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Solvability of differential equations of order $2 < \alpha \leq 3$ involving the p-Laplacian operator with boundary conditions

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Abstract

In this paper, we study the existence of solutions for non-linear fractional differential equations of order $2 < \alpha \leq 3$ involving the p-Laplacian operator with various boundary value conditions including an anti-periodic case. By using the Banach contraction mapping principle, we prove that, under certain conditions, the suggested non-linear fractional boundary value problem involving the p-Laplacian operator has a unique solution for both cases of $0 < p < 1$ and $p \geq 2$. Finally, we illustrate our results with some examples.

Keywords: p-Laplacian operators; fractional derivative; fractional integral; Caputo fractional derivative; boundary value problem; Caputo fractional boundary value problem; anti-periodic boundary value problem

1 Introduction

Recently, boundary value problems for fractional differential equations have gained popularity among researchers since they have many applications in biophysics, blood flow phenomena, aerodynamics, polymer rheology, viscoelasticity, thermodynamics, electrodynamics of complex medium, capacitor theory, electrical circuits, electro-analytical chemistry, control theory (see [1–8] and [9]). Various results on boundary value problems for fractional differential equations have appeared in the literature (see [10–13] and [14]). Also, solvability of fractional differential equations with anti-periodic boundary value problems have been considered by different authors ([15, 16] and [17]). On the other hand, in studying turbulent flow in a porous medium, Leibenson introduced the concept of p-Laplacian operator [18] and it was used for fractional boundary value problems in [19] and [20]. In this paper, we consider boundary value problems for fractional differential equations of order $2 < \alpha \leq 3$ involving the p-Laplacian operator with various boundary conditions including an anti-periodic case.

Now, we present basic definitions and results that will be needed in the rest of the paper. More detailed information about the theory of fractional calculus and fractional differential equations can be found in [1, 2, 4] and [21].

It is well known that the beta function $B(t,s)$ has the following integral representation:

$$B(t,s) = \int_0^1 \tau^{t-1}(1-\tau)^{s-1}\,d\tau, \quad t,s > 0.$$

Moreover, $B(t,s)$ can be expressed in terms of $\Gamma(t)$, the gamma function, as follows:

$$B(t,s) = \frac{\Gamma(t)\Gamma(s)}{\Gamma(t+s)}. \tag{1}$$

Definition 1 Let α be a positive real number. Then the fractional integral of $x : (0,\infty) \to \mathbb{R}$ is defined by

$$I_{0^+}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{x(s)}{(t-s)^{1-\alpha}}\, ds.$$

Definition 2 Let α be a positive real number. Then the Caputo fractional derivative of $x : (0,\infty) \to \mathbb{R}$ is defined by

$$C_{0^+}^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{x^{(n)}(s)}{(t-s)^{\alpha+1-n}}\, ds,$$

where

$$n = \begin{cases} [\alpha]+1 & \text{if } \alpha \notin \mathbb{N}_0, \\ \alpha & \text{if } \alpha \in \mathbb{N}_0 \end{cases} \tag{2}$$

and $[\alpha]$ denotes the greatest integer less than or equal to α.

Recall that $C^n([a,b],\mathbb{R})$ is the space of all real-valued functions $x(t)$ which have continuous derivatives up to order $n-1$ on $[a,b]$.

In the following lemmas, we give some auxiliary results which will be used in the sequel.

Lemma 1 [1] *Let* $\alpha > 0$ *and* $y(t) \in C^n([0,1],\mathbb{R})$. *Then*

$$\left(I_{0^+}^{\alpha} C_{0^+}^{\alpha} x\right)(t) = x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(0)}{k!} t^k,$$

where n is given in (2).

On the other hand, the operator $\varphi_p(s) = |s|^{p-2}s$, where $p > 1$, is called the p-Laplacian operator. It is easy to see that $\varphi_p^{-1} = \varphi_q$, where $\frac{1}{p} + \frac{1}{q} = 1$. The following properties of the p-Laplacian operator will play an important role in the rest of the paper.

Lemma 2 *Let* φ_p *be a p-Laplacian operator.*

(i) *If* $1 < p < 2$, $xy > 0$, *and* $|x|, |y| \geq m > 0$, *then*

$$\left|\varphi_p(x) - \varphi_p(y)\right| \leq (p-1)m^{p-2}|x-y|. \tag{3}$$

(ii) *If* $p \geq 2$ *and* $|x|, |y| \leq M$, *then*

$$\left|\varphi_p(x) - \varphi_p(y)\right| \leq (p-1)M^{p-2}|x-y|. \tag{4}$$

In this paper, we focus on the solvability of the following non-linear fractional differential equations of order $\alpha \in (2,3]$ involving the p-Laplacian operator with boundary conditions:

$$\begin{cases} (\varphi_p(C^{\alpha}_{0^+}x(t)))' = f(t,x(t)), \\ x(0) = a_0 x(1), \\ x'(0) = a_1 x'(1), \\ x''(0) = a_2 x''(1), \end{cases} \tag{5}$$

where $a_i \neq 1$, $i = 0,1,2$, $t \in [0,1]$, $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$ and $x(t) \in C^2([0,1], \mathbb{R})$.

Remark 1 For $a_i = -1$, $i = 0,1,2$, the boundary value problem given in (5) becomes anti-periodic.

Lemma 3 *Assume that* $\alpha \in (2,3]$, $a_0, a_1, a_2 \neq 1$, $t \in [0,1]$ *and* $h \in C([0,1])$. *Then the solution* $x(t)$ *of the boundary value problem*

$$\begin{cases} (\varphi_p(C^{\alpha}_{0^+}x(t)))' = h(t), \\ x(0) = a_0 x(1), \\ x'(0) = a_1 x'(1), \\ x''(0) = a_2 x''(1) \end{cases} \tag{6}$$

can be represented by the following integral equation:

$$\begin{aligned} x(t) = {} & \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \varphi_q\left(\int_0^{\tau} h(s)\,ds\right) d\tau \\ & \times \frac{A_0}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} \varphi_q\left(\int_0^{\tau} h(s)\,ds\right) d\tau \\ & + \left[\frac{A_0 A_1 + t A_1}{\Gamma(\alpha-1)}\right] \int_0^1 (1-\tau)^{\alpha-2} \varphi_q\left(\int_0^{\tau} h(s)\,ds\right) d\tau \\ & + \left[\frac{2A_1 A_2 (A_0 + t) + A_2 (t^2 + A_0)}{2\Gamma(\alpha-2)}\right] \int_0^1 (1-\tau)^{\alpha-3} \varphi_q\left(\int_0^{\tau} h(s)\,ds\right) d\tau, \end{aligned} \tag{7}$$

where $A_0 = \frac{a_0}{1-a_0}$, $A_1 = \frac{a_1}{1-a_1}$ *and* $A_2 = \frac{a_2}{1-a_2}$.

Proof Using (6) and the fact that $\varphi_p(C^{\alpha}_{0^+}x(0)) = 0$, we have

$$\varphi_p\bigl(C^{\alpha}_{0^+}x(t)\bigr) = \int_0^t h(s)\,ds,$$

or equivalently,

$$C^{\alpha}_{0^+}x(t) = \varphi_q\left(\int_0^t h(s)\,ds\right), \tag{8}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Applying the fractional integral operator $I^{\alpha}_{0^+}$ to both sides of (8), we get

$$x(t) - x(0) - x'(0)t - \frac{x''(0)}{2} t^2 = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \varphi_q\left(\int_0^{\tau} h(s)\,ds\right) d\tau,$$

or equivalently,

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \varphi_q \left(\int_0^\tau h(s)\,ds \right) d\tau + x(0) + x'(0)t + \frac{x''(0)}{2} t^2, \tag{9}$$

$$x'(t) = \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-\tau)^{\alpha-2} \varphi_q \left(\int_0^\tau h(s)\,ds \right) d\tau + x'(0) + x''(0)t, \tag{10}$$

and

$$x''(t) = \frac{1}{\Gamma(\alpha-2)} \int_0^t (t-\tau)^{\alpha-3} \varphi_q \left(\int_0^\tau h(s)\,ds \right) d\tau + x''(0). \tag{11}$$

Taking $t = 1$ on both sides of (9), (10) and (11), we have

$$x(1) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} \varphi_q \left(\int_0^\tau h(s)\,ds \right) d\tau + x(0) + x'(0) + \frac{x''(0)}{2}, \tag{12}$$

$$x'(1) = \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-\tau)^{\alpha-2} \varphi_q \left(\int_0^\tau h(s)\,ds \right) d\tau + x'(0) + x''(0), \tag{13}$$

$$x''(1) = \frac{1}{\Gamma(\alpha-2)} \int_0^1 (1-\tau)^{\alpha-3} \varphi_q \left(\int_0^\tau h(s)\,ds \right) d\tau + x''(0). \tag{14}$$

Using equations (12), (13), (14) and the boundary value conditions

$$\begin{aligned} x(0) &= a_0 x(1), \\ x'(0) &= a_1 x'(1), \\ x''(0) &= a_2 x''(1), \end{aligned}$$

we can get that

$$x''(0) = \frac{A_2}{\Gamma(\alpha-2)} \int_0^1 (1-\tau)^{\alpha-3} \varphi_q \left(\int_0^\tau h(s)\,ds \right) d\tau, \tag{15}$$

$$\begin{aligned} x'(0) = {} & \frac{A_1}{\Gamma(\alpha-1)} \int_0^1 (1-\tau)^{\alpha-2} \varphi_q \left(\int_0^\tau h(s)\,ds \right) d\tau \\ & + \frac{A_2 A_1}{\Gamma(\alpha-2)} \int_0^1 (1-\tau)^{\alpha-3} \varphi_q \left(\int_0^\tau h(s)\,ds \right) d\tau \end{aligned} \tag{16}$$

and

$$\begin{aligned} x(0) = {} & \frac{A_0}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} \varphi_q \left(\int_0^\tau h(s)\,ds \right) d\tau \\ & + \frac{A_0 A_1}{\Gamma(\alpha-1)} \int_0^1 (1-\tau)^{\alpha-2} \varphi_q \left(\int_0^\tau h(s)\,ds \right) d\tau \\ & + \frac{A_0 A_1 A_2}{\Gamma(\alpha-2)} \int_0^1 (1-\tau)^{\alpha-3} \varphi_q \left(\int_0^\tau h(s)\,ds \right) d\tau \\ & \times \frac{A_0 A_2}{2\Gamma(\alpha-2)} \int_0^1 (1-\tau)^{\alpha-3} \varphi_q \left(\int_0^\tau h(s)\,ds \right) d\tau. \end{aligned} \tag{17}$$

Substituting (15), (16) and (17) into (9) gives (7) and this completes the proof. □

2 Solvability of the fractional boundary value problem

This section is devoted to the solvability of the fractional boundary value problem given in (5). First, we obtain conditions for existence and uniqueness of the solution $x(t)$ of the fractional boundary value problem given in (5). Then, each result obtained here is illustrated by examples.

Recall that $C[0,1]$, the space of continuous functions on $[0,1]$ is a Banach space with the norm $\|x\| = \max_{t\in[0,1]} |x(t)|$. Now consider $T_i : C[0,1] \to C[0,1]$, $i = 0,1$, with

$$T_0x(t) := \varphi_q\left(\int_0^t f\bigl(s,x(s)\bigr)\,ds\right)$$

and

$$\begin{aligned}T_1x(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}x(\tau)\,d\tau \\ &\quad + \frac{A_0}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1}x(\tau)\,d\tau \\ &\quad + \left(\frac{A_0A_1 + tA_1}{\Gamma(\alpha-1)}\right)\int_0^1 (1-\tau)^{\alpha-2}x(\tau)\,d\tau \\ &\quad + \left(\frac{2A_1A_2(A_0+t) + A_2(t^2+A_0)}{2\Gamma(\alpha-2)}\right)\int_0^1 (1-\tau)^{\alpha-3}x(\tau)\,d\tau.\end{aligned}$$

Then the operator $T : C[0,1] \to C[0,1]$, defined by $T = T_1 \circ T_0$, is continuous and compact.

Theorem 1 *Suppose* $1 < q < 2$, $a_0, a_1, a_2 \neq 1$, *and the following conditions hold*: $\exists\lambda > 0$, $0 < \delta < \frac{2}{2-q}$ *and* d *with*

$$\begin{aligned}0 < d &< \bigl[\bigl(2\lambda^{2-q}\Gamma\bigl(\delta(q-2)+\alpha+2\bigr)\bigr) \\ &\quad /\bigl((q-1)\Gamma\bigl(\delta(q-2)+2\bigr)\bigl(1+|A_0|\bigr)\bigl[2\bigl(1+|A_1|\bigl(\delta(q-2)+\alpha+1\bigr)\bigr) \\ &\quad + |A_2|\bigl(1+2|A_1|\bigr)\bigl(\delta(q-2)+\alpha\bigr)\bigl(\delta(q-2)+\alpha+1\bigr)\bigr]\bigr)\bigr]\end{aligned} \tag{18}$$

such that

$$\lambda\delta t^{\delta-1} \le f(t,x) \quad \text{for any } (t,x) \in (0,1] \times \mathbb{R} \tag{19}$$

and

$$\bigl|f(t,x) - f(t,y)\bigr| \le d|x-y| \quad \text{for } t \in [0,1] \text{ and } x,y \in \mathbb{R}. \tag{20}$$

Then boundary value problem (5) *has a unique solution.*

Proof Using inequality (19), we get

$$\lambda t^\delta \le \int_0^t f(s,x)\,ds \quad \text{for any } (t,x) \in [0,1] \times \mathbb{R}.$$

By (3) and (20), we have

$$\begin{aligned}
&\bigl|T_0x(t) - T_0y(t)\bigr| \\
&\quad = \left|\varphi_q\left(\int_0^t f\bigl(s,x(s)\bigr)\,ds\right) - \varphi_q\left(\int_0^t f\bigl(s,y(s)\bigr)\,ds\right)\right| \\
&\quad \le (q-1)\bigl(\lambda t^{\delta}\bigr)^{q-2}\left|\int_0^t f\bigl(s,x(s)\bigr)\,ds - \int_0^t f\bigl(s,y(s)\bigr)\,ds\right| \\
&\quad \le (q-1)\lambda^{q-2}t^{\delta(q-2)}\int_0^t \bigl|f\bigl(s,x(s)\bigr) - f\bigl(s,y(s)\bigr)\bigr|\,ds \\
&\quad \le d(q-1)\lambda^{q-2}t^{\delta(q-2)}\int_0^t \bigl|x(s) - y(s)\bigr|\,ds \\
&\quad \le d(q-1)\lambda^{q-2}t^{\delta(q-2)+1}\|x-y\|.
\end{aligned} \tag{21}$$

Moreover,

$$\begin{aligned}
&\bigl|Tx(t) - Ty(t)\bigr| \\
&\quad = \bigl|T_1\bigl(T_0\bigl(x(t)\bigr)\bigr) - T_1\bigl(T_0\bigl(y(t)\bigr)\bigr)\bigr| \\
&\quad = \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\bigl((T_0x)(\tau) - (T_0y)(\tau)\bigr)\,d\tau\right. \\
&\qquad + \frac{A_0}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1}\bigl((T_0x)(\tau) - (T_0y)(\tau)\bigr)\,d\tau \\
&\qquad + \left(\frac{A_1(A_0+t)}{\Gamma(\alpha-1)}\right)\int_0^1 (1-\tau)^{\alpha-2}\bigl((T_0x)(\tau) - (T_0y)(\tau)\bigr)\,d\tau \\
&\qquad + \left(\frac{2A_1A_2(A_0+t) + A_2(t^2+A_0)}{2\Gamma(\alpha-2)}\right) \\
&\qquad \times \left.\int_0^1 (1-\tau)^{\alpha-3}\bigl((T_0x)(\tau) - (T_0y)(\tau)\bigr)\,d\tau\right|.
\end{aligned} \tag{22}$$

Finally, substituting (21) in (22), we get

$$\begin{aligned}
&\bigl|Tx(t) - Ty(t)\bigr| \\
&\quad \le d(q-1)\lambda^{q-2}\|x-y\|\left[\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\tau^{\delta(q-2)+1}\,d\tau\right. \\
&\qquad + \left|\frac{A_0}{\Gamma(\alpha)}\right|\int_0^1 (1-\tau)^{\alpha-1}\tau^{\delta(q-2)+1}\,d\tau \\
&\qquad + \left|\frac{A_1(A_0+t)}{\Gamma(\alpha-1)}\right|\int_0^1 (1-\tau)^{\alpha-2}\tau^{\delta(q-2)+1}\,d\tau \\
&\qquad + \left.\left|\frac{2A_1A_2(A_0+t) + A_2(t^2+A_0)}{2\Gamma(\alpha-2)}\right|\int_0^1 (1-\tau)^{\alpha-3}\tau^{\delta(q-2)+1}\,d\tau\right].
\end{aligned} \tag{23}$$

Using the equality

$$\int_0^t (t-\tau)^{\alpha-1}\tau^{\delta(q-2)+1}\,d\tau = t^{\delta(q-2)+\alpha+1}\int_0^1 (1-\tau)^{\alpha-1}\tau^{\delta(q-2)+1}\,d\tau$$

in (23) one can write that

$$\begin{aligned}
\bigl|Tx(t)-Ty(t)\bigr| &\leq d(q-1)\lambda^{q-2}\|x-y\|\biggl[\frac{t^{\delta(q-2)+\alpha+1}}{\Gamma(\alpha)}\int_0^1(1-\tau)^{\alpha-1}\tau^{\delta(q-2)+1}\,d\tau \\
&\quad+\left|\frac{A_0}{\Gamma(\alpha)}\right|\int_0^1(1-\tau)^{\alpha-1}\tau^{\delta(q-2)+1}\,d\tau \\
&\quad+\left|\frac{(A_0A_1+tA_1)}{\Gamma(\alpha-1)}\right|\int_0^1(1-\tau)^{\alpha-2}\tau^{\delta(q-2)+1}\,d\tau \\
&\quad+\left|\frac{2A_1A_2(A_0+t)+A_2(t^2+A_0)}{2\Gamma(\alpha-2)}\right|\int_0^1(1-\tau)^{\alpha-3}\tau^{\delta(q-2)+1}\,d\tau\biggr] \\
&= d(q-1)\lambda^{q-2}\|x-y\|B\bigl(\delta(q-2)+2,\alpha\bigr) \\
&\quad\times\biggl[\frac{t^{\delta(q-2)+\alpha+1}}{\Gamma(\alpha)}+\frac{|A_0|}{\Gamma(\alpha)}+\frac{|(A_0A_1+tA_1)|}{\Gamma(\alpha)}\bigl(\delta(q-2)+\alpha+1\bigr) \\
&\quad+\frac{|2A_1A_2(A_0+t)+A_2(t^2+A_0)|}{2\Gamma(\alpha)}\bigl(\delta(q-2)+\alpha\bigr)\bigl(\delta(q-2)+\alpha+1\bigr)\biggr].
\end{aligned}$$

Using (1), we get that

$$\begin{aligned}
&\bigl|Tx(t)-Ty(t)\bigr| \\
&\quad\leq d(q-1)\lambda^{q-2}\|x-y\|\frac{\Gamma(\delta(q-2)+2)\Gamma(\alpha)}{\Gamma(\delta(q-2)+\alpha+2)} \\
&\qquad\times\biggl[\frac{1}{\Gamma(\alpha)}+\frac{|A_0|}{\Gamma(\alpha)}+\frac{(|A_0||A_1|+|A_1|)}{\Gamma(\alpha)}\bigl(\delta(q-2)+\alpha+1\bigr) \\
&\qquad+\frac{(2|A_1||A_2|(|A_0|+1)+|A_2|(1+|A_0|))}{2\Gamma(\alpha)}\bigl(\delta(q-2)+\alpha\bigr)\bigl(\delta(q-2)+\alpha+1\bigr)\biggr] \\
&\quad\leq d(q-1)\lambda^{q-2}\|x-y\|\frac{\Gamma(\delta(q-2)+2)}{2\Gamma(\delta(q-2)+2+\alpha)} \\
&\qquad\times\bigl[2\bigl(1+|A_0|\bigr)+2|A_1|\bigl(1+|A_0|\bigr)\bigl(\delta(q-2)+\alpha+1\bigr) \\
&\qquad+|A_2|\bigl(1+|A_0|\bigr)\bigl(1+2|A_1|\bigr)\bigl(\delta(q-2)+\alpha\bigr)\bigl(\delta(q-2)+\alpha+1\bigr)\bigr] \\
&\quad\leq d(q-1)\lambda^{q-2}\|x-y\|\frac{\Gamma(\delta(q-2)+2)}{2\Gamma(\delta(q-2)+2+\alpha)} \\
&\qquad\times\bigl(1+|A_0|\bigr)\bigl[2\bigl(1+|A_1|\bigl(\delta(q-2)+\alpha+1\bigr)\bigr) \\
&\qquad+|A_2|\bigl(1+2|A_1|\bigr)\bigl(\delta(q-2)+\alpha\bigr)\bigl(\delta(q-2)+\alpha+1\bigr)\bigr] \\
&\quad= K\|x-y\|,
\end{aligned}$$

where

$$\begin{aligned}
K &= d(q-1)\lambda^{q-2}\frac{\Gamma(\delta(q-2)+2)}{2\Gamma(\delta(q-2)+\alpha+2)}\bigl(1+|A_0|\bigr)\bigl[2\bigl(1+|A_1|\bigl(\delta(q-2)+\alpha+1\bigr)\bigr) \\
&\quad+|A_2|\bigl(1+2|A_1|\bigr)\bigl(\delta(q-2)+\alpha\bigr)\bigl(\delta(q-2)+\alpha+1\bigr)\bigr].
\end{aligned}\tag{24}$$

Combining (24) with (18) implies that $0 < K < 1$, therefore T is a contraction. As a consequence of the Banach contraction mapping theorem [22] the boundary value problem given in (5) has a unique solution. □

Theorem 2 *Suppose* $1 < q < 2$, $a_0, a_1, a_2 \neq 1$ *and the following conditions hold for a fixed,* $\exists \lambda > 0$, $0 < \delta < \frac{2}{2-q}$ *and* d *with*

$$\begin{aligned} 0 < d < {} & \bigl[\bigl(2\lambda^{2-q}\Gamma\bigl(\delta(q-2)+\alpha+2\bigr)\bigr) \\ & /\bigl((q-1)\Gamma\bigl(\delta(q-2)+2\bigr)\bigl(1+|A_0|\bigr)\bigl[2\bigl(1+|A_1|\bigl(\delta(q-2)+\alpha+1\bigr)\bigr) \\ & + |A_2|\bigl(1+2|A_1|\bigr)\bigl(\delta(q-2)+\alpha\bigr)\bigl(\delta(q-2)+\alpha+1\bigr)\bigr]\bigr)\bigr] \end{aligned}$$

such that

$$f(t,x) \leq -\lambda\delta t^{\delta-1} \quad \textit{for any } (t,x) \in (0,1] \times \mathbb{R}$$

and

$$\bigl|f(t,x) - f(t,y)\bigr| \leq d|x-y| \quad \textit{for } t \in [0,1] \textit{ and } x,y \in \mathbb{R}.$$

Then boundary value problem (5) *has a unique solution.*

Proof The inequality $f(t,x) \leq -\lambda\delta t^{\delta-1}$ implies that $\lambda\delta t^{\delta-1} \leq -f(t,x)$. Therefore replace $f(t,x)$ by $-f(t,x)$ in the proof of Theorem 1. □

Theorem 3 *Suppose* $q > 2$, $a_0, a_1, a_2 \neq 1$, *there exists a non-negative function* $g(x) \in L[0,1]$ *with* $M := \int_0^1 g(\tau)\,d\tau \geq 0$ *such that*

$$\bigl|f(t,x)\bigr| \leq g(t) \quad \textit{for any } (t,x) \in [0,1] \times \mathbb{R} \tag{25}$$

and there exists a constant d *with*

$$0 < d < \frac{2\Gamma(\alpha+2)}{(q-1)M^{q-2}(1+|A_0|)[2(1+|A_1|(\alpha+1)) + |A_2|(1+2|A_1|)\alpha(\alpha+1)]} \tag{26}$$

and

$$\bigl|f(t,x) - f(t,y)\bigr| \leq d|x-y| \quad \textit{for } t \in [0,1] \textit{ and } x,y \in \mathbb{R}.$$

Then boundary value problem (5) *has a unique solution.*

Proof Using (25), we get that

$$\int_0^t \bigl|f\bigl(\tau, x(\tau)\bigr)\bigr|\,d\tau \leq \int_0^1 g(\tau)\,d\tau = M \tag{27}$$

for all $t \in [0,1]$. By the definition of the operator T_0, one can write that

$$\bigl|T_0x(t) - T_0y(t)\bigr| = \biggl|\varphi_q\biggl(\int_0^t f\bigl(s,x(s)\bigr)\,ds\biggr) - \varphi_q\biggl(\int_0^t f\bigl(s,y(s)\bigr)\,ds\biggr)\biggr|. \tag{28}$$

As a consequence of (4), (27) and (28), we have

$$\begin{aligned}
\bigl|T_0x(t) - T_0y(t)\bigr| &\le (q-1)M^{q-2}\left|\int_0^t f\bigl(s,x(s)\bigr)\,ds - \int_0^t f\bigl(s,y(s)\bigr)\,ds\right| \\
&\le (q-1)M^{q-2}\int_0^t \bigl|f\bigl(s,x(s)\bigr) - f\bigl(s,y(s)\bigr)\bigr|\,ds \\
&\le d(q-1)M^{q-2}\int_0^t \bigl|x(s) - y(s)\bigr|\,ds \\
&\le d(q-1)M^{q-2}t\|x-y\|.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\bigl|Tx(t) - Ty(t)\bigr| &= \bigl|T_1\bigl(T_0\bigl(x(t)\bigr)\bigr) - T_1\bigl(T_0\bigl(y(t)\bigr)\bigr)\bigr| \\
&= \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\bigl((T_0x)(\tau) - (T_0y)(\tau)\bigr)\,d\tau\right. \\
&\quad + \frac{A_0}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1}\bigl((T_0x)(\tau) - (T_0y)(\tau)\bigr)\,d\tau \\
&\quad + \frac{A_1(A_0+t)}{\Gamma(\alpha-1)}\int_0^1 (1-\tau)^{\alpha-2}\bigl((T_0x)(\tau) - (T_0y)(\tau)\bigr)\,d\tau \\
&\quad + \left|\frac{2A_1A_2(A_0+t) + A_2(t^2+A_0)}{2\Gamma(\alpha-2)}\right| \\
&\quad \left.\times \int_0^1 (1-\tau)^{\alpha-3}\bigl((T_0x)(\tau) - (T_0y)(\tau)\bigr)\,d\tau\right|, \\
\bigl|Tx(t) - Ty(t)\bigr| &\le d(q-1)M^{q-2}\|x-y\|\left[\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\tau\,d\tau\right. \\
&\quad + \frac{|A_0|}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1}\tau\,d\tau + \left|\frac{A_1(A_0+1)}{\Gamma(\alpha-1)}\right|\int_0^1 (1-\tau)^{\alpha-2}\tau\,d\tau \\
&\quad \left.+ \frac{|2A_1A_2(A_0+t) + A_2(t^2+A_0)|}{2\Gamma(\alpha-2)}\int_0^1 (1-\tau)^{\alpha-3}\tau\,d\tau\right] \\
&\le d(q-1)M^{q-2}\|x-y\|\left[\frac{t^{\alpha+1}}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1}\tau\,d\tau\right. \\
&\quad + \frac{|A_0|}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1}\tau\,d\tau + \frac{|A_1|(|A_0|+1)}{\Gamma(\alpha-1)}\int_0^1 (1-\tau)^{\alpha-2}\tau\,d\tau \\
&\quad \left.+ \frac{|A_2|(|A_0|+1)(1+2|A_1|)}{2\Gamma(\alpha-2)}\int_0^1 (1-\tau)^{\alpha-3}\tau\,d\tau\right] \\
&\le d(q-1)M^{q-2}B(\alpha,2)\|x-y\|\left[\frac{1}{\Gamma(\alpha)} + \frac{|A_0|}{\Gamma(\alpha)} + \frac{|A_1|(|A_0|+1)(\alpha+1)}{\Gamma(\alpha)}\right. \\
&\quad \left.+ \frac{|A_2|(|A_0|+1)(1+2|A_1|)\alpha(\alpha+1)}{2\Gamma(\alpha)}\right] \\
&\le \frac{d(q-1)M^{q-2}\|x-y\|}{\Gamma(\alpha+2)}\left[\bigl(1+|A_0|\bigr)\bigl(1+|A_1|(\alpha+1)\bigr)\right. \\
&\quad \left.+ \frac{|A_2|(|A_0|+1)(1+2|A_1|)\alpha(\alpha+1)}{2}\right]
\end{aligned}$$

$$\begin{aligned}&\leq \frac{d(q-1)M^{q-2}}{\Gamma(\alpha+2)}\bigl(1+|A_0|\bigr)\biggl[\frac{2(1+|A_1|(\alpha+1))}{2}\\&\quad+\frac{|A_2|(1+2|A_1|)\alpha(\alpha+1)}{2}\biggr]\|x-y\|\\&\leq K\|x-y\|,\end{aligned}$$

where

$$K=\frac{d(q-1)M^{q-2}}{2\Gamma(\alpha+2)}\bigl(1+|A_0|\bigr)\bigl[2\bigl(1+|A_1|(\alpha+1)\bigr)+|A_2|\bigl(1+2|A_1|\bigr)\alpha(\alpha+1)\bigr].$$

By (26), we get $K<1$, which implies that T is a contraction, therefore the boundary value problem given in (5) has a unique solution. □

In the present part, we illustrate our results by examples.

Example 1 Consider the following anti-periodic boundary value problem:

$$\begin{cases}(\varphi_{\frac{7}{3}}(C_0^{\frac{5}{2}}x(t)))'=4t^2(2+\sin(\frac{\sqrt{\pi}x}{16}+\omega)), & t\in(0,1),\\ x(0)=-x(1),\\ x'(0)=-x'(1),\\ x''(0)=-x''(1),\end{cases}\tag{29}$$

where

$$p=\frac{7}{3},\qquad \alpha=\frac{5}{2},\quad \text{and}\quad a_0=a_1=a_2=-1.$$

Then $q=\frac{7}{4}$, $|A_0|=|A_1|=|A_2|=\frac{1}{2}$ and take $\delta=4$, $\lambda=1$ and $d=\frac{\sqrt{\pi}}{4}$. Obviously,

$$\begin{aligned}&\bigl[\bigl(2\lambda^{2-q}\Gamma\bigl(\delta(q-2)+\alpha+2\bigr)\bigr)\\&\quad/\bigl((q-1)\Gamma\bigl(\delta(q-2)+2\bigr)\bigl(1+|A_0|\bigr)\bigl[2\bigl(1+|A_1|\bigl(\delta(q-2)+\alpha+1\bigr)\bigr)\\&\quad+|A_2|\bigl(1+2|A_1|\bigr)\bigl(\delta(q-2)+\alpha\bigr)\bigl(\delta(q-2)+\alpha+1\bigr)\bigr]\bigr)\bigr]\\&\quad=\frac{64\Gamma(\frac{7}{2})}{297}=\frac{40\sqrt{\pi}}{99}>\frac{\sqrt{\pi}}{4}=d.\end{aligned}$$

On the other hand,

$$\delta\lambda t^{\delta-1}=4t^3\leq 4t^2\biggl(2+\sin\biggl(\frac{\sqrt{\pi}x}{16}+\omega\biggr)\biggr)=f(t,x).$$

Finally,

$$\begin{aligned}\bigl|f(t,x)-f(t,y)\bigr|&=\biggl|4t^2\biggl(2+\sin\biggl(\frac{\sqrt{\pi}x}{16}+\omega\biggr)\biggr)-4t^2\biggl(2+\sin\biggl(\frac{\sqrt{\pi}y}{16}+\omega\biggr)\biggr)\biggr|\\&=4t^2\biggl|\sin\biggl(\frac{\sqrt{\pi}x}{16}+\omega\biggr)-\sin\biggl(\frac{\sqrt{\pi}y}{16}+\omega\biggr)\biggr|\end{aligned}$$

$$\leq 4\left|\left(\frac{\sqrt{\pi}x}{16}+\omega\right)-\left(\frac{\sqrt{\pi}y}{16}+\omega\right)\right|$$
$$=\frac{\sqrt{\pi}}{4}|x-y|.$$

Therefore, as a consequence of Theorem 1, the boundary value problem given in (29) has a unique solution.

Example 2 Consider the following anti-periodic boundary value problem:

$$\begin{cases} (\varphi_{\frac{7}{3}}(C_0^{\frac{5}{2}}x(t)))' = -2t(2+\sin(\frac{x}{2\sqrt{\pi}}+\omega)), \quad t\in(0,1),\\ x(0)=-x(1),\\ x'(0)=-x'(1),\\ x''(0)=-x''(1), \end{cases} \tag{30}$$

where

$$p=\frac{7}{3}, \qquad \alpha=\frac{5}{2} \quad \text{and} \quad a_0=a_1=a_2=-1.$$

Then, obviously, $q=\frac{7}{4}$, $|A_0|=|A_1|=|A_2|=\frac{1}{2}$. Taking $\delta=2$, $\lambda=1$ and $d=\frac{1}{\sqrt{\pi}}$, we have

$$\begin{aligned}
&\big[\big(2\lambda^{2-q}\Gamma\big(\delta(q-2)+\alpha+2\big)\big)\\
&\quad /\big((q-1)\Gamma\big(\delta(q-2)+2\big)\big(1+|A_0|\big)\big[2\big(1+|A_1|\big(\delta(q-2)+\alpha+1\big)\big)\\
&\quad +|A_2|\big(1+2|A_1|\big)\big(\delta(q-2)+\alpha\big)\big(\delta(q-2)+\alpha+1\big)\big]\big)\big]\\
&=\frac{16\Gamma(4)}{9\Gamma(\frac{3}{2})[2(1+\frac{3}{2})+6]}=\frac{64}{33\sqrt{\pi}}>\frac{1}{\sqrt{\pi}}=d.
\end{aligned}$$

On the other hand,

$$f(t,x)=-2t\left(2+\sin\left(\frac{x}{2\sqrt{\pi}}+\omega\right)\right)\leq -2t=-\delta\lambda t^{\delta-1}.$$

Finally,

$$\begin{aligned}
\big|f(t,x)-f(t,y)\big| &= \left|-2t\left(2+\sin\left(\frac{x}{2\sqrt{\pi}}+\omega\right)\right)+2t\left(2+\sin\left(\frac{y}{2\sqrt{\pi}}+\omega\right)\right)\right|\\
&\leq 2t\left|\sin\left(\frac{x}{2\sqrt{\pi}}+\omega\right)-\sin\left(\frac{y}{2\sqrt{\pi}}+\omega\right)\right|\\
&\leq 2\left|\left(\frac{x}{2\sqrt{\pi}}+\omega\right)-\left(\frac{y}{2\sqrt{\pi}}+\omega\right)\right| = \frac{1}{\sqrt{\pi}}|x-y|.
\end{aligned}$$

Therefore, as a consequence of Theorem 2, the boundary value problem given in (30) has a unique solution.

Example 3 Now consider the following boundary value problem:

$$\begin{cases} (\varphi_{\frac{9}{5}}(C_0^{\frac{7}{3}}x(t)))' = \sin^2(\frac{\sqrt{\pi}x}{40} + \omega), & t \in (0,1), \\ x(0) = -\frac{1}{5}x(1), \\ x(0)' = \frac{1}{2}x'(1), \\ x(0)'' = \frac{1}{2}x''(1), \end{cases} \tag{31}$$

where

$$p = \frac{9}{5}, \qquad \alpha = \frac{7}{3}, \qquad a_0 = -\frac{1}{5} \quad \text{and} \quad a_1 = a_2 = \frac{1}{2}.$$

Then $q = \frac{9}{4}$, $|A_0| = \frac{1}{6}$ and $|A_1| = |A_1| = 1$. Also, taking $d = \frac{\sqrt{\pi}}{20}$ and $g(t) = 1$, we have

$$M = 1$$

and

$$\left[\frac{2\Gamma(\alpha+2)}{(q-1)M^{q-2}(1+|A_0|)[2(1+|A_1|(\alpha+1)) + |A_2|(1+2|A_1|)\alpha(\alpha+1)]}\right]$$
$$= \frac{48\Gamma(\frac{13}{3})}{35[2(1+\frac{10}{3}) + (3\frac{7}{3}\frac{10}{3})]} = \frac{4\Gamma(\frac{4}{3})}{9} > \frac{\sqrt{\pi}}{20}.$$

On the other hand,

$$\left|f(t,x) - f(t,y)\right| \leq \left|\sin^2\left(\frac{\sqrt{\pi}x}{40} + \omega\right) - \sin^2\left(\frac{\sqrt{\pi}y}{40} + \omega\right)\right| \leq \frac{\sqrt{\pi}}{20}|x-y|$$
$$\text{for } t \in [0,1] \text{ and } x, y \in \mathbb{R};$$

therefore, by Theorem 3, the boundary value problem given in (31) has a unique solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors completed the paper together. All authors read and approved the final manuscript.

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