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Growth and poles of solutions of systems of complex composite functional equations

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Abstract
In this paper, we investigate the growth of transcendental meromorphic solutions of some types of systems of complex functional equations and obtain the lower bounds for Nevanlinna lower order for meromorphic solutions of such equations. Our results are improvement of the previous theorems given by Gao, Zheng and Chen. Some examples are also given to illustrate our results.

MSC: 39A50; 30D35

Keywords: system; *q*-shift; difference equation

1 Introduction and main results

Throughout this paper, the term 'meromorphic' will always mean meromorphic in the complex plane $\mathbb{C}$. Considering a meromorphic function f, we shall assume that readers are familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions such as $m(r,f)$, $N(r,f)$, $T(r,f)$, the first and second main theorems, lemma on the logarithmic derivatives *etc.* of Nevanlinna theory (see Hayman [1], Yang [2] and Yi and Yang [3]). We also use $\rho(f)$, $\mu(f)$, $\lambda(f)$ and $\lambda(\frac{1}{f})$ to denote the order, the lower order, the exponent of convergence of zeros and the exponent of convergence of poles of $f(z)$, respectively, and $S(r,f)$ to denote any quantity satisfying $S(r,f) = o(T(r,f))$ for all r outside a possible exceptional set of finite logarithmic measure $\lim_{r\to\infty}\int_{[1,r)\cap E}\frac{dt}{t} < \infty$.

Recently, there have been a number of papers focusing on the growth of solutions of difference equations, value distribution and uniqueness of differences analogues of Nevanlinna's theory (including [4–9]). Based on these results given in [10–12], people obtained many interesting theorems in the fields of complex analysis.

In 2003, Silvennoinen [13] studied the growth and existence of meromorphic solutions of functional equations of the form $f(p(z)) = R(z,f(z))$ and obtained the following result.

Theorem 1.1 [13] *Let f be a non-constant meromorphic solution of the equation*

$$f\bigl(p(z)\bigr) = R\bigl(z,f(z)\bigr) = \frac{\sum_{i=0}^{m_1} a_i(z)f(z)^i}{\sum_{j=0}^{n_1} b_j(z)f(z)^j},$$

where $p(z)$ is an entire function, a_i, b_j are small meromorphic functions with respect to f. Then $p(z)$ is a polynomial.

In 2012, Gao [14] studied the problem when the above equation is replaced by the following system of function equations:

$$\begin{cases} f_1(p(z)) = R_1(z,f_2(z)) = \frac{\sum_{i=0}^{m_1} a_i(z)f_2(z)^i}{\sum_{j=0}^{n_1} b_j(z)f_2(z)^j}, \\ f_2(p(z)) = R_2(z,f_1(z)) = \frac{\sum_{i=0}^{m_1} c_i(z)f_1(z)^i}{\sum_{j=0}^{n_1} d_j(z)f_1(z)^j}, \end{cases} \tag{1}$$

where $p(z)$ is an entire function, $R_1(z,f_2(z))$, $R_2(z,f_1(z))$ are irreducible rational functions, the coefficients are small functions; and he obtained the following.

Theorem 1.2 [14, Theorem 1] *Let* (f_1,f_2) *be a non-constant meromorphic solution of system* (1). *Then* $p(z)$ *is a polynomial.*

After his works, Gao [15, 16], Xu *et al.* [17] further investigated the growth and existence of meromorphic solutions of some types of systems of complex functional equations and obtained a series of results (see [15, 16, 18, 19]). Inspired by the ideas of Refs. [14–16, 20, 21], we investigate some properties of solutions of some types of systems of complex functional equations and obtain the following results.

The first theorem is about meromorphic solutions with few zeros and poles of a type of system of complex functional equations.

Theorem 1.3 *Let* $c_j \in \mathbb{C} \setminus \{0\}$ *and suppose that* f_1, f_2 *are a pair of non-rational meromorphic solutions of the system*

$$\begin{cases} \prod_{j=1}^{n_1} f_1(z+c_j) = \frac{a_0(z)+a_1(z)f_2(z)+\cdots+a_{p_2}(z)f_2(z)^{p_2}}{b_0(z)+b_1(z)f_2(z)+\cdots+b_{q_2}(z)f_2(z)^{q_2}}, \\ \prod_{j=1}^{n_2} f_2(z+c_j) = \frac{e_0(z)+e_1(z)f_1(z)+\cdots+e_{p_1}(z)f_1(z)^{p_1}}{d_0(z)+d_1(z)f_1(z)+\cdots+d_{q_1}(z)f_1(z)^{q_1}}, \end{cases} \tag{2}$$

with the coefficients $a_i(z)$, $b_i(z)$, $e_i(z)$, $d_i(z)$ *being small functions with respect to* f_1, f_2 *and* $a_{p_2}(z)b_{q_2}(z)e_{p_1}(z)d_{q_1}(z) \not\equiv 0$. *If*

$$\max\{\lambda(f_t), \lambda(1/f_t)\} < \rho(f_t), \quad t = 1, 2, \tag{3}$$

then system (2) *is of the form*

$$\begin{cases} \prod_{j=1}^{n_1} f_1(z+c_j) = c_2(z)f_2(z)^{s_2}, \\ \prod_{j=1}^{n_2} f_2(z+c_j) = c_1(z)f_1(z)^{s_1}, \end{cases}$$

where $c_1(z)$, $c_2(z)$ *are meromorphic functions*, $T(r,c_1) + T(r,c_2) = S(r,f_1) + S(r,f_2)$, $s_1, s_2 \in \mathbf{Z}$.

Theorem 1.4 *Suppose that* (f_1,f_2) *are a pair of transcendental meromorphic solutions of the system of q-shift difference equations*

$$\begin{cases} \sum_{j=1}^{n} a_j^1(z)f_1(q^j z + c_j) = \sum_{i=0}^{d_1} b_i^1(z)f_2(z)^i, \\ \sum_{j=1}^{n} a_j^2(z)f_2(q^j z + c_j) = \sum_{i=0}^{d_2} b_i^2(z)f_1(z)^i, \end{cases} \tag{4}$$

where $c_j \in \mathbb{C} \setminus \{0\}$, $q \in \mathbb{C}$, $|q| > 1$, $d_1 d_2 \geq 2$ and the coefficients $a_j^t(z)$, $b_i^t(z)$ $(t = 1, 2)$ are rational functions. If f_t $(t = 1, 2)$ are entire or have finitely many poles, then there exist constants $K_t > 0$ $(t = 1, 2)$ and $r_0 > 0$ such that for all $r \geq r_0$,

$$\log M(r, f_t) \geq K_t (d_1 d_2)^{\frac{\log r}{2n \log |q|}}, \quad t = 1, 2.$$

Theorem 1.5 *Suppose that (f_1, f_2) are a pair of transcendental meromorphic solutions of the system of q-shift difference equations*

$$\begin{cases} \sum_{j=1}^{n_1} a_j^1(z) f_1(q^j z + c_j) = \frac{P_2(z, f_2(z))}{Q_2(z, f_2(z))}, \\ \sum_{j=1}^{n_2} a_j^2(z) f_2(q^j z + c_j) = \frac{P_1(z, f_1(z))}{Q_1(z, f_1(z))}, \end{cases} \tag{5}$$

where $c_j \in \mathbb{C} \setminus \{0\}$, $q \in \mathbb{C}$, $|q| > 1$, the coefficients $a_j^t(z)$, $t = 1, 2$, are rational functions, and P_t, Q_t are relatively prime polynomials in f_t over the field of rational functions satisfying $p_t = \deg_{f_t} P_t$, $l_t = \deg_{f_t} Q_t$, $d_t = p_t - l_t \geq 2$, $t = 1, 2$. If f_t $(t = 1, 2)$ have infinitely many poles, then for sufficiently large r,

$$n(r, f_t) \geq K_t (d_1 d_2)^{\frac{\log r}{(n_1 + n_2) \log |q|}}, \quad t = 1, 2,$$

and

$$\mu(f_1) + \mu(f_2) \geq \frac{2(\log d_1 + \log d_2)}{(n_1 + n_2) \log |q|}.$$

Remark 1.1 Since system (4) is a particular case of system (5), from the conclusions of Theorem 1.5, we can get the following result.

Under the assumptions of Theorem 1.4. If f_t $(t = 1, 2)$ have infinitely many poles, then there exist constants $K_t > 0$ $(t = 1, 2)$ and $r_0 > 0$ such that for all $r \geq r_0$,

$$n(r, f_t) \geq K_t (d_1 d_2)^{\frac{\log r}{2n \log |q|}}, \quad t = 1, 2,$$

and

$$\mu(f_1) + \mu(f_2) \geq \frac{\log d_1 + \log d_2}{n \log |q|}.$$

Example 1.1 The function $(f_1(z), f_2(z)) = (\frac{e^z}{z}, \frac{e^z}{-z})$ satisfies the system of the form

$$\begin{cases} \sum_{j=1}^{n} \frac{2^j z + c_j}{e^{c_j} z^{2^j}} f_1(2^j z + c_j) = \sum_{j=1}^{n} f_2(z)^{2^j}, \\ \sum_{j=1}^{n} \frac{-(2^j z + c_j)}{e^{c_j} z^{2^j}} f_2(2^j z + c_j) = \sum_{j=1}^{n} f_1(z)^{2^j}, \end{cases}$$

with rational coefficients, where $|q| = 2 > 1$, $d_1 = d_2 = 2^n$ and $c_j \in \mathbb{C}$. Since $n < 2^n = d_1 = d_2$ for all $n \in \mathbb{N}_+$, we have $\log M(r, f_t) = r - \log r \geq \frac{1}{2} r = \frac{1}{2} (d_1 d_2)^{\frac{\log r}{2n \log |q|}}$ $(r \to \infty)$ and $\mu(f_t) = \sigma(f_t) = 1 = \frac{\log(d_1 d_2)}{2n \log |q|}$ for $t = 1, 2$. This shows that the conclusion of Theorem 1.4 is sharp and the equality in the consequent result $\mu(f_1) + \mu(f_2) \geq \frac{2(\log d_1 + \log d_2)}{(n_1 + n_2) \log |q|}$ of Remark 1.1 can be arrived.

Let q, c_j be stated as in Theorem 1.5, set

$$F_1(z;f_1,n_1,q,c_j) = \frac{\sum_{\lambda^1\in I_1} d_{\lambda^1}(z)f_1(qz+c_1)^{i_{\lambda_1^1}} f_1(q^2z+c_2)^{i_{\lambda_2^1}}\cdots f_1(q^{n_1}z+c_{n_1})^{i_{\lambda_{n_1}^1}}}{\sum_{\mu^1\in J_1} e_{\mu^1}(z)f_1(qz+c_1)^{j_{\mu_1^1}} f_1(q^2z+c_2)^{j_{\mu_2^1}}\cdots f_1(q^{n_1}z+c_{n_1})^{j_{\mu_{n_1}^1}}},$$

$$F_2(z;f_2,n_2,q,c_j) = \frac{\sum_{\lambda^2\in I_2} d_{\lambda^2}(z)f_2(qz+c_1)^{i_{\lambda_1^2}} f_2(q^2z+c_2)^{i_{\lambda_2^2}}\cdots f_2(q^{n_2}z+c_{n_2})^{i_{\lambda_{n_2}^2}}}{\sum_{\mu^2\in J_2} e_{\mu^2}(z)f_2(qz+c_1)^{j_{\mu_1^2}} f_2(q^2z+c_2)^{j_{\mu_2^2}}\cdots f_2(q^{n_2}z+c_{n_2})^{j_{\mu_{n_2}^2}}}.$$

Now, we will investigate the lower order of meromorphic solutions of a type of system of complex function equations and obtain a result as follows.

Theorem 1.6 *Suppose that* (f_1,f_2) *are a pair of transcendental meromorphic solutions of the system of q-difference equations*

$$\begin{cases} F_1(z;f_1,n_1,q,c_j) = \frac{\sum_{j=0}^{s_1} a_j^1(z)f_2(z)^j}{\sum_{j=0}^{l_1} b_j^1(z)f_2(z)^j}, \\ F_2(z;f_2,n_2,q,c_j) = \frac{\sum_{j=0}^{s_2} a_j^2(z)f_1(z)^j}{\sum_{j=0}^{l_2} b_j^2(z)f_1(z)^j}, \end{cases} \tag{6}$$

where $I_t = \{(i_{\lambda_1^t}, i_{\lambda_2^t}, \ldots, i_{\lambda_{n_t}^t})\}$, $J_t = \{j_{\mu_1^t}, j_{\mu_2^t}, \ldots, j_{\mu_{n_t}^t}\}$ *are finite index sets satisfying*

$$\max_{\lambda^t,\mu^t}\{i_{\lambda_1^t} + i_{\lambda_2^t} + \cdots + i_{\lambda_{n_t}^t}, j_{\mu_1^t} + j_{\mu_2^t} + \cdots + j_{\mu_{n_t}^t}\} = \sigma_t, \quad t = 1,2,$$

and $d_t = \max\{s_t, l_t\} \geq 2$, $t = 1,2$, *and all coefficients of* (6) *are of growth* $S(r,f_1)$, $S(r,f_2)$. *If*

$$d_1d_2 > 4n_1n_2\sigma_1\sigma_2, \tag{7}$$

then for sufficiently large r,

$$T(r,f_t) \geq K_t\left(\frac{d_1d_2}{4n_1n_2\sigma_1\sigma_2}\right)^{\frac{\log r}{(n_1+n_2)\log|q|}}, \quad t = 1,2,$$

where $K_t > 0$ *are constants. Thus, the lower order of* f_1, f_2 *satisfy*

$$\mu(f_1) + \mu(f_2) \geq \frac{2(\log d_1d_2 - \log 4n_1n_2\sigma_1\sigma_2)}{(n_1+n_2)\log|q|}.$$

Example 1.2 The functions $(f_1(z), f_2(z)) = (e^{z^2}, e^{-z^2})$ satisfy the system of function equations

$$\begin{cases} \frac{f_2(4z+c_2)+f_2(2z+c_1)f_2(4z+c_2)}{f_2(2z+c_1)} = \frac{a_1f_1(z)^4+a_0}{f_1(z)^{16}}, \\ \frac{f_1(2z+c_1)+f_1(2z+c_1)f_1(4z+c_2)}{f_1(4z+c_2)} = \frac{b_1f_2(z)^{16}+b_0}{f_2(z)^4}, \end{cases}$$

with small function coefficients

$$a_1 = e^{-8zc_2-4zc_1}e^{c_2^2-c_1^2}, \qquad a_0 = e^{-8zc_2-c_2^2}, \qquad b_1 = e^{4zc_1+c_1^2-8zc_2-c_2^2}, \qquad b_0 = e^{4zc_1+c_1^2},$$

where $q = n_1 = n_2 = \sigma_1 = \sigma_2 = 2$, $d_1 = d_2 = 16$, $d_1 d_2 = 256 > 4n_1 n_2 \sigma_1 \sigma_2$, $c_1, c_2 \in \mathbb{C}$ and a_1, a_0, b_1, b_0 are small functions of f_1, f_2. We have $\mu(f_t) = \sigma(f_t) = 2$, $t = 1,2$ and

$$\mu(f_1) + \mu(f_2) = 4 > 1 = \frac{2(\log d_1 d_2 - \log 4n_1 n_2 \sigma_1 \sigma_2)}{(n_1 + n_2)\log|q|}.$$

This shows that Theorem 1.6 may hold.

2 The proof of Theorem 1.3

Denote $G_t(z) = \prod_{j=1}^{n_t} f_t(z + c_j)$, $t = 1,2$. By applying Valiron-Mohon'ko theorem [22] to (2), we have

$$\begin{aligned} T(r, G_1) &= \max\{p_2, q_2\} T(r, f_2) + S(r, f_1) + S(r, f_2), \\ T(r, G_2) &= \max\{p_1, q_1\} T(r, f_1) + S(r, f_1) + S(r, f_2). \end{aligned} \tag{8}$$

From (3), we can take constants ξ_t, δ_t such that

$$\max\{\lambda(f_t), \lambda(1/f_t)\} < \xi_t < \delta_t < \rho(f_t), \quad t = 1,2,$$

then we have

$$T\left(r, \frac{f_t'}{f_t}\right) = \overline{N}(r, f_t) + \overline{N}\left(r, \frac{1}{f_t}\right) + S(r, f_t) = O\left(r^{\xi_t}\right) + S(r, f_t), \quad t = 1,2.$$

From (8) and the definitions of G_t ($t = 1,2$), similar to the above argument, we have

$$\begin{aligned} T\left(r, \frac{G_t'}{G_t}\right) &= N\left(r, \frac{G_t'}{G_t}\right) + m\left(r, \frac{G_t'}{G_t}\right) \\ &\le n_t \overline{N}(r + C, f_t) + n_t \overline{N}\left(r, \frac{1}{f_t}\right) + S(r, f_1) + S(r, f_2) \\ &= O\left(r^{\xi_t}\right) + S(r, f_1) + S(r, f_2), \end{aligned}$$

where $C := \max\{|c_i|, |c_j|, i = 1,2,\ldots,n_1; j = 1,2,\ldots,n_2\}$. From (3), we know that zeros and poles are Borel exceptions of f_t ($t = 1,2$), and from [23, Satz 13.4], we have that f_t ($t = 1,2$) is of regular growth. Hence, there exists $r_0 > 0$ such that $T(r, f_t) > r^{\delta_t}$ for $r > r_0$. So, we can get that

$$T\left(r, \frac{G_t'}{G_t}\right) = S(r, f_1) + S(r, f_2), \qquad T\left(r, \frac{f_t'}{f_t}\right) = S(r, f_1) + S(r, f_2), \quad t = 1,2.$$

Now, we rewrite system (2) as

$$\begin{cases} \frac{b_{p_2}(z)}{a_{p_2}(z)} G_1(z) = \frac{P_2(z, f_2)}{Q_2(z, f_2)} = u_2(z, f_2), \\ \frac{d_{p_1}(z)}{e_{p_1}(z)} G_2(z) = \frac{P_1(z, f_1)}{Q_1(z, f_1)} = u_1(z, f_1), \end{cases} \tag{9}$$

without loss of generality, assume that P_t, Q_t are monic polynomials in f_t with coefficients of growth $S(r, f_1)$, $S(r, f_2)$. Set $F_t := \frac{f_t'}{f_t}$, $U_t := \frac{u_t'}{u_t}$, $t = 1,2$. From (9), we have $T(r, U_t) =$

$S(r,f_1) + S(r,f_2)$. And because

$$\begin{cases} \frac{P_2'Q_2-P_2Q_2'}{Q_2^2} = u_2' = U_2u_2 = \frac{U_2P_2}{Q_2}, \\ \frac{P_1'Q_1-P_1Q_1'}{Q_1^2} = u_1' = U_1u_1 = \frac{U_1P_1}{Q_1}, \end{cases}$$

it follows that

$$\begin{cases} P_2'Q_2 - P_2Q_2' = U_2P_2Q_2, \\ P_1'Q_1 - P_1Q_1' = U_1P_1Q_1. \end{cases}$$

Substituting $f_t' = F_tf_t$, $t = 1,2$, to the above equalities and comparing the leading coefficients, we can get

$$(p_t - q_t)F_t = U_t, \quad t = 1,2.$$

Solving the above equations, we get

$$u_t = \pi_t\bigl(f_t(z)\bigr)^{p_2-q_2}, \quad \pi_t \in \mathbb{C}, t = 1,2. \tag{10}$$

From (9) and (10), it follows that

$$\begin{cases} G_1(z) = \pi_2\frac{a_{p_2}(z)}{b_{p_2}(z)}(f_2(z))^{p_2-q_2}, \\ G_2(z) = \pi_1\frac{e_{p_1}(z)}{d_{p_1}(z)}(f_1(z))^{p_1-q_1}. \end{cases}$$

Thus, we complete the proof of Theorem 1.3.

3 Proofs of Theorems 1.4 and 1.5

3.1 The proof of Theorem 1.4

Because the coefficients $a_j^t(z)$, $b_i^t(z)$ $(t = 1,2)$ are rational functions, we can rewrite (4) as follows:

$$\begin{cases} \sum_{j=1}^n A_j^1(z)f_1(q^jz + c_j) = \sum_{i=0}^{d_1} B_i^1(z)f_2(z)^i, \\ \sum_{j=1}^n A_j^2(z)f_2(q^jz + c_j) = \sum_{i=0}^{d_2} B_i^2(z)f_1(z)^i, \end{cases} \tag{11}$$

where the coefficients $A_j^t(z)$, $B_i^t(z)$ $(t = 1,2)$ are polynomials. We will consider two cases as follows.

Case 1. Since (f_1,f_2) are a pair of solutions of system (4) or (11) and f_t, $t = 1,2$, are transcendental entire, set $p_i^t = \deg A_j^t$ $(j = 1,2,\ldots,n)$, $q_i^t = \deg B_i^t$ $(i = 0,1,\ldots,d_i)$, $t = 1,2$, and $C := \max\{|c_i|,|c_j|, i = 1,2,\ldots,n_1; j = 1,2,\ldots,n_2\}$. Taking $m_t = \max\{p_1^t,\ldots,p_n^t\} + 1$, and from $|q| > 1$ and $M(r,f_t(q^jz + c_j)) \leq M(|q|^jr + |c_j|,f_t)$, we have that

$$\begin{cases} M(r,\sum_{i=0}^{d_1} B_i^1(z)f_2(z)^i) = M(r,\sum_{j=1}^n A_j^1(z)f_1(q^jz + c_j)) \leq nr^{m_1}M(|q|^nr + C,f_1), \\ M(r,\sum_{i=0}^{d_2} B_i^2(z)f_1(z)^i) = M(r,\sum_{j=1}^n A_j^2(z)f_2(q^jz + c_j)) \leq nr^{m_2}M(|q|^nr + C,f_2), \end{cases} \tag{12}$$

when r is sufficiently large. Since B_i^t $(i = 0,1,\ldots,d_t; t = 1,2)$ are polynomials and f_t $(t = 1,2)$ are transcendental entire functions, we have $M(r,\sum_{i=0}^{d_1-1} B_i^1f_2(z)^i) = o(M(r,f_2(z)^{d_1}))$ and

$M(r, \sum_{i=0}^{d_2-1} B_i^2 f_1(z)^i) = o(M(r, f_1(z)^{d_2}))$. Then, for sufficiently large r, it follows that

$$\begin{cases} M(r, \sum_{i=0}^{d_1} B_i^1(z) f_2(z)^i) \geq \frac{1}{2} M(r, B_{d_1}^1 f_2(z)^{d_1}), \\ M(r, \sum_{i=0}^{d_2} B_i^2(z) f_1(z)^i) \geq \frac{1}{2} M(r, B_{d_2}^2 f_1(z)^{d_2}). \end{cases} \tag{13}$$

From (12) and (13), for sufficiently large r it follows that

$$\begin{cases} \log M(|q|^n r + C, f_1) \geq d_1 \log M(r, f_2) + g_1(r), \\ \log M(|q|^n r + C, f_2) \geq d_2 \log M(r, f_1) + g_2(r), \end{cases} \tag{14}$$

where $|g_t(r)| < K_t \log r$, $t = 1, 2$, for some constants $K_t > 0$. From (14), for sufficiently large r, we get

$$\log M\bigl(|q|^{2n} r + C + C|q|^n, f_1\bigr) \geq d_1 d_2 \log M(r, f_1) + g_1\bigl(|q|^n r + C\bigr) + d_1 g_2(r). \tag{15}$$

Iterating (15), we have

$$\log M\left(|q|^{2nk} r + C \sum_{\nu=0}^{2k-1} |q|^{\nu n}, f_1\right) \geq (d_1 d_2)^k \log M(r, f_1) + E_k^1(r) + E_k^2(r) \quad (k \in \mathbb{N}), \tag{16}$$

where

$$\begin{aligned} \bigl|E_k^1(r)\bigr| &= \left|(d_1 d_2)^{k-1} g_1\bigl(|q|^n r + C\bigr) + \cdots + g_1\left(|q|^{(2k-1)n} r + C \sum_{\nu=0}^{2k-2} |q|^{\nu n}\right)\right| \\ &\leq K_1 (d_1 d_2)^{k-1} \sum_{j=0}^{k-1} \frac{\log |q|^{(2j+1)n} r + C \sum_{\nu=0}^{2j-1} \log |q|^{\nu n}}{(d_1 d_2)^j} \\ &\leq K_1 (d_1 d_2)^{k-1} \sum_{j=0}^{\infty} \frac{\log |q|^{(2j+1)n} r + C \sum_{\nu=0}^{2j-1} \log |q|^{\nu n}}{(d_1 d_2)^j}, \end{aligned}$$

and

$$\begin{aligned} \bigl|E_k^2(r)\bigr| &= \left|d_1 (d_1 d_2)^{k-1} g_2(r) + \cdots + d_1 g_2\left(|q|^{(2k-2)n} r + C \sum_{\nu=0}^{2k-3} |q|^{\nu n}\right)\right| \\ &\leq K_2 d_1 (d_1 d_2)^{k-1} \sum_{j=0}^{k-1} \frac{\log |q|^{2(j-1)n} r + C \sum_{\nu=0}^{2j-3} |q|^{\nu n}}{(d_1 d_2)^j} \\ &\leq K_2 d_1 (d_1 d_2)^{k-1} \sum_{j=0}^{\infty} \frac{\log |q|^{2(j-1)n} r + C \sum_{\nu=0}^{2j-3} |q|^{\nu n}}{(d_1 d_2)^j}. \end{aligned}$$

Observe that $|q| > 1$, then for sufficiently large r, we have

$$\begin{aligned} \log |q|^{(2j+1)n} r + C \sum_{\nu=0}^{2j} |q|^{\nu n} &\leq \log |q|^{(2j+1)n} + \log r + \log(2j+1)C + \log |q|^{2jn} \\ &\leq 2j(2j+1)n^2 \bigl(\log |q|\bigr)^2 \log j \log C \log r. \end{aligned}$$

And since $d_1d_2 \geq 2$, it follows that the series $\sum_{i=0}^{\infty} \frac{2j(2j+1)n^2(\log|q|)^2 \log j}{(d_1d_2)^i}$ is convergent. Thus, for sufficiently large r, we have

$$\left|E_k^t(r)\right| \leq K_t'(d_1d_2)^k \log r, \quad t = 1, 2, \tag{17}$$

where $K_t' > 0$ $(t = 1, 2)$ are some constants. Since f_1 is a transcendental entire function, for sufficiently large r, we have

$$\log M(r, f_1) \geq 3K' \log r, \tag{18}$$

where $K' > \max\{K_1', K_2'\}$. Hence, from (16)-(17), there exists $r_0 \geq e$ such that for $r \geq r_0$, we have

$$\log M\left(|q|^{2nk} r + C \sum_{\nu=0}^{2k-1} |q|^{\nu n}, f_1\right) \geq K'(d_1d_2)^k \log r. \tag{19}$$

Thus, for each sufficiently large R, there exists $k \in \mathbb{N}$ such that

$$R \in \left[|q|^{2nk} r_0 + C \sum_{\nu=0}^{2k-1} |q|^{\nu n}, |q|^{2n(k+1)} r_0 + C \sum_{\nu=0}^{2k+1} |q|^{\nu n}\right),$$

i.e.,

$$k > \frac{\log R + \log(|q|^n - 1) - \log r_0 - \log C - 4n \log|q|}{2n \log|q|}. \tag{20}$$

From (19) and (20), we have

$$\begin{aligned} \log M(R, f_1) &\geq \log M\left(|q|^{2nk} r_0 + C \sum_{\nu=0}^{2k-1} |q|^{\nu n}, f_1\right) \\ &\geq K'(d_1d_2)^k \log r_0 \\ &\geq K''(d_1d_2)^{\frac{\log R}{2n \log|q|}}, \end{aligned} \tag{21}$$

where

$$K'' = K'(d_1d_2)^{\frac{\log(|q|^n - 1) - \log r_0 - \log C - 4n \log|q|}{2n \log|q|}}.$$

Similar to the above argument, we can get that there exist constants $K > 0$ and $r_0 > 0$ such that for all $r \geq r_0$,

$$\log M(r, f_2) \geq K(d_1d_2)^{\frac{\log r}{2n \log|q|}}. \tag{22}$$

Case 2. Suppose that (f_1, f_2) are a pair of solutions of system (4) and f_t $(t = 1, 2)$ are meromorphic with finitely many poles. Then there exist polynomials $P_t(z)$ such that $g_t(z) = P_t(z)f_t(z)$ $(t = 1, 2)$ are entire functions. Substituting $f_t(z) = \frac{g_t(z)}{P_t(z)}$ into (11) and again multiplying away the denominators, we can get a system similar to (11). By using the same argument as in the above, we can get that for sufficiently large r,

$$\log M(r,f_t) = \log M(r,g_t) + O(1) \geq \left(K_t'' - \varepsilon\right)(d_1d_2)^{\frac{\log r}{2n\log|q|}} \geq K_t'''(d_1d_2)^{\frac{\log r}{2n\log|q|}},$$

where K_t''' (> 0) $(t = 1,2)$ are some constants.

From Case 1 and Case 2, this completes the proof of Theorem 1.4.

3.2 The proof of Theorem 1.5

Since the coefficients of $P_t(z,f_t(z))$, $Q_t(z,f_t(z))$ are rational functions, we can choose a sufficiently large constant R (> 0) such that the coefficients of $P_t(z,f_t(z))$, $Q_t(z,f_t(z))$ $(t = 1,2)$ have no zeros or poles in $\{z \in \mathbb{C} : |z| > R\}$. Assume that (f_1,f_2) is a solution of system (5) and f_t $(t = 1,2)$ are transcendental, since f_t $(t = 1,2)$ have infinitely many poles. Thus, without loss of generality, we choose a pole z_0 of f_1 of multiplicity $\mu \geq 1$ satisfying $|z_0| > R$. Since $d_1 = s_1 - t_1 \geq 2$, then the right-hand side of the second equation in (5) has a pole of multiplicity $d_1\mu$ at z_0. Therefore, there exists at least one index $j_1 \in \{1,2,\dots,n_2\}$ such that $q^{j_1}z_0 + c_{j_1}$ is a pole of f_2 of multiplicity $\mu_1' \geq d_1\mu$. If $|q^{j_1}z_0 + c_{j_1}| \leq R$, this process will be terminated and we have to choose another pole z_0 of f_1 in the way we did above. If $|q^{j_1}z_0 + c_{j_1}| > R$, since $d_2 = s_2 - t_2 \geq 2$, then the right-hand side of the first equation in (5) has a pole of multiplicity $d_2\mu_1' \geq d_1d_2\mu$. Therefore, there exists at least one index $j_1' \in \{1,2,\dots,n_1\}$ such that $q^{j_1'}(q^{j_1}z_0 + c_{j_1}) + c_{j_1'}$ is a pole of f_1 of multiplicity $\mu_1 \geq d_2\mu_1' \geq d_1d_2\mu$.

We proceed to follow the step above, we can get a sequence

$$\{\zeta_k\}_{k=1}^{\infty} := \left\{\prod_{i=1}^{k} q^{j_i+j_i'}z_0 + \sum_{s=1}^{k}\prod_{i=s+1}^{k} q^{j_i+j_i'}\left(q^{j_s'}c_{j_s} + c_{j_s'}\right)\right\}_{k=1}^{\infty},$$

where ζ_k is a pole of f_1 with multiplicity μ_k, $j_s \in \{1,2,\dots,n_2\}$ and $j_s' \in \{1,2,\dots,n_1\}$. From the above discussion, we can get $\mu_k \geq (d_1d_2)^k\mu$. Obviously, we have $|\zeta_k| \to \infty$ as $k \to \infty$. Then there exists a positive integer $k_0 \in \mathbb{N}_+$ such that for sufficiently large k $(\geq k_0)$,

$$\begin{aligned}\mu(d_1d_2)^k &\leq \mu\left[1 + d_1d_2 + \cdots + (d_1d_2)^k\right] \leq n\left(|\zeta_k|,f_1\right)\\ &\leq n\left(|q|^{(n_1+n_2)k}|z_0| + C\left(|q|^{n_1} + 1\right)\sum_{i=0}^{k-1}|q|^{i(n_1+n_2)},f_1\right),\end{aligned} \tag{23}$$

where $C := \max\{|c_i|,|c_j|, i = 1,2,\dots,n_1; j = 1,2,\dots,n_2\}$. Thus, for each sufficiently large r, there exists $k \in \mathbb{N}_+$ such that $r \in [\eta_k,\eta_{k+1})$, where $\eta_k := |q|^{(n_1+n_2)k}|z_0| + C(|q|^{n_1} + 1) \times \sum_{i=0}^{k-1}|q|^{i(n_1+n_2)}$, it follows that

$$k > \frac{\log r - \log|z_0| - \log C - \log(|q|^{n_1} + 1) - (n_1 + n_2)\log|q| + \log(|q|^{n_1+n_2} - 1)}{(n_1 + n_2)\log|q|}. \tag{24}$$

From (23) and (24), we have

$$\begin{aligned}n(r,f_1) &\geq \mu(d_1d_2)^k \geq \mu(d_1d_2)^{\frac{\log r - \log|z_0| - \log C - \log(|q|^{n_1}+1) - (n_1+n_2)\log|q| + \log(|q|^{n_1+n_2}-1)}{(n_1+n_2)\log|q|}}\\ &\geq K_5 d^{\frac{\log r}{(n_1+n_2)\log|q|}},\end{aligned}$$

where

$$K_5 = \mu(d_1d_2)^{\frac{-\log|z_0|-\log C-\log(|q|^{n_1}+1)-(n_1+n_2)\log|q|+\log(|q|^{n_1+n_2}-1)}{(n_1+n_2)\log|q|}}.$$

And there exists $r_0 > 0$ and for all $r \geq r_0$, we have

$$K_5(d_1d_2)^{\frac{\log r}{(n_1+n_2)\log|q|}} \leq n(r,f_1) \leq \frac{1}{\log 2}T(2r,f_1).$$

Similar to the above discussion, we can get that there exists $r_0 > 0$ and for all $r \geq r_0$, we have

$$K_5'(d_1d_2)^{\frac{\log r}{(n_1+n_2)\log|q|}} \leq n(r,f_2) \leq \frac{1}{\log 2}T(2r,f_2).$$

From these inequalities, we can get $\mu(f_1) + \mu(f_2) \geq \frac{2(\log d_1+\log d_2)}{(n_1+n_2)\log|q|}$ easily.

Thus, the proof of Theorem 1.5 is completed.

4 Proof of Theorem 1.6

Lemma 4.1 [21, Lemma 2] *Let $f_1, f_2, \ldots, f_n$ be meromorphic functions. Then*

$$T\left(r, \sum_{\lambda\in I} f_1^{i_{\lambda_1}} f_2^{i_{\lambda_2}} \cdots f_n^{i_{\lambda_n}}\right) \leq \sigma \sum_{i=1}^{n} T(r,f_i) + \log s,$$

where $I = \{i_{\lambda_1}, i_{\lambda_2}, \ldots, i_{\lambda_n}\}$ is an index set consisting of s elements, and $\sigma = \max_{\lambda\in I}\{i_{\lambda_1} + i_{\lambda_2} + \cdots + i_{\lambda_n}\}$.

Proof of Theorem 1.6 From $|q| > 1$, $c_j \in \mathbb{C}$ and [24, p.249], we have $T(r, f_t(q^jz + c_j)) = T(|q|^jr + |c_j|, f_t) + O(1)$, $t = 1, 2$. For any given ε ($0 < \varepsilon < \frac{\sqrt{d_1d_2}-\sqrt{4n_1n_2\sigma_1\sigma_2}}{\sqrt{d_1d_2}+\sqrt{4n_1n_2\sigma_1\sigma_2}}$), applying Valiron-Mohon'ko theorem [22] and Lemma 4.1 to (6), it follows that

$$\begin{cases} d_1(1-\varepsilon)T(r,f_2) \leq d_1T(r,f_2) + S(r,f_2) \leq 2\sigma_1\sum_{j=1}^{n_1} T(|q|^jr + C, f_1) + S(r,f_1) \\ \qquad \leq 2n_1\sigma_1(1+\varepsilon)T(|q|^{n_1}r + C, f_1), \\ d_2(1-\varepsilon)T(r,f_1) \leq d_2T(r,f_1) + S(r,f_1) \leq 2\sigma_2\sum_{j=1}^{n_2} T(|q|^jr + C, f_2) + S(r,f_2) \\ \qquad \leq 2n_2\sigma_2(1+\varepsilon)T(|q|^{n_2}r + C, f_2), \end{cases} \tag{25}$$

outside of a possible exceptional set of finite linear measure. Then from (25) there exists $r_0 > 0$ such that

$$\begin{cases} T(|q|^{n_1+n_2}r + C(|q|^{n_1}+1), f_1) \geq \frac{d_1d_2(1-\varepsilon)^2}{4n_1n_2\sigma_1\sigma_2(1+\varepsilon)^2}T(r,f_1), \\ T(|q|^{n_1+n_2}r + C(|q|^{n_2}+1), f_2) \geq \frac{d_1d_2(1-\varepsilon)^2}{4n_1n_2\sigma_1\sigma_2(1+\varepsilon)^2}T(r,f_2), \end{cases} \tag{26}$$

holds for all $r > r_0$. Iterating (26), for any $k \in \mathbb{N}_+$ and $r \geq r_0$, we have

$$\begin{cases} T(|q|^{(n_1+n_2)k}r + C(|q|^{n_1}+1)\sum_{i=0}^{k-1}|q|^{i(n_1+n_2)}, f_1) \geq (\frac{d_1d_2(1-\varepsilon)^2}{4n_1n_2\sigma_1\sigma_2(1+\varepsilon)^2})^kT(r,f_1), \\ T(|q|^{(n_1+n_2)k}r + C(|q|^{n_2}+1)\sum_{i=0}^{k-1}|q|^{i(n_1+n_2)}, f_2) \geq (\frac{d_1d_2(1-\varepsilon)^2}{4n_1n_2\sigma_1\sigma_2(1+\varepsilon)^2})^kT(r,f_2). \end{cases}$$

By employing the same argument as in the proof of Theorem 1.5, for sufficiently large ϱ, from the above inequalities, we can get

$$\begin{cases} T(\varrho,f_1) \geq K_6 T(r_0,f_1)(\frac{d_1d_2(1-\varepsilon)^2}{4n_1n_2\sigma_1\sigma_2(1+\varepsilon)^2})^{\frac{\log\varrho}{(n_1+n_2)\log|q|}}, \\ T(\varrho,f_2) \geq K_6' T(r_0,f_2)(\frac{d_1d_2(1-\varepsilon)^2}{4n_1n_2\sigma_1\sigma_2(1+\varepsilon)^2})^{\frac{\log\varrho}{(n_1+n_2)\log|q|}}, \end{cases} \tag{27}$$

where

$$K_6 = \left(\frac{d_1d_2(1-\varepsilon)^2}{4n_1n_2\sigma_1\sigma_2(1+\varepsilon)^2}\right)^{\frac{-\log|z_0|-\log C-\log(|q|^{n_1}+1)-(n_1+n_2)\log|q|+\log(|q|^{n_1+n_2}-1)}{(n_1+n_2)\log|q|}},$$

$$K_6' = \left(\frac{d_1d_2(1-\varepsilon)^2}{4n_1n_2\sigma_1\sigma_2(1+\varepsilon)^2}\right)^{\frac{-\log|z_0|-\log C-\log(|q|^{n_2}+1)-(n_1+n_2)\log|q|+\log(|q|^{n_1+n_2}-1)}{(n_1+n_2)\log|q|}}.$$

Letting $\varepsilon \to 0$, from (27) we have

$$\begin{cases} T(\varrho,f_1) \geq K_7(\frac{d_1d_2}{4n_1n_2\sigma_1\sigma_2})^{\frac{\log\varrho}{(n_1+n_2)\log|q|}}, \\ T(\varrho,f_2) \geq K_7'(\frac{d_1d_2}{4n_1n_2\sigma_1\sigma_2})^{\frac{\log\varrho}{(n_1+n_2)\log|q|}}, \end{cases} \tag{28}$$

where K_6, K_6' are constants satisfying

$$K_7 = T(r_0,f_1)\left(\frac{d_1d_2}{4n_1n_2\sigma_1\sigma_2}\right)^{\frac{-\log|z_0|-\log C-\log(|q|^{n_1}+1)-(n_1+n_2)\log|q|+\log(|q|^{n_1+n_2}-1)}{(n_1+n_2)\log|q|}},$$

$$K_7' = T(r_0,f_2)\left(\frac{d_1d_2}{4n_1n_2\sigma_1\sigma_2}\right)^{\frac{-\log|z_0|-\log C-\log(|q|^{n_2}+1)-(n_1+n_2)\log|q|+\log(|q|^{n_1+n_2}-1)}{(n_1+n_2)\log|q|}}.$$

Thus, from (28) the lower order of f_1, f_2 satisfy

$$\mu(f_1) + \mu(f_2) \geq \frac{2\log(d_1d_2) - 2\log(4n_1n_2\sigma_1\sigma_2)}{(n_1+n_2)\log|q|}.$$

Hence, we complete the proof of Theorem 1.6. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
HW, HYX completed the main part of this article, HW, YH, HYX corrected the main theorems. All authors read and approved the final manuscript.

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References

1. Hayman, WK: Meromorphic Functions. Clarendon, Oxford (1964)
2. Yang, L: Value Distribution Theory. Springer, Berlin (1993)
3. Yi, HX, Yang, CC: Uniqueness Theory of Meromorphic Functions. Kluwer Academic, Dordrecht (2003). Chinese original: Science Press, Beijing (1995)
4. Chen, ZX, Huang, ZB, Zheng, XM: On properties of difference polynomials. Acta Math. Sci. **31B**(2), 627-633 (2011)
5. Halburd, RG, Korhonen, RJ: Nevanlinna theory for the difference operator. Ann. Acad. Sci. Fenn., Math. **31**, 463-478 (2006)
6. Heittokangas, J, Korhonen, RJ, Laine, I, Rieppo, J, Zhang, JL: Value sharing results for shifts of meromorphic functions, and sufficient conditions for periodicity. J. Math. Anal. Appl. **355**, 352-363 (2009)
7. Laine, I, Yang, CC: Value distribution of difference polynomials. Proc. Jpn. Acad., Ser. A, Math. Sci. **83**, 148-151 (2007)
8. Liu, K, Yang, LZ: Value distribution of the difference operator. Arch. Math. **92**, 270-278 (2009)
9. Zhang, JL, Korhonen, RJ: On the Nevanlinna characteristic of $f(qz)$ and its applications. J. Math. Anal. Appl. **369**, 537-544 (2010)
10. Barnett, DC, Halburd, RG, Korhonen, RJ, Morgan, W: Nevanlinna theory for the q-difference operator and meromorphic solutions of q-difference equations. Proc. R. Soc. Edinb., Sect. A, Math. **137**, 457-474 (2007)
11. Chiang, YM, Feng, SJ: On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane. Ramanujan J. **16**, 105-129 (2008)
12. Halburd, RG, Korhonen, RJ: Difference analogue of the lemma on the logarithmic derivative with applications to difference equations. J. Math. Anal. Appl. **314**, 477-487 (2006)
13. Silvennoinen, H: Meromorphic solutions of some composite functional equations. Ann. Acad. Sci. Fenn., Math. Diss. **13**, 14-20 (2003)
14. Gao, LY: On meromorphic solutions of a type of system of composite functional equations. Acta Math. Sci. **32B**(2), 800-806 (2012)
15. Gao, LY: Systems of complex difference equations of Malmquist type. Acta Math. Sin. **55**, 293-300 (2012)
16. Gao, LY: Estimates of N-function and m-function of meromorphic solutions of systems of complex difference equations. Acta Math. Sci. **32B**(4), 1495-1502 (2012)
17. Xu, HY, Liu, BX, Tang, KZ: Some properties of meromorphic solutions of systems of complex q-shift difference equations. Abstr. Appl. Anal. **2013**, Article ID 680956 (2013)
18. Xu, HY, Cao, TB, Liu, BX: The growth of solutions of systems of complex q-shift difference equations. Adv. Differ. Equ. **2012**, 216 (2012)
19. Xu, HY, Xuan, ZX: Some properties of solutions of a class of systems of complex q-shift difference equations. Adv. Differ. Equ. **2013**, 271 (2013)
20. Heittokangas, J, Korhonen, RJ, Laine, I, Rieppo, J, Tohge, K: Complex difference equations of Malmquist type. Comput. Methods Funct. Theory **1**(1), 27-39 (2001)
21. Zheng, XM, Chen, ZX: Some properties of meromorphic solutions of q-difference equations. J. Math. Anal. Appl. **361**, 472-480 (2010)
22. Laine, I: Nevanlinna Theory and Complex Differential Equations. de Gruyter, Berlin (1993)
23. Jank, G, Volkmann, L: Einführung in die Theorie der ganzen und meromorphen Funktionen mit Anwendungen auf Diffrentialgleichungen. Birkhäuser, Basel (1985)
24. Bergweiler, W, Ishizaki, K, Yanagihara, N: Meromorphic solutions of some functional equations. Methods Appl. Anal. **5**(3), 248-259 (1998) Correction: Methods Appl. Anal. **6**(4), 617-618 (1999)

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