\bigl(t, v, v', \ldots, v^{(n-1)}\bigr)|_{t=1} - \sum_{i=1}^{\infty} a_i I_{0^+}^{\alpha} f\bigl(t, v, v', \ldots, v^{(n-1)}\bigr)|_{t=\xi_i} \Biggr] > 0, \\ &v \cdot \Biggl[I_{0^+}^{\beta} g\bigl(t, u, u', \ldots, u^{(n-1)}\bigr)|_{t=1} - \sum_{i=1}^{\infty} b_i I_{0^+}^{\beta} g\bigl(t, u, u', \ldots, u^{(n-1)}\bigr)|_{t=\eta_i} \Biggr] > 0, \end{aligned}$$

or

$$\begin{aligned} &u \cdot \Biggl[I_{0^+}^{\alpha} f\bigl(t, v, v', \ldots, v^{(n-1)}\bigr)|_{t=1} - \sum_{i=1}^{\infty} a_i I_{0^+}^{\alpha} f\bigl(t, v, v', \ldots, v^{(n-1)}\bigr)|_{t=\xi_i} \Biggr] < 0, \\ &v \cdot \Biggl[I_{0^+}^{\beta} g\bigl(t, u, u', \ldots, u^{(n-1)}\bigr)|_{t=1} - \sum_{i=1}^{\infty} b_i I_{0^+}^{\beta} g\bigl(t, u, u', \ldots, u^{(n-1)}\bigr)|_{t=\eta_i} \Biggr] < 0. \end{aligned}$$

Then BVP (1.1) *has at least a solution in Y provided that*

$$\max\Biggl\{ 2a \sum_{i=1}^{n} \|\varphi_i\|_\infty, a \sum_{i=1}^{n} \|\varphi_i\|_\infty + b \sum_{i=1}^{n} \|\psi_i\|_\infty, 2b \sum_{i=1}^{n} \|\psi_i\|_\infty \Biggr\} < 1. \tag{3.8}$$

Proof Let

$$\Omega_1 = \bigl\{(u, v) \in \operatorname{dom} L \setminus \operatorname{Ker} L : L(u, v) = \lambda N(u, v), \lambda \in (0, 1)\bigr\}.$$

For $L(u, v) = \lambda N(u, v) \in \operatorname{Im} L = \operatorname{Ker} Q$, by the definition of $\operatorname{Ker} Q$, hence

$$\begin{aligned} &I_{0^+}^{\alpha} f\bigl(t, v, v', \ldots, v^{(n-1)}\bigr)|_{t=1} - \sum_{i=1}^{\infty} a_i I_{0^+}^{\alpha} f\bigl(t, v, v', \ldots, v^{(n-1)}\bigr)|_{t=\xi_i} = 0, \\ &I_{0^+}^{\beta} g\bigl(t, u, u', \ldots, u^{(n-1)}\bigr)|_{t=1} - \sum_{i=1}^{\infty} b_i I_{0^+}^{\beta} g\bigl(t, u, u', \ldots, u^{(n-1)}\bigr)|_{t=\eta_i} = 0. \end{aligned}$$

From (H3), there exist $t_0, t_1 \in (0,1)$ such that $|u(t_0)| \le A$ and $|v(t_1)| \le A$. According to $L_1 u = \lambda N_1 v$, $u \in \operatorname{dom} L_1 \setminus \operatorname{Ker} L_1$, that is, $D_{0^+}^{\alpha} u = \lambda N_1 v$, we have

$$u(t) = \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\bigl(s, v(s), v'(s), \ldots, v^{(n-1)}(s)\bigr)\, ds + c_0.$$

Substituting $t = t_0$ into the above equation, we get

$$u(t_0) = \frac{\lambda}{\Gamma(\alpha)} \int_0^{t_0} (t_0 - s)^{\alpha-1} f\bigl(s, v(s), v'(s), \ldots, v^{(n-1)}(s)\bigr)\, ds + c_0.$$

Furthermore, we get

$$\begin{aligned} u(t) - u(t_0) = {} & \frac{\lambda}{\Gamma(\alpha)} \int_0^{t} (t - s)^{\alpha-1} f\bigl(s, v(s), v'(s), \ldots, v^{(n-1)}(s)\bigr)\, ds \\ & - \frac{\lambda}{\Gamma(\alpha)} \int_0^{t_0} (t_0 - s)^{\alpha-1} f\bigl(s, v(s), v'(s), \ldots, v^{(n-1)}(s)\bigr)\, ds. \end{aligned}$$

Together with $|u(t_0)| \leq A$, we have

$$\begin{aligned} \bigl|u(0)\bigr| & \leq \bigl|u(t_0)\bigr| + \left| \frac{\lambda}{\Gamma(\alpha)} \int_0^{t_0} (t_0 - s)^{\alpha-1} f\bigl(s, v(s), v'(s), \ldots, v^{(n-1)}(s)\bigr)\, ds \right| \\ & \leq A + \frac{1}{\Gamma(\alpha)} \int_0^{t_0} (t_0 - s)^{\alpha-1} \bigl| f\bigl(s, v(s), v'(s), \ldots, v^{(n-1)}(s)\bigr) \bigr|\, ds \\ & \leq A + \frac{1}{\Gamma(\alpha)} \int_0^{t_0} (t_0 - s)^{\alpha-1} \left(\varphi_0(t) + \sum_{i=1}^{n} \varphi_i(t) \bigl|v^{(i-1)}\bigr| \right) ds \\ & \leq A + \frac{1}{\Gamma(\alpha)} \left(\bigl\|\varphi_0(t)\bigr\|_\infty + \sum_{i=1}^{n} \bigl\|\varphi_i(t)\bigr\|_\infty \bigl\|v^{(i-1)}\bigr\|_\infty \right) \cdot \int_0^{t_0} (t_0 - s)^{\alpha-1}\, ds \\ & \leq A + \frac{1}{\Gamma(\alpha+1)} \bigl\|\varphi_0(t)\bigr\|_\infty + \frac{1}{\Gamma(\alpha+1)} \sum_{i=1}^{n} \bigl\|\varphi_i(t)\bigr\|_\infty \bigl\|v^{(i-1)}\bigr\|_\infty \\ & \leq A + \frac{1}{\Gamma(\alpha+1)} \bigl\|\varphi_0(t)\bigr\|_\infty + \|v\|_X \cdot \frac{1}{\Gamma(\alpha+1)} \sum_{i=1}^{n} \bigl\|\varphi_i(t)\bigr\|_\infty \\ & \leq A + a\bigl\|\varphi_0(t)\bigr\|_\infty + \|v\|_X \cdot a \sum_{i=1}^{n} \bigl\|\varphi_i(t)\bigr\|_\infty. \end{aligned} \tag{3.9}$$

By similar arguments, we obtain

$$\bigl|v(0)\bigr| \leq A + b\bigl\|\psi_0(t)\bigr\|_\infty + \|u\|_X \cdot b \sum_{i=1}^{n} \bigl\|\psi_i(t)\bigr\|_\infty. \tag{3.10}$$

For $(u,v) \in \Omega_1$, by (3.7), we have

$$\begin{aligned} \bigl\|(u,v)\bigr\|_Y & = \bigl\|P(u,v) + (I-P)(u,v)\bigr\|_Y \\ & \leq \bigl\|P(u,v)\bigr\|_Y + \bigl\|(I-P)(u,v)\bigr\|_Y \\ & \leq \max\bigl\{ \bigl|u(0)\bigr| + a\|N_1 v\|_\infty, \bigl|u(0)\bigr| + b\|N_2 u\|_\infty, \\ & \qquad \bigl|v(0)\bigr| + a\|N_1 v\|_\infty, \bigl|v(0)\bigr| + b\|N_2 u\|_\infty \bigr\}. \end{aligned}$$

The following proof is divided into four cases.

Case 1. $\|(u,v)\|_Y \leq |u(0)| + a\|N_1 v\|_\infty$. By (3.9) and ($H_1$), we have

$$\begin{aligned}
\|(u,v)\|_Y &\leq |u(0)| + a\|N_1 v\|_\infty \\
&\leq A + a\|\varphi_0\|_\infty + \|v\|_X \cdot a\sum_{i=1}^{n} \|\varphi_i\|_\infty + a\|f(t,v,\ldots,v^{(N-1)})\|_\infty \\
&\leq A + a\|\varphi_0\|_\infty + \|v\|_X \cdot a\sum_{i=1}^{n} \|\varphi_i\|_\infty + a\left(\|\varphi_0\|_\infty + \|v\|_X \cdot \sum_{i=1}^{n} \|\varphi_i\|_\infty\right) \\
&= A + 2a\left(\|\varphi_0\|_\infty + \|v\|_X \cdot \sum_{i=1}^{n} \|\varphi_i\|_\infty\right).
\end{aligned} \tag{3.11}$$

According to (3.11) and the definition of $\|(u,v)\|_Y$, we can derive

$$\|v\|_X \leq \|(u,v)\|_Y \leq A + 2a\left(\|\varphi_0\|_\infty + \|v\|_X \cdot \sum_{i=1}^{n} \|\varphi_i\|_\infty\right).$$

By (3.8), we have

$$\|v\|_X \leq \frac{A + 2a\|\varphi_0\|_\infty}{1 - 2a\sum_{i=1}^{n} \|\varphi_i\|_\infty} := M.$$

From (3.11), we see that Ω_1 is bounded.

Case 2. $\|(u,v)\|_Y \leq |v(0)| + b\|N_2 u\|_\infty$. Similar to the above argument, we can also prove that Ω_1 is bounded. Here, we omit it.

Case 3. $\|(u,v)\|_Y \leq |u(0)| + b\|N_2 u\|_\infty$. From (3.10) and (H2), we obtain

$$\begin{aligned}
\|(u,v)\|_Y &\leq |u(0)| + b\|N_2 u\|_\infty \\
&\leq A + a\|\varphi_0\|_\infty + \|v\|_X \cdot a\sum_{i=1}^{n} \|\varphi_i\|_\infty + b\|g(t,u,u',\ldots,u^{(n-1)})\|_\infty \\
&\leq A + a\|\varphi_0\|_\infty + \|v\|_X \cdot a\sum_{i=1}^{n} \|\varphi_i\|_\infty + b\left(\|\psi_0\|_\infty + \|u\|_X \cdot \sum_{i=1}^{n} \|\psi_i\|_\infty\right) \\
&\leq A + a\|\varphi_0\|_\infty + b\|\psi_0\|_\infty + \left[a\sum_{i=1}^{n} \|\varphi_i\|_\infty + b\sum_{i=1}^{n} \|\psi_i\|_\infty\right] \cdot \|(u,v)\|_Y.
\end{aligned}$$

By (3.8), we get

$$\|(u,v)\|_Y \leq \left[1 - a\sum_{i=1}^{n} \|\varphi_i\|_\infty - b\sum_{i=1}^{n} \|\psi_i\|_\infty\right]^{-1} \left(A + a\|\varphi_0\|_\infty + b\|\psi_0\|_\infty\right) := M,$$

that is, Ω_1 is bounded.

Case 4. $\|(u,v)\|_\infty \leq |v(0)| + a\|N_1 v\|_\infty$. We can prove that Ω_1 is bounded too. The proof is similar to the case 2. Here, we omit it.

According the above arguments, we have proved that Ω_1 is bounded.

Let

$$\Omega_2 = \big\{(u,v) \in \operatorname{Ker} L : N(u,v) \in \operatorname{Im} L\big\}.$$

Let $(u,v) \in \operatorname{Ker} L$, so we have $u = c_0$, $v = d_0$. In view of $N(u,v) = (N_1 v, N_2 u) \in \operatorname{Im} L = \operatorname{Ker} Q$, we have $Q_1(N_1 v) = 0$, $Q_2(N_2 u) = 0$, that is,

$$I_{0^+}^{\alpha} f(t, d_0, 0, \ldots, 0)|_{t=1} - \sum_{i=1}^{\infty} a_i I_{0^+}^{\alpha} f(t, d_0, 0, \ldots, 0)|_{t=\xi_i} = 0,$$

$$I_{0^+}^{\beta} g(t, c_0, 0, \ldots, 0)|_{t=1} - \sum_{i=1}^{\infty} b_i I_{0^+}^{\beta} g(t, c_0, 0, \ldots, 0)|_{t=\eta_i} = 0.$$

By (H2), there exist constants $t_0, t_1 \in [0,1]$ such that

$$\big|u(t_0)\big| = |c_0| \le A, \qquad \big|v(t_1)\big| = |d_0| \le A.$$

Therefore, Ω_2 is bounded.

Let

$$\Omega_3 = \big\{(u,v) \in \operatorname{Ker} L : \lambda(u,v) + (1-\lambda) QN(u,v) = (0,0), \lambda \in [0,1]\big\}.$$

For $(u,v) \in \operatorname{Ker} L$, we have $u = c_0$ and $v = d_0$. By the definition of the set Ω_3, we have

$$\lambda c_0 + (1-\lambda) Q_1 N_1(d_0) = 0, \qquad \lambda d_0 + (1-\lambda) Q_2 N_2(c_0) = 0. \tag{3.12}$$

If $\lambda = 0$, similar to the proof of the boundedness of Ω_2, we have $|c_0| \le A$ and $|d_0| \le A$. If $\lambda = 1$, we have $c_0 = d_0 = 0$. If $\lambda \in (0,1)$, we also have $|c_0| \le A$ and $|d_0| \le A$. Otherwise, if $|c_0| > A$ or $|d_0| > A$, in view of the first part of (H3), we obtain

$$\lambda c_0^2 + (1-\lambda) c_0 \cdot Q_1 N_1(d_0) > 0, \qquad \lambda d_0^2 + (1-\lambda) d_0 \cdot Q_2 N_2(c_0) > 0,$$

which contradict (3.12). Thus, Ω_3 is bounded.

If the second part of (H_3) holds, then we can prove the set

$$\Omega_3' = \big\{(u,v) \in \operatorname{Ker} L : -\lambda(u,v) + (1-\lambda) QN(u,v) = (0,0), \lambda \in [0,1]\big\}$$

is bounded.

Finally, let Ω be a bounded open set of Y, such that $\bigcup_{i=1}^{3} \overline{\Omega}_i \subset \Omega$. By Lemma 3.3, N is L-compact on Ω. Then by the above arguments, we get:

(1) $Lu \neq \lambda Nu$, for every $(u,v) \in [(\operatorname{dom} L \setminus KerL) \cap \partial\Omega] \times (0,1)$.
(2) $N(u,v) \notin \operatorname{Im} L$ for every $(u,v) \in \operatorname{Ker} L \cap \partial\Omega$.
(3) Let $H((u,v),\lambda) = \pm\lambda I(u,v) + (1-\lambda) JQN(u,v)$, where I is the identical operator. Via the homotopy property of degree, we obtain

$$\begin{aligned} \deg(JQN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) &= \deg\big(H(\cdot,0), \Omega \cap \operatorname{Ker} L, 0\big) \\ &= \deg\big(H(\cdot,1), \Omega \cap \operatorname{Ker} L, 0\big) \end{aligned}$$

$$= \deg(I, \Omega \cap \operatorname{Ker} L, 0)$$
$$= 1 \neq 0.$$

Applying Theorem 2.1, we conclude that $L(u,v) = N(u,v)$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$. □

4 Example

Let us consider the following coupled system of fractional differential equations at resonance:

$$\begin{cases} D_{0^+}^{2.4}u(t) = f(t,v,v',v''), \quad 0 < t < 1, \\ D_{0^+}^{2.7}v(t) = g(t,u,u',u''), \quad 0 < t < 1, \\ u'(0) = u''(0) = 0 = v'(0) = v''(1), \\ u(1) = \sum_{i=1}^{\infty} \frac{1}{2^i} u(\frac{1}{2i}), \qquad v(1) = \sum_{i=1}^{\infty} \frac{2}{3^i} v(\frac{1}{2i+1}), \end{cases} \tag{4.1}$$

where

$$f(t,x_1,x_2,x_3) = \frac{t}{2} + \frac{1}{15}e^{-|x_1|} + \sin^2 x_2 + \cos x_3 + 1,$$
$$g(t,y_1,y_2,y_3) = t + \frac{\arctan y_1}{25} + \frac{\sin(y_2+y_3)}{30} + \frac{1}{2}.$$

Corresponding to BVP (1.1), we have $\alpha = 2.4$, $\beta = 2.7$, $n = 3$, $a = (\Gamma(\alpha - n + 2))^{-1} = (\Gamma(1.4))^{-1} \approx 1.13$, $b = (\Gamma(\beta - n + 2))^{-1} = (\Gamma(1.7))^{-1} \approx 1.10$, $a_i = \frac{1}{2^i}$, $b_i = \frac{2}{3^i}$, $\xi_i = \frac{1}{2i}$, $\eta_i = \frac{1}{2i+1}$, $i = 1, 2, \ldots$. We can get

$$\sum_{i=1}^{\infty} a_i\xi_i^{\alpha} = \sum_{i=1}^{\infty} \frac{1}{2^i}\frac{1}{(2i)^{2.4}} \approx 0.106 \neq 1, \qquad \sum_{i=1}^{\infty} b_i\eta_i^{\beta} = \sum_{i=1}^{\infty} \frac{2}{3^i}\frac{1}{(2i+1)^{2.7}} \approx 0.037 \neq 1,$$

which implies (H1) holds. We choose $\varphi_0(t) = \frac{t}{2} + 4$, $\psi_0(t) = t + 2$, $\varphi_i = \psi_i = 0$, $i = 1, 2, 3$. Then we can verify (H2) and (3.8) hold. Take $A = 12$, then the condition (H3) holds. Hence, from Theorem 3.1, BVP (4.1) has at least one solution.

Competing interests
The author declares that he has no competing interests.

Author's contributions
Only the author contributed to the writing of this paper. The author read and approved the final manuscript.

Acknowledgements
The research was supported by the Science Foundation of Shandong Jiaotong University (Z201429).

Received: 21 March 2016 Accepted: 22 July 2016 Published online: 02 August 2016

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