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A fourth order block-hexagonal grid approximation for the solution of Laplace's equation with singularities

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Abstract

The hexagonal grid version of the block-grid method, which is a difference-analytical method, has been applied for the solution of Laplace's equation with Dirichlet boundary conditions, in a special type of polygon with corner singularities. It has been justified that in this polygon, when the boundary functions away from the singular corners are from the Hölder classes $C^{4,\lambda}$, $0<\lambda<1$, the uniform error is of order $O(h^4)$, h is the step size, when the hexagonal grid is applied in the 'nonsingular' part of the domain. Moreover, in each of the finite neighborhoods of the singular corners ('singular' parts), the approximate solution is defined as a quadrature approximation of the integral representation of the harmonic function, and the errors of any order derivatives are estimated. Numerical results are presented in order to demonstrate the theoretical results obtained.

Keywords: hexagonal grid; Laplace's equation; singularity problem; block-grid method

1 Introduction

It is well known that angular singularities arise in many applied problems when the solution of Laplace's equation is considered, and that finite-difference and finite-element methods may become less accurate when singularities are not taken into account. In the last two decades, for the solution of singularity problems, various combined and highly accurate methods have been proposed (see [1–8], and references therein).

Among these methods the block-grid method (BGM), presented in [6–8], on polygons with interior angles $\alpha_j\pi$, $j=1,2,\dots,N$, where $\alpha_j\in\{\frac{1}{2},1,\frac{3}{2},2\}$ (staircase polygons), requires the finite neighborhood of the singular vertices to be covered by sectors (blocks), and the remaining part of the domain by overlapping rectangles ('nonsingular' part). The finite-difference method with square grids is used for the approximate solution in the 'nonsingular' part, and in the blocks the integral representations of the harmonic function are approximated by the exponentially convergent mid-point quadrature rule (see [9]). Finally these subsystems are connected together by constructing an appropriate order matching operator. BGM is a highly accurate method not only for the approximation of the solution, but also for the approximation of its derivatives around singular points.

In this paper, the fourth order BGM is extended and justified for the Dirichlet problem of Laplace's equation on polygons with interior angles $\alpha_j\pi$, where $\alpha_j\in\{\frac{1}{3},\frac{2}{3},1,2\}$ (nonstaircase), by gluing with the matching operator the 7-point approximation on a hexagonal grid in the 'nonsingular' part and the approximation of the integral representations around the singular points (on 'singular' parts).

An advantage of using the hexagonal grid version of BGM in this domain is that a highly accurate approximation on the irregular grids is not required as in [8]. Thus the realization of the total system of algebraic equations becomes simpler. This may not be the case for this type of domain when square or rectangular grids are applied.

Furthermore it is justified that, when the boundary functions on the sides except the adjacent sides of the singular vertices are given in $C^{4,\lambda}$, $0 < \lambda < 1$, the proposed hexagonal grid version of BGM has an accuracy of $O(h^4)$, h is the mesh step. The same order of accuracy is obtained for the 9-point scheme on a square grid (see [10, 11]).

Finally in the last section of the paper, numerical experiments are demonstrated to support the theoretical results obtained.

2 Boundary value problem on a special type of polygon

Let D be an open simply connected polygon, γ_j, $j = 1, 2, \ldots, N$, be its sides, including the ends, enumerated counterclockwise ($\gamma_0 \equiv \gamma_N$, $\gamma_1 \equiv \gamma_{N+1}$), and let $\alpha_j\pi$, $\alpha_j \in \{\frac{1}{3}, \frac{2}{3}, 1, 2\}$, be the interior angles formed by the sides γ_{j-1} and γ_j. Furthermore, let $\dot{\gamma}_j = \gamma_{j-1} \cap \gamma_j$ be the jth vertex of D, $\gamma = \bigcup_{j=1}^{N} \gamma_j$ be the boundary of D; s is the arclength measured along the boundary of D in the positive direction, and s_j is the value of s at $\dot{\gamma}_j$. We denote by r_j, θ_j the polar system of coordinates with poles in $\dot{\gamma}_j$ and the angle θ_j is taken counterclockwise from the side γ_j.

Consider the boundary value problem

$$\Delta u = 0 \quad \text{on } D, \tag{1}$$

$$u = \varphi_j \quad \text{on } \gamma_j, j = 1, 2, \ldots, N, \tag{2}$$

where $\Delta \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$, φ_j, $j = 1, 2, \ldots, N$, are given functions and

$$\varphi_j \in C^{4,\lambda}(\gamma_j), \quad 0 < \lambda < 1, 1 \le j \le N. \tag{3}$$

In addition, at the vertices $\dot{\gamma}_j$, for $\alpha_j = 1/3$, the following conjugation conditions are satisfied:

$$\varphi_{j-1}^{(3p)}(s_j) = \varphi_j^{(3p)}(s_j), \quad p = 0, 1. \tag{4}$$

No compatibility conditions are required at the vertices for $\alpha_j \neq 1/3$. Moreover, it is required that when $\alpha_j \neq 1/3$, the boundary functions on γ_{j-1} and γ_j are given as algebraic polynomials of the arclength s measured along γ.

Let $E = \{j : \alpha_j \neq 1/3, 1 \le j \le N\}$. We construct two fixed block sectors in the neighborhood of $\dot{\gamma}_j$, $j \in E$, denoted by $T_j^i = T_j(r_{ji}) \subset D$, $i = 1, 2$, where $0 < r_{j2} < r_{j1} < \min\{s_{j+1} - s_j, s_j - s_{j-1}\}$, $T_j(r) = \{(r_j, \theta_j) : 0 < r_j < r, 0 < \theta_j < \alpha_j\pi\}$. On the closed sector $\overline{T}_j^1$, $j \in E$, we consider the carrier function $Q_j(r_j, \theta_j)$ in the form given in [12], which satisfies the boundary conditions (2) on $\gamma_{j-1} \cap \overline{T}_j^1$ and $\gamma_j \cap \overline{T}_j^1$, $j \in E$.

We set (see [12])

$$R_j(r_j,\theta_j,\eta) = \frac{1}{\alpha_j}\sum_{k=0}^{1}(-1)^k R\left(\left(\frac{r}{r_{j2}}\right)^{1/\alpha_j}, \frac{\theta}{\alpha_j}, (-1)^k\frac{\eta}{\alpha_j}\right), \quad j \in E,$$

where

$$R(r,\theta,\eta) = \frac{1-r^2}{2\pi(1-2r\cos(\theta-\eta)+r^2)}$$

is the kernel of the Poisson integral for a unit circle.

The following lemma acts as a basis for the approximation of the solution around the vertices $\dot{\gamma}_j$ with angle $\alpha_j\pi$, $\alpha_j \neq 1/3$.

Lemma 2.1 ([12]) *The solution u of problem* (1)-(4) *can be represented on $\overline{T}_j^2 \backslash V_j$, $j \in E$, in the form*

$$u(r_j,\theta_j) = Q_j(r_j,\theta_j) + \int_0^{\alpha_j\pi}\big(u(r_{j2},\eta) - Q_j(r_{j2},\eta)\big)R_j(r_j,\theta_j,\eta)\,d\eta, \tag{5}$$

where V_j is the curvilinear part of the boundary of the sector T_j^2.

For the approximation of problem (1), (2) in the domain $\overline{D}$, we apply the hexagonal grid version of the block-grid method (see [6–8]). The application of this method first of all requires the construction of two more sectors T_j^3 and T_j^4, where $0 < r_{j4} < r_{j3} < r_{j2}$. Let $D_T = D\backslash(\bigcup_{j\in E}\overline{T}_j^4)$. The following steps are taken for the realization:

(1) We blockade the singular corners $\dot{\gamma}_j$, $j \in E$, by the double sectors $T_j^i(r_{ji})$, $i = 2,3$, with $T_k^2 \cap T_l^3 = \emptyset$, $k \neq l$, $k,l \in E$, and cover the polygon D by overlapping parallelograms denoted by D_l', $l = 1,2,\ldots,M$, and sectors T_j^3, $j \in E$, such that the distance from $\overline{D}_l'$ to $\dot{\gamma}_j$ is not less that r_{j4} for all $l = 1,2,\ldots,M$.
(2) On the parallelograms $\overline{D}_l'$, $l = 1,2,\ldots,M$, we use the 7-point scheme for the hexagonal grid with step size $h_l \leq h$, h a parameter, for the approximation of Laplace's equation, and the singular part $\overline{T}_j^3$, $j \in E$, is approximated by using the harmonic function defined in Lemma 2.1.
(3) The fourth order matching operator in a hexagonal grid is applied for connecting the subsystems.
(4) Schwarz's alternating procedure is used for solving the finite-difference system formed for Laplace's equation on the parallelograms covering D_T.

Let $D_l' \subset D_T$, $l = 1,2,\ldots,M$, be open fixed parallelograms and $D \subset (\bigcup_{l=1}^M D_l') \cup (\bigcup_{j\in E} T_j^3) \subset D$. We denote by η_l the boundary of D_l', $l = 1,2,\ldots M$, by V_j the curvilinear part of the boundary of the sector T_j^2, and let $t_j = (\bigcup_{l=1}^M \eta_l) \cap \overline{T}_j^3$. For the arrangement of the parallelograms D_l', $l = 1,2,\ldots,M$, it is required that any point P lying on $\eta_l \cap D_T$, $1 \leq l \leq M$, or lying on $V_j \cap D$, $j \in E$, falls inside at least on of the parallelograms $D_{l(P)}'$, $1 \leq l(P) \leq M$, depending on P, where the distance from P to $D_T \cap \eta_{l(P)}$ is not less than some constant κ_0 independent of P.

Let $h \in (0,\kappa_0/4]$ be a parameter, and define a hexagonal grid on D_l', $1 \leq l \leq M$, with maximal positive step $h_l \leq h$, such that the boundary η_l lies entirely on the grid lines. Let

D'_{lh} be the set of grid nodes on D'_l, η_l^h be the set of nodes on η_l, and let $\overline{D}'_{lh} = D'_{lh} \cup \eta_l^h$. Furthermore, η_{l0}^h denotes the set of nodes on $\eta_l \cap D_T$, $\eta_{l1}^h = \eta_l^h \backslash \eta_{l0}^h$, and t_j^h denotes the set of nodes on t_j.

We also specify a natural number $n \geq [\ln^{1+\chi} h^{-1}] + 1$, where $\chi > 0$ is a fixed number, and the quantities $n(j) = \max\{4, [\alpha_j n]\}$, $\beta_j = \alpha_j \pi / n(j)$, and $\theta_j^m = (m - 1/2)\beta_j$, $j \in E$, $1 \leq m \leq n(j)$. On the arc V_j we choose the points (r_{j2}, θ_j^m), $1 \leq m \leq n(j)$, and denote the set of these points by V_j^n. Finally, we have

$$\omega^{h,n} = \left(\bigcup_{l=1}^{M} \eta_{l0}^h \right) \cup \left(\bigcup_{j \in E} V_j^n \right), \qquad \overline{D}_*^{h,n} = \omega^{h,n} \cup \left(\bigcup_{l=1}^{M} \overline{D}'_{lh} \right).$$

For the approximation of the solution at the points of the set $\omega^{h,n}$ we use the fourth order linear matching operator S^4 constructed in [8], which can be represented as follows:

$$S^4(u_h, \varphi) = \sum_{k=0}^{16} \lambda_k u_h(P_k), \tag{6}$$

where $\varphi = \{\varphi_j\}_{j=1}^N$,

$$\lambda_k \geq 0, \qquad \sum_{k=0}^{16} \lambda_k = 1. \tag{7}$$

Consider the system of difference equations

$$u_h = S u_h \quad \text{on } D'_{lh}, \tag{8}$$

$$u_h = \varphi \quad \text{on } \eta_{l1}^h, \tag{9}$$

$$u_h(r_j, \theta_j) = Q_j(r_j, \theta_j) + \beta_j \sum_{k=1}^{n(j)} R_j\bigl(r_j, \theta_j, \theta_j^k\bigr)\bigl[u_h\bigl(r_{j2}, \theta_j^k\bigr) - Q_j\bigl(r_{j2}, \theta_j^k\bigr)\bigr] \quad \text{on } t_j^h, \tag{10}$$

$$u_h = S^4(u_h, \varphi) \quad \text{on } \omega^{h,n}, \tag{11}$$

where $1 \leq l \leq M$, $j \in E$, and the operator S is defined as

$$\begin{aligned} Su(x,y) = \frac{1}{6}\biggl(& u(x+h, y) + u\biggl(x + \frac{h}{2}, y + \frac{\sqrt{3}}{2}h\biggr) + u\biggl(x - \frac{h}{2}, y + \frac{\sqrt{3}}{2}h\biggr) \\ & + u(x-h, y) + u\biggl(x - \frac{h}{2}, y - \frac{\sqrt{3}}{2}h\biggr) + u\biggl(x + \frac{h}{2}, y - \frac{\sqrt{3}}{2}h\biggr)\biggr). \end{aligned}$$

The solution of this system is the approximation of the solution of problem (1), (2) on $\overline{D}_*^{h,n}$.

Theorem 2.2 *There is a natural number n_0 such that for all $n \geq n_0$ the system of equations* (8)-(11) *has a unique solution.*

Proof Taking into account the corresponding homogeneous system of system (8)-(10), the proof follows by analogy to Lemma 2 in [6]. □

Now consider the sector $T_j^* = T_j(r_j^*)$, where $r_j^* = (r_{j2} + r_{j3})/2$, $j \in E$. Let u_h be the solution of the system of equations (8)-(11). The function

$$U_h(r_j,\theta_j) = Q_j(r_j,\theta_j) + \beta_j \sum_{q=1}^{n(j)} R_j\bigl(r_j,\theta_j,\theta_j^q\bigr)\bigl(u_h\bigl(r_{j2},\theta_j^q\bigr) - Q_j\bigl(r_{j2},\theta_j^q\bigr)\bigr) \tag{12}$$

defined on T_j^* is an approximate solution of problem (1), (2), on the closed block $\overline{T}_j^3$, $j \in E$.

3 Error analysis of the 7-point approximation on the special parallelogram

Let D' be one of the parallelograms covering the 'nonsingular' part of the polygon D defined in Section 2. The boundaries of the parallelogram D' are denoted by γ_j', enumerated counterclockwise starting from left, including the ends, $\dot{\gamma}_j' = \gamma_{j-1}' \cap \gamma_j'$, $j = 1,2,3,4$, denotes the vertices of D', $\gamma' = \bigcup_{j=1}^4 \gamma_j'$, and $\overline{D}' = D' \cup \gamma'$. Furthermore $\gamma \cap \gamma' \neq \emptyset$, but the vertices $\dot{\gamma}_m'$ with an interior angle of $\alpha_m\pi \neq \pi/3$ are located either inside of D, or on the interior of a side γ_m of D, $1 \leq m \leq N$. We define the open parallelogram D' as $D' = \{(x,y) : 0 < y < \sqrt{3}a/2, d - y/\sqrt{3} < x < e - y/\sqrt{3}\}$. The boundary value problem (1)-(4) is considered on D':

$$\Delta v = 0 \quad \text{on } D', \tag{13}$$

$$v = \psi_j \quad \text{on } \gamma_j', j = 1,2,3,4, \tag{14}$$

where ψ_j are the values of the solution of problem (1)-(4) on γ'.

Let $h > 0$, where $(e-d)/h \geq 2$, $a/\sqrt{3}h \geq 2$ are integers. We assign to D' a hexagonal grid of the form $D_h' = \{(x,y) \in D' : x = \frac{h}{2}(1-l) + kh, y = l\frac{\sqrt{3}h}{2}, k,l = 0,\pm1,\pm2,\pm3,\ldots\}$. Let γ_{jh}' be the set of nodes on the interior of γ_j', and

$$\dot{\gamma}_{jh}' = \gamma_{j-1}' \cap \gamma_j', \qquad \gamma_h' = \bigcup_{j=1}^4 \gamma_{jh}', \quad j = 1,2,3,4,$$

$$\overline{D}_h' = D_h' \cup \gamma_h'.$$

We consider the system of finite-difference equations:

$$v_h = Sv_h \quad \text{on } D_h', \tag{15}$$

$$v_h = \psi_j \quad \text{on } \gamma_{jh}', j = 1,2,3,4, \tag{16}$$

where

$$\begin{aligned} Sv(x,y) = \frac{1}{6}\biggl(& v(x+h,y) + v\biggl(x+\frac{h}{2}, y+\frac{\sqrt{3}}{2}h\biggr) + v\biggl(x-\frac{h}{2}, y+\frac{\sqrt{3}}{2}h\biggr) \\ & + v(x-h,y) + v\biggl(x-\frac{h}{2}, y-\frac{\sqrt{3}}{2}h\biggr) + v\biggl(x+\frac{h}{2}, y-\frac{\sqrt{3}}{2}h\biggr)\biggr). \end{aligned} \tag{17}$$

Since (17) has nonnegative coefficients and their sum is equal to 1, the solution of system (15), (16) exists and is unique (see [13]).

Everywhere below we will denote constants which are independent of h and of the cofactors on their right by $c, c_0, c_1, \ldots$, generally using the same notation for different constants for simplicity.

Lemma 3.1 *Let*

$$\psi_j(s) \in C^{4,\lambda}\left(\gamma_j'\right), \quad 0 < \lambda < 1 \tag{18}$$

and

$$\psi_{j-1}^{(3p)}(s_j) = \psi_j^{(3p)}(s_j), \quad p = 0, 1, \tag{19}$$

be satisfied on the vertices whose interior angles are $\alpha_j\pi = \pi/3$, *where* $j = 1, 2, 3, 4$. *Then the solution of problem* (13), (14)

$$v \in C^{4,\lambda}\left(\overline{D}'\right). \tag{20}$$

Proof The closed parallelogram $\overline{D}'$ lies inside the polygon D defined in Section 2, and the vertices $\dot{\gamma}_m'$ with an interior angle of $\alpha_m\pi \neq \pi/3$ are located either inside of D or on the interior of a side γ_m of D, $1 \leq m \leq N$. Since the boundary functions (14), by the definition of the boundary functions φ_j in problem (1), (2) satisfy conditions (3), (4), from the results in [14], (20) follows. □

Let $D_{h,k}'$ be the set of nodes whose distance from the point $P \in D_h'$ to γ_h' is $\frac{\sqrt{3}}{2}kh$, $1 \leq k \leq a^*$, where $a^* = [\frac{2d_t}{\sqrt{3}h}]$, $[c]$ denotes the integer part of c, and d_t is the minimum of the half-lengths of the sides of the parallelogram.

Lemma 3.2 *Let* $w_h^k \neq \text{const.}$ *be the solution of the system of equations*

$$\begin{aligned}
&w_h^k = Sw_h^k + f_h^k \quad \text{on } D_{h,k}',\\
&w_h^k = Sw_h^k \quad \text{on } D_h' \setminus D_{h,k}',\\
&w_h^k = 0 \quad \text{on } \gamma_h',
\end{aligned}$$

and $z_h^k \neq \text{const.}$ *be the solution of the system of equations*

$$\begin{aligned}
&z_h^k = Sz_h^k + g_h^k \quad \text{on } D_{h,k}',\\
&z_h^k = Sz_h^k \quad \text{on } D_h' \setminus D_{h,k}',\\
&z_h^k = 0 \quad \text{on } \gamma_h',
\end{aligned}$$

where $1 \leq k \leq a^*$. *If* $|f_h^k| \leq g_h^k$, *then*

$$\left|w_h^k\right| \leq z_h^k, \quad 1 \leq k \leq a^*. \tag{21}$$

Proof The proof follows analogously to the proof of the comparison theorem given in [13]. □

Lemma 3.3 *Let v be the trace of the solution of problem* (13), (14) *on $\overline{D}_h'$, and v_h be the solution of system* (15), (16). *If*

$$\psi_j(s) \in C^{4,\lambda}(\gamma_j'), \quad 0 < \lambda < 1, j = 1,2,3,4$$

and

$$\psi_{j-1}^{(3p)}(s_j) = \psi_j^{(3p)}(s_j), \quad p = 0,1,$$

on the vertices with an interior angle of $\alpha_j\pi = \pi/3$, $j = 1,2,3,4$, then

$$\max_{\overline{D}_h'} |v - v_h| \leq ch^4. \tag{22}$$

Proof Let $\epsilon_h = v_h - v$ on $\overline{D}_h'$. Clearly

$$\epsilon_h = S\epsilon_h + (Sv - v) \quad \text{on } D_h', \tag{23}$$

$$\epsilon_h = 0 \quad \text{on } \gamma_h'. \tag{24}$$

Let D_{1h}' contain the set of nodes whose distance from the boundary γ' is $\frac{\sqrt{3}h}{2}$, and hence for $(x,y) \in D_{1h}'$, $(x + sH, y + sK) \in \overline{D}'$ for $0 \leq s \leq 1$, $H = \pm\frac{h}{2}, \pm h$, $K = 0, \pm\frac{\sqrt{3}h}{2}$, $H^2 + K^2 > 0$, and $D_{2h}' = D_h' \backslash D_{1h}'$.

Moreover, let

$$\epsilon_h = \epsilon_h^1 + \epsilon_h^2. \tag{25}$$

We rewrite problem (23), (24) as

$$\begin{aligned} &\epsilon_h^1 = S\epsilon_h^1 + (Sv - v) \quad \text{on } D_{1h}', \\ &\epsilon_h^1 = S\epsilon_h^1 \quad \text{on } D_{2h}', \\ &\epsilon_h^1 = 0 \quad \text{on } \gamma_h' \end{aligned} \tag{26}$$

and

$$\begin{aligned} &\epsilon_h^2 = S\epsilon_h^2 \quad \text{on } D_{1h}', \\ &\epsilon_h^2 = S\epsilon_h^2 + (Sv - v) \quad \text{on } D_{2h}', \\ &\epsilon_h^2 = 0 \quad \text{on } \gamma_h'. \end{aligned} \tag{27}$$

In order to obtain an estimation for $Sv - v$ on D_{1h}', we use Taylor's formula. On the basis of Lemma 3.1, we have

$$|Sv - v| \leq c_3 h^4 \quad \text{on } D_{1h}'. \tag{28}$$

Since at least two values of ϵ_h^1 in $S\epsilon_h^1$ are lying on the boundary γ_h', on which $\epsilon_h^1 = 0$, from (26), (28), and the maximum principle (see [13]), we obtain

$$\max_{\overline{D}_h'} |\epsilon_h^1| \leq \frac{2}{3} \max_{\overline{D}_h'} |\epsilon_h^1| + c_3 h^4.$$

Hence

$$\max_{\overline{D}'_h}\left|\epsilon_h^1\right| \le c_4 h^4, \tag{29}$$

where $c_4 = 3c_3$.

Next, we consider the estimation of ϵ_h^2. Let $D'_{2h,k}$ be the set of nodes whose distance from the point $P \in D'_{2h}$ to γ'_h is $\frac{\sqrt{3}}{2}kh$, $2 \le k \le a^*$, where $a^* = [\frac{2d_t}{\sqrt{3}h}]$, $[c]$ denotes the integer part of c, and d_t is the minimum of the half-lengths of the sides of the parallelogram. Furthermore, $D'_{2h,1} \equiv D'_{1h}$ and $D'_{2h,0} \equiv \gamma'_h$. Since the vertices with $\alpha_j = \frac{1}{3}$ of the parallelogram D' are never used as a node of the hexagonal grid for the estimation of $|Sv - v|$ on $D'_{2h,k}$, $2 \le k \le a^*$, we use the inequality

$$\max_{p+q=6}\left|\frac{\partial^6 v(x,y)}{\partial x^p\,\partial y^q}\right| \le c_0\rho^{\lambda-2} \quad \text{on } \overline{D}'\backslash\gamma'_m,$$

for the sixth order derivatives, where ρ is the distance from $(x,y) \in D'$ to γ'_m. Hence, we obtain

$$|Sv - v| \le c_5 h^6/(kh)^{2-\lambda} \quad \text{on } D'_{2h,k}, 2 \le k \le a^*. \tag{30}$$

Consider a majorant function of the form

$$Y_k = \begin{cases} 3m & \text{if } P \in D'_{2h,m}, 0 \le m \le k, \\ 3k & \text{if } P \in D'_{2h,m}, m > k. \end{cases} \tag{31}$$

Hence Y_k is a solution of the finite-difference problem

$$\begin{aligned} &Y_k = SY_k + \mu_k \quad \text{on } D'_{2h,k}, \\ &Y_k = SY_k \quad \text{on } D'_h\backslash D'_{2h,k}, \\ &Y_k = 0 \quad \text{on } \gamma'_h, \end{aligned} \tag{32}$$

where $1 \le \mu_k \le 3$, $1 \le k \le a^*$.

We represent the solution of system (27) as the sum of the solution of the following subsystems:

$$\begin{aligned} &\epsilon_{h,k}^2 = S\epsilon_{h,k}^2 + \mu'_k \quad \text{on } D'_{2h,k}, \\ &\epsilon_{h,k}^2 = S\epsilon_{h,k}^2 \quad \text{on } D'_h\backslash D'_{2h,k}, \\ &\epsilon_{h,k}^2 = 0 \quad \text{on } \gamma'_h, \end{aligned} \tag{33}$$

where $1 \le k \le a^*$, $\mu'_k = 0$ when $k = 1$ and $|\mu'_k| \le c_6\frac{h^{4+\lambda}}{k^{2-\lambda}}$ when $k = 2, 3, \ldots, a^*$.

By (32), (33), and Lemma 3.2, it follows that

$$\left|\epsilon_{h,k}^2\right| \le c_6\frac{h^{4+\lambda}}{k^{2-\lambda}} Y_k. \tag{34}$$

Hence, by taking (33) and (34) into consideration, we have

$$\begin{aligned}\max_{D'_h}\left|\epsilon_h^2\right| &\leq \sum_{k=1}^{a^*}\epsilon_{h,k}^2 \leq \sum_{k=1}^{a^*} c_6\frac{h^{4+\lambda}}{k^{2-\lambda}}Y_k \\ &\leq 3c_6h^{4+\lambda}\sum_{k=1}^{a^*}\frac{1}{k^{1-\lambda}} \leq c_7h^4. \end{aligned}\tag{35}$$

On the basis of (25), (29), and (35), we have estimation (22). □

4 Error analysis of the hexagonal block-grid equations

Let

$$\epsilon_h = u_h - u,\tag{36}$$

where u_h is the solution of system (8)-(11), and u is the trace of the solution of problem (1), (2) on $\overline{D}_*^{h,n}$. It is easy to show that (36) satisfies the system of equations

$$\begin{aligned}&\epsilon_h = S\epsilon_h + r_h^1 \quad \text{on } D'_{lh},\\ &\epsilon_h = 0 \quad \text{on } \eta_{l1}\cap\gamma_m,\\ &\epsilon_h(r_j,\theta_j) = \beta_j\sum_{k=1}^{n(j)}R_j\left(r_j,\theta_j,\theta_j^k\right)\epsilon_h\left(r_{j2},\theta_j^k\right) + r_{jh}^2 \quad \text{on } t_j^h,\\ &\epsilon_h = S^4(\epsilon_h,0) + r_h^3 \quad \text{on } \omega^{h,n},\end{aligned}\tag{37}$$

where $1\leq m\leq N$, $1\leq l\leq M$, $j\in E$, and

$$r_h^1 = Su - u \quad \text{on } \bigcup_{l=1}^{M} D'_{lh},\tag{38}$$

$$\begin{aligned}r_{jh}^2 = {}&\beta_j\sum_{k=1}^{n(j)}R_j\left(r_j,\theta_j,\theta_j^k\right)\left[u\left(r_{j2},\theta_j^k\right) - Q_j\left(r_{j2},\theta_j^k\right)\right]\\ &- \left(u(r_j,\theta_j) - Q_j(r_j,\theta_j)\right) \quad \text{on } \bigcup_{j\in E} t_j^h,\end{aligned}\tag{39}$$

$$r_h^3 = S^4(u,0) - u \quad \text{on } \omega^{h,n}.\tag{40}$$

Lemma 4.1 *Let the boundary functions φ_j, $j = 1,2,3,4$, in problem* (1), (2) *satisfy conditions* (3), (4). *Then*

$$\max_{\omega^{h,n}}\left|r_h^3\right| \leq c_5h^4,\tag{41}$$

where $\varphi = \bigcup_{j=1}^4\varphi_j$.

Proof The function $S^4(u,\varphi)$ is defined as (3.14) in [8]. Keeping in mind the positioning of the points in $\omega^{h,n}$, conditions (3), (4), and estimation (4.64) in [14], it follows that the fourth order partial derivatives of the exact solution of problem (1), (2) are bounded on D_T. Then estimation (41) follows from the construction of the operator S^4. □

Lemma 4.2 *There exists a natural number n_0 such that for all $n \geq \max\{n_0, [\ln^{1+\chi} h^{-1}] + 1\}$, $\chi > 0$ being a fixed number,*

$$\max_{j \in E} |r_{jh}^2| \leq c_6 h^4.$$

Proof The proof follows by analogy to the proof of Lemma 6.2 in [7]. □

Theorem 4.3 *Assume that conditions* (3), (4) *hold. Then there exists a natural number n_0 such that for all $n \geq \max\{n_0, [\ln^{1+\chi} h^{-1}] + 1\}$, $\chi > 0$ being a fixed number,*

$$\max_{\overline{D}_*^{h,n}} |u_h - u| \leq ch^4. \tag{42}$$

Proof Consider an arbitrary parallelogram D'_{l^*} and let $t_{l^*j}^h = \overline{D}'_{l^*} \cap t_j^h$. Assume that $t_{l^*j}^h \neq \emptyset$, z_h is the solution of system (37), and r_h^1, r_{jh}^2, r_h^3 are defined in the same way as (38)-(40) on D'_{l^*}, but are zero on $\overline{D}_*^{h,n} \backslash D'_{l^*}$. Hence,

$$V = \max_{\overline{D}_*^{h,n}} |z_h| = \max_{\overline{D}'_{l^*}} |z_h|. \tag{43}$$

We represent the function z_h as

$$z_h = \sum_{k=1}^{4} z_h^k, \tag{44}$$

where

$$\begin{aligned} z_h^2 &= S z_h^2 + r_h^1 \quad \text{on } D'_{l^*}, \\ z_h^2 &= 0 \quad \text{on } \eta_{l^*1}^h \cap \gamma_m, \\ z_h^2 &= 0 \quad \text{on } t_{l^*j}^h, \\ z_h^2 &= 0 \quad \text{on } \omega^{h,n} \cap \overline{D}'_{l^*}, \end{aligned} \tag{45}$$

$$\begin{aligned} z_h^3 &= S z_h^3 \quad \text{on } D'_{l^*}, \\ z_h^3 &= 0 \quad \text{on } \eta_{l^*1}^h \cap \gamma_m, \\ z_h^3 &= r_{jh}^2 \quad \text{on } t_{l^*j}^h, \\ z_h^3 &= 0 \quad \text{on } \omega^{h,n} \cap \overline{D}'_{l^*}, \end{aligned} \tag{46}$$

$$\begin{aligned} z_h^4 &= S z_h^4 \quad \text{on } D'_{l^*}, \\ z_h^4 &= 0 \quad \text{on } \eta_{l^*1}^h \cap \gamma_m, \\ z_h^4 &= 0 \quad \text{on } t_{l^*j}^h, \\ z_h^4 &= r_h^3 \quad \text{on } \omega^{h,n} \cap \overline{D}'_{l^*} \end{aligned} \tag{47}$$

and

$$z_h^k = 0, \quad k = 2,3,4 \text{ on } \overline{D}_*^{h,n} \backslash D'_{l^*}. \tag{48}$$

Hence by (44)-(48), z_h^1 satisfies the system of equations

$$\begin{aligned}
&z_h^1 = S z_h^1 \quad \text{on } D'_l,\\
&z_h^1 = 0 \quad \text{on } \eta_{l1}^h \cap \gamma_m,\\
&z_h^1 = \beta_j \sum_{k=1}^{n(j)} R_j\left(r_j,\theta_j,\theta_j^k\right) \sum_{k=1}^{4} z_h^k\left(r_{j2},\theta_j^k\right) \quad \text{on } t_{lj}^h,\\
&z_h^1 = S^4\left(\sum_{k=1}^{4} z_h^k\right) \quad \text{on } \omega^{h,n},
\end{aligned} \tag{49}$$

where $1 \le m \le N$, $1 \le l \le M$, $j \in E$, and the functions z_h^k, $k = 2,3,4$, are assumed to be known.

As the solution of system (45), z_h^2, is the error function of the finite-difference solution with step size $h_{l^*} \le h$ of system (15), (16), by (48), the maximum principle and Lemma 3.3, we have

$$V_2 = \max_{\overline{D}_*^{h,n}} \left|z_h^2\right| \le c_9 h^4. \tag{50}$$

Also, for the solutions of systems (46) and (47), as the operator S has coefficients which are nonnegative and their sum does not exceed 1, by the maximum principle, (48), Lemma 4.1, and Lemma 4.2, we obtain the inequalities

$$V_3 = \max_{\overline{D}_*^{h,n}} \left|z_h^3\right| \le c_{10} h^4, \tag{51}$$

$$V_4 = \max_{\overline{D}_*^{h,n}} \left|z_h^4\right| \le c_{11} h^4. \tag{52}$$

Now we consider the solution of v_h^1. Taking into consideration (49), the gluing condition of D'_l, $l = 1,2,\ldots,M$, and T_j^2, $j \in E$, for all $n \ge \max\{n_0, [\ln^{1+\chi} h^{-1}] + 1\}$ we have the inequality

$$V_1 = \max_{\overline{D}_*^{h,n}} \left|z_h^1\right| \le \lambda^* V + \sum_{k=2}^{4} \max_{\overline{D}_*^{h,n}} \left|z_h^k\right|, \tag{53}$$

where $0 < \lambda^* < 1$. By (43), (44), (50), (51), (52), and (53), we have

$$V = \max_{\overline{D}_*^{h,n}} |z_h| \le c h^4.$$

Hence (42) follows. □

For the approximation of (12), we consider the following theorem.

Theorem 4.4 *Let u_h be the solution of the system of equations* (8)-(11) *and let an approximate solution of problem* (1), (2) *be found on the blocks $\overline{T}_j^3$, $j \in E$, by* (12). *There is a natural* *number* n_0 *such that for all* $n \geq \max\{n_0, [\ln^{1+\chi} h^{-1}]\}$, $\chi > 0$ *being a fixed number, the following estimations hold*:

For $\alpha_j = 1$, $p \geq 1$,

$$\left|\frac{\partial^p}{\partial x^{p-q}\,\partial y^q}\big(U_h(r_j,\theta_j) - u(r_j,\theta_j)\big)\right| \leq c_p h^4 \quad \text{on } \overline{T}_j^3.$$

For $\alpha_j = \frac{2}{3}, 1, 2$, $0 \leq p \leq \frac{1}{\alpha_j}$,

$$\left|\frac{\partial^p}{\partial x^{p-q}\,\partial y^q}\big(U_h(r_j,\theta_j) - u(r_j,\theta_j)\big)\right| \leq c_p h^4 r_j^{1/\alpha_j - p} \quad \text{on } \overline{T}_j^3.$$

For $\alpha_j = \frac{2}{3}, 2$, $p > \frac{1}{\alpha_j}$,

$$\left|\frac{\partial^p}{\partial x^{p-q}\,\partial y^q}\big(U_h(r_j,\theta_j) - u(r_j,\theta_j)\big)\right| \leq c_p h^4 / r_j^{p-1/\alpha_j} \quad \text{on } \overline{T}_j^3 \backslash \dot{\gamma}_j,$$

where $j \in E$, $0 \leq q \leq p$, *and* c_p, $p = 0, 1, \ldots$, *are constants independent of* r_j, θ_j, *and* h.

Proof By taking estimation (42) into account, the proof follows by analogy to the proof of Theorem 6.4 in [7]. □

5 Numerical results

Two examples have been solved in order to test the effectiveness of the proposed method. In Example 5.1, it is assumed that there is a slit in the domain D, thus causing a strong singularity at the origin. The vertex $\dot{\gamma}_1$ containing the singularity has an interior angle of $\alpha_1\pi = 2\pi$. The exact solution of this problem is assumed to be known. In Example 5.2, we consider a problem with two singularities. The vertices which contain the singularities have interior angles of $\alpha_j\pi = \frac{2}{3}\pi$, $j = 2, 4$. In this example, the exact solution is not known.

After separating the 'singular' part in Example 5.1, the remaining part of the domain is covered by 5 overlapping parallelograms, whereas in Example 5.2, the 'nonsingular' part of the domain is covered by only two parallelograms. For the solution of the block-grid equations, Schwarz's alternating method is used. In each Schwarz iteration the system of equations on the parallelograms are solved by the block Gauss-Seidel method. The carrier function is constructed for each example, taking into consideration the boundary conditions given on the adjacent sides of the vertices in the 'singular' parts. Furthermore, the derivatives are approximated in the 'singular' parts for both of the examples.

The results are provided in Tables 1-5, and Figures 1-5.

Example 5.1 Consider the open parallelogram $D = \{(x,y) : -\frac{\sqrt{3}}{2} < y < \frac{\sqrt{3}}{2}, -1 - \frac{y}{\sqrt{3}} < x < 1 - \frac{y}{\sqrt{3}}\}$. We assume that there is a slit along the straight line $y = 0$, $0 \leq x \leq 1$. Let γ_j, $j =$

Table 1 Results obtained for the slit problem

(h^{-1}, n)	$\Vert u - u_h\Vert_{\overline{D}_{NS}}$	$\Vert u - u_h\Vert_{\overline{D}_S}$	$R^m_{D_{NS}}$	$R^m_{D_S}$
(16, 70)	5.924280×10^{-5}	5.191270×10^{-7}		
(32, 70)	3.910378×10^{-6}	4.794595×10^{-8}	15.1501	10.8273
(64, 110)	2.478126×10^{-7}	2.558563×10^{-9}	15.7796	18.7394
(128, 130)	1.56560×10^{-8}	1.27915×10^{-10}	15.8286	20.0021

Table 2 Results obtained for first derivative of the slit problem

(h^{-1},n)	(16, 70)	(32, 70)	(64, 110)	(128, 130)
$\lVert\epsilon_h^{(1)}\rVert_{\overline{D}_S}$	7.89831×10^{-7}	9.78871×10^{-8}	4.29502×10^{-9}	2.94108×10^{-10}

Table 3 Results obtained for second derivative of the slit problem

(h^{-1},n)	(16, 70)	(32, 70)	(64, 110)	(128, 130)
$\lVert\epsilon_h^{(2)}\rVert_{\overline{D}_S}$	3.7119×10^{-6}	9.736×10^{-7}	2.03211×10^{-8}	9.30597×10^{-10}

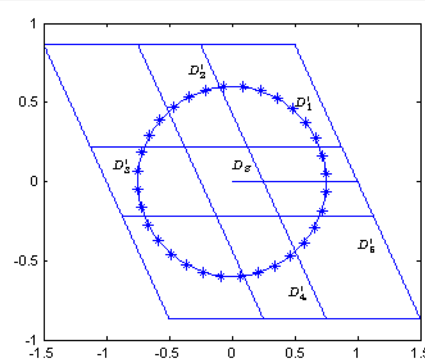
Table 4 Order of convergence of Example 5.2

2^{-m}	2^{-5}	2^{-6}
$\widetilde{R}^{m}_{p^{1}_{NS}}$	16.257	15.9884
$\widetilde{R}^{m}_{p^{2}_{NS}}$	16.2387	16.0086
$\widetilde{R}^{m}_{p^{1}_{S}}$	19.3268	12.7771
$\widetilde{R}^{m}_{p^{2}_{S}}$	18.2604	14.0755

Table 5 Order of convergence of derivatives in 'singular' parts of Example 5.2

2^{-m}	2^{-5}	2^{-6}
$\widetilde{R}^{m}_{p^{1}_{S}}$	13.8404	19.6426
$\widetilde{R}^{m}_{p^{2}_{S}}$	13.7489	19.6505

Figure 1 Domain of the slit problem.



$1, 2, \ldots, 7$, be the sides of D, including the ends, enumerated counterclockwise starting from the upper side of the slit ($\gamma_0 \equiv \gamma_7$), $\gamma = \bigcup_{j=1}^{7} \gamma_j$, and $\dot{\gamma}_j = \gamma_j \cap \gamma_{j-1}$ be the vertices of D. Let $(r,\theta) \equiv (r_1,\theta_1)$ be a polar system of coordinates with pole in $\dot{\gamma}_1$, where the angle θ is taken counterclockwise from the side γ_1.

We consider the boundary value problem

$$\begin{aligned} &\Delta u = 0 \quad \text{on } D, \\ &u = \varphi_j \quad \text{on } \gamma_j, j = 1, 2, \ldots, 7, \end{aligned} \tag{54}$$

where φ_j is the value of the function $v(r,\theta) = 0.5r^{1/2}\sin\frac{\theta}{2} + 0.8r^{3/2}\sin\frac{3\theta}{2} + 2r^2\cos 2\theta + 2.5r^3\cos 3\theta + 2\theta$ on γ_j.

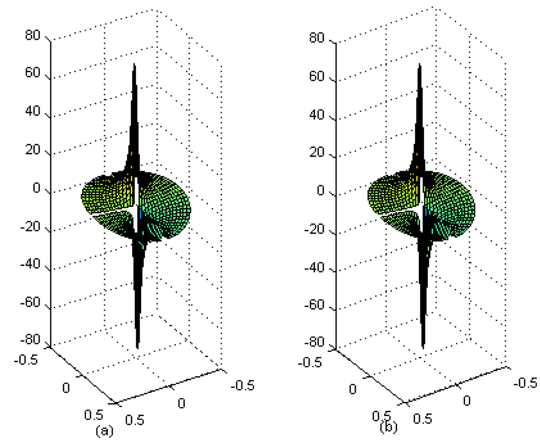


Figure 2 **Approximate solution (a) and exact solution (b) of $\frac{\partial u}{\partial x}$, respectively, using polar coordinates.**

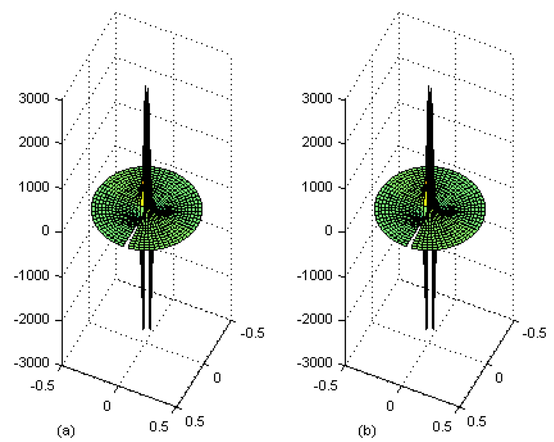
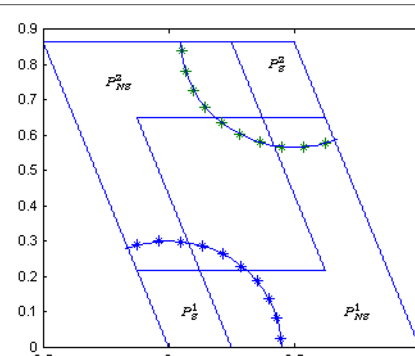


Figure 3 **Approximate solution (a) and exact solution (b) of $\frac{\partial^2 u}{\partial x^2}$, respectively, using polar coordinates.**

Figure 4 **Domain of the problem with two singularities.**



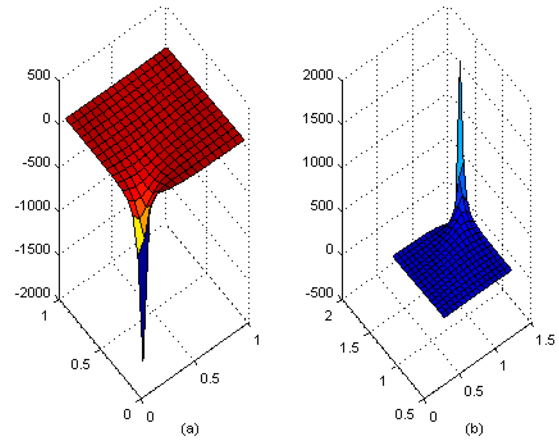


Figure 5 $\frac{\partial^2 U_h}{\partial x^2}$ **in 'singular' part** P_S^1 **shown by (a) and in** P_S^2 **shown by (b).**

As $\varphi_0 = 2x^2 + 2.5x^3 + 4\pi$ and $\varphi_1 = 2x^2 + 2.5x^3$, we obtain the carrier function in the form

$$\begin{aligned} Q_1(r,\theta) = {} & 2\theta + 2\bigl(\xi_2(r,\theta) + \xi_2(r, 2\pi - \theta)\bigr) \\ & + 2.5\bigl(\xi_3(r,\theta) + \xi_3(r, 2\pi - \theta)\bigr), \end{aligned}$$

where $\xi_2(r,\theta) = r^2((2\pi - \theta)\cos 2(2\pi - \theta) + \ln r \sin 2(2\pi - \theta))/2\pi$ and $\xi_3(r,\theta) = r^3((2\pi - \theta)\cos 3(2\pi - \theta) + \ln r \sin 3(2\pi - \theta))/2\pi$. The following notation is used in the table of results. Let D_l', $l = 1, 2, \ldots, 5$, be the open overlapping parallelograms, $D_{NS} = \bigcup_{l=1}^5 \overline{D}_l'$ be the 'nonsingular' part, and $D_S = \overline{D} \backslash D_{NS}$ denote the 'singular' part of D (see Figure 1). In Table 1, the values are obtained in the maximum norm of the difference between the exact and the approximate solutions, for the values of $h = 2^{-k}$, $k = 4, 5, 6, 7$, and n, which is the number of quadrature nodes on V_j. The order of convergence, $R_D^m = \frac{\|v - v_{2^{-m}}\|_D}{\|v - v_{2^{-(m+1)}}\|_D}$ has also been included. We also present the error obtained between the derivatives of the exact and the block-grid solutions $\epsilon_h^{(1)} = r^{1/2}(\frac{\partial u}{\partial x} - \frac{\partial U_h}{\partial x})$ and $\epsilon_h^{(2)} = r^{3/2}(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 U_h}{\partial x^2})$, in the maximum norm, in Tables 2 and 3, respectively. Figures 2 and 3 illustrate the shapes of the derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial^2 u}{\partial x^2}$ of the approximate (a) and the exact (b) solutions. These figures also demonstrate the highly accurate approximation of the derivatives.

Example 5.2 Let P be the open parallelogram $P = \{(x,y) : 0 < y < \frac{\sqrt{3}}{2}, -\frac{y}{\sqrt{3}} < x < 1 - \frac{y}{\sqrt{3}}\}$, let γ_j, $j = 1, 2, 3, 4$, be the sides of P, including the ends, enumerated counterclockwise starting from left ($\gamma_0 \equiv \gamma_4$, $\gamma_1 \equiv \gamma_5$), $\gamma = \bigcup_{j=1}^4 \gamma_j$, and $\dot{\gamma}_j = \gamma_j \cap \gamma_{j-1}$ be the vertices of P. We look at a problem with two corner singularities at the vertices $\dot{\gamma}_2$ and $\dot{\gamma}_4$, where $\alpha_j \pi = \frac{2}{3}\pi$, $j = 2, 4$. The two 'singular' corners of P are covered by sectors and these areas are denoted by P_S^i, $i = 1, 2$, and two overlapping parallelograms cover the 'nonsingular' part of the domain, denoted by P_{NS}^i, $i = 1, 2$ (see Figure 4).

We consider the boundary value problem

$$\begin{aligned} & \Delta u = 0 \quad \text{on } P, \\ & u = 0 \quad \text{on } \gamma_j, j = 1, 4, \qquad (55) \\ & u = 1 \quad \text{on } \gamma_j, j = 2, 3. \end{aligned}$$

The carrier functions constructed for each singularity are $Q_2(r_2,\theta_2) = 1 - \frac{3\theta_2}{2\pi}$ and $Q_4(r_4,\theta_4) = \frac{3\theta_4}{2\pi}$. We have checked the accuracy of the obtained approximate results u_h by looking at the order of convergence using the formula $\widetilde{R}_P^m = \frac{\|u_{2^{-m}} - u_{2^{-m+1}}\|_P}{\|u_{2^{-m-1}} - u_{2^{-m}}\|_P}$, which corresponds to 2^4, for the pairs $(h,n) = (2^{-4},80)$, $(2^{-5},100)$, $(2^{-6},100)$, $(2^{-7},90)$. The results are presented in Table 4. Moreover, $\frac{\partial^2 u}{\partial x^2}$ has been approximated in the 'singular' part, where u is the unknown exact solution of problem (55). The results are presented in Table 5 and illustrated further in Figure 5.

6 Conclusion

A fourth order square and hexagonal grid version of the block-grid method, for the solution of the boundary value problem of Laplace's equation on staircase polygons, with interior angles $\alpha_j \in \{\frac{1}{2},1,\frac{3}{2},2\}$, is extended for the polygons with interior angles $\alpha_j\pi$, $\alpha_j \in \{\frac{1}{3},\frac{2}{3},1,2\}$, by constructing and justifying the block-hexagonal grid method. Moreover, the smoothness requirement on the boundary functions away from the singular vertices (outside of the 'singular' parts) is lowered down from the Hölder classes $C^{6,\lambda}$, $0 < \lambda < 1$, as in [8], to $C^{4,\lambda}$, $0 < \lambda < 1$, which was proved for the 9-point scheme on square grids (see [10, 11]).

The proposed version of the BGM can be applied for the mixed boundary value problem of Laplace's equation on the above mentioned polygons. Furthermore, by this method any order derivatives of the solution can be highly approximated on the 'singular' parts, which are difficult to obtain in other numerical methods.

This method can also be used for the solution of the biharmonic equation by representing the problem with two problems for the Laplace and Poisson equations.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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