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Existence and growth of meromorphic solutions of some nonlinear q-difference equations

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Abstract

In this paper, we investigate the existence of transcendental meromorphic solutions of order zero of some nonlinear q-difference equations and some more general equations. We also give some estimates on the growth of transcendental meromorphic solutions of these equations. Some examples are given to illustrate the sharpness of some of our results.

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1 Introduction and main results

In this paper, a meromorphic function means being meromorphic in the whole complex plane. We also assume that the readers are familiar with the usual notations of Nevanlinna theory (see, *e.g.*, [1–3]). Especially, we use notations $\sigma(f)$ and $\mu(f)$ for the order and the lower order of a meromorphic function f. We also denote by $S(r,f)$ any quantity satisfying $S(r,f) = o(T(r,f))$ for all r outside of a possible exceptional set of finite logarithmic measure. Moreover, the standard definitions of logarithmic measure and logarithmic density can be found in [4].

In last two decades, there has been a renewed interest in the complex analytic properties of complex differences and meromorphic solutions of complex difference equations owing to the introduction of Nevanlinna theory in this field. And the study of complex q-differences and q-difference equations is an important component of the study of complex differences and difference equations.

The original study of complex nonlinear q-difference equations can be derived from the study of the nonautonomous Schröder equation

$$f(qz) = R\big(z,f(z)\big) \tag{1.1}$$

by Valiron [5], which is closely related to the equations in complex dynamic systems. Indeed, Ritt [6] is an earlier classical paper on the autonomous Schröder equation

$$f(qz) = R\big(f(z)\big).$$

And in the important collection [7] of research problems, Rubel posed the question: What can be said about the more general equation (1.1)?

Recently, a number of papers (including [8–23]) focus on complex differences and complex q-differences. These papers also investigate the existence and the growth of meromorphic solutions of complex difference equations and complex q-difference equations.

In particular, Yang-Laine [21] pointed out some similarities between results on the existence and uniqueness of finite order entire solutions of the nonlinear differential equations and differential-difference equations. They obtained the following results.

Theorem A *Let $p(z)$, $q(z)$ be polynomials. Then a nonlinear difference equation*

$$f^2(z) + q(z)f(z+1) = p(z) \tag{1.2}$$

has no transcendental entire solutions of finite order.

Theorem B *Let $n \geq 4$ be an integer, $M(z,f)$ be a linear differential-difference polynomial of f, not vanishing identically, and h be a meromorphic function of finite order. Then the differential-difference equation*

$$f^n(z) + M(z,f) = h(z) \tag{1.3}$$

possesses at most one admissible transcendental entire solution of finite order such that all coefficients of $M(z,f)$ are small functions of f. If such a solution f exists, then f is of the same order as h.

Some generalizations of Theorems A and B can be found in Peng-Chen [19], and we omit those results here.

In this paper, we consider a similar problem on the existence and the growth of transcendental meromorphic solutions of complex q-difference equations (resp. differential-q-difference equations) instead of complex difference equation (1.2) (resp. differential-difference equation (1.3)). And the involved equations are more general than (1.1). Moreover, the fact that all meromorphic solutions of the Riccati q-difference equation and linear q-difference equation, both with rational coefficients, are of order zero, shows that it is of great importance to investigate meromorphic solutions of order zero of q-difference equations.

Theorem 1.1 *Let $s(z)$ $(\not\equiv 0)$, $t(z)$ be rational functions, $q \in \mathbb{C}\backslash\{0,1\}$, and n, m be positive integers such that $n \neq m$.*

(i) *If $n > m$, then the nonlinear q-difference equation*

$$f^n(z) + s(z)f(qz)f\left(q^2z\right)\cdots f\left(q^mz\right) = t(z) \tag{1.4}$$

has no transcendental meromorphic solutions of order zero. Furthermore, when $|q| > 1$, if there exists a transcendental meromorphic solution f of positive order of (1.4), then

$$\sigma(f) \geq \mu(f) \geq \frac{\log n - \log m}{m \log |q|}.$$

(ii) *If $n < m$, then nonlinear q-difference equation* (1.4) *has no transcendental entire solutions of order zero.*

Corollary 1.1 *Let $s(z)$ ($\not\equiv 0$), $t(z)$ be rational functions, $q \in \mathbb{C}\backslash\{0,1\}$, and $n \geq 2$ be an integer. Then the nonlinear q-difference equation*

$$f^n(z) + s(z)f(qz) = t(z) \tag{1.5}$$

has no transcendental meromorphic solutions of order zero. Furthermore, when $|q| > 1$, if there exists a transcendental meromorphic solution f of positive order of (1.5), *then*

$$\sigma(f) = \mu(f) = \frac{\log n}{\log |q|}.$$

Remark 1.1 Wittich [24] and Ishizaki [18] had earlier treated equation (1.5) of the case $n = 1$, which is the first order linear q-difference equation.

Remark 1.2 Equation (1.4) may have meromorphic solutions of order zero, when $n = m$. For example, the function

$$f(z) = \frac{1}{\prod_{i=0}^{\infty}(1 - q^i z)}, \quad 0 < |q| < 1$$

is a transcendental meromorphic function of order zero (see Ramis [20]), and it satisfies the nonlinear q-difference equation

$$f^2(z) - \frac{1}{(1-z)^2(1-qz)} f(qz) f\left(q^2 z\right) = 0,$$

where $n = m = 2$.

Remark 1.3 Equations (1.4) and (1.5) may have meromorphic solutions of positive orders, when $|q| > 1$. For example, the function

$$f(z) = \frac{e^{z^2}}{z}$$

satisfies the nonlinear q-difference equation

$$f^{20}(z) - \frac{8}{z^{18}} f(2z) f(4z) = 0,$$

where $n = 20 > 2 = m$, $q = 2$ and $\sigma(f) = \mu(f) = 2 > \log_4 10 = \frac{\log n - \log m}{m \log |q|}$. And the function

$$f(z) = \frac{\cos z}{z}$$

satisfies the nonlinear q-difference equation

$$f^2(z) - \frac{1}{z} f(2z) = \frac{1}{2z^2},$$

where $n = 2$, $q = 2$ and $\sigma(f) = \mu(f) = 1 = \frac{\log n}{\log|q|}$. The above two examples show that the estimates on the growth of meromorphic solutions of equations (1.4) and (1.5) are sharp.

In the following, we consider the existence of entire solutions of order zero of a type of differential-q-difference equation, which includes equations (1.4) and (1.5) as its special cases. We define a differential-q-difference polynomial in f, which is a finite sum of products of f, derivatives of f and of their q-shifts, with all meromorphic coefficients of these monomials of growth $S(r,f)$. Concretely, we denote a differential-q-difference polynomial in f by

$$U_q(z,f) = \sum_{\lambda\in J} b_\lambda(z) \prod_{i=1}^{k_\lambda} f^{(i)}(z)^{m_{\lambda,i}} \prod_{j=1}^{l_\lambda} f^{(j)}\left(q^{\mu_{\lambda,j}} z\right)^{n_{\lambda,j}},$$

where J is a finite set of indices, $b_\lambda(z)$, $\lambda \in J$ are meromorphic functions of growth $S(r,f)$, and $q \in \mathbb{C}\backslash\{0,1\}$. And we denote the degree of $U_q(z,f)$ by

$$\deg_f U_q(z,f) = \max_{\lambda\in J}\left\{\sum_{i=1}^{k_\lambda} m_{\lambda,i} + \sum_{j=1}^{l_\lambda} n_{\lambda,j}\right\}.$$

In particular, if each monomial of $U_q(z,f)$ is of the same degree, then we call $U_q(z,f)$ a homogeneous differential-q-difference polynomial in f.

Theorem 1.2 *Let n, m be integers such that $n > 2m > 0$, $U_q(z,f)$ $(\not\equiv 0)$ be a homogeneous differential-q-difference polynomial in f of degree m, with all meromorphic coefficients of growth $S(r,f)$, and $t(z)$ be a rational function. Then the differential-q-difference equation*

$$f^n(z) + U_q(z,f) = t(z) \tag{1.6}$$

has no transcendental entire solutions of order zero.

Corollary 1.2 *Let $n \geq 3$ be an integer, $U_q(z,f)$ $(\not\equiv 0)$ be a linear differential-q-difference polynomial in f, with all meromorphic coefficients of growth $S(r,f)$, and $t(z)$ be a rational function. Then differential-q-difference equation* (1.6) *has no transcendental entire solutions of order zero.*

Corollary 1.3 *Let $s_j(z)$, $j = 1,\dots,m+1$ be rational functions, not all vanishing identically, $q \in \mathbb{C}\backslash\{0,1\}$, and let n (≥ 3), m be positive integers. Then the nonlinear q-difference equation*

$$f^n(z) + \sum_{j=1}^{m} s_j(z) f\left(q^j z\right) = s_{m+1}(z) \tag{1.7}$$

has no transcendental entire solutions of order zero. Furthermore, if $n > m$, then nonlinear q-difference equation (1.7) *has no transcendental meromorphic solutions of order zero, and any transcendental meromorphic solution f of positive order of* (1.7) *satisfies $\sigma(f) \geq \mu(f) \geq \frac{\log n - \log m}{m \log|q|}$.*

Remark 1.4 The results concerning the existence of meromorphic solutions of order zero in Theorem 1.1(i) and Corollary 1.1 are not only special cases of Theorem 1.2 and Corollary 1.2 respectively, but also more precise.

Remark 1.5 Clearly, equations (1.4)-(1.7) can have rational solutions.

2 Preliminary lemmas

Lemma 2.1 (See [22]) *Let f be a non-constant zero-order meromorphic function and $q \in \mathbb{C}\setminus\{0\}$. Then*

$$T\bigl(r,f(qz)\bigr) = \bigl(1+o(1)\bigr)T(r,f)$$

on a set of lower logarithmic density 1.

The following two lemmas can be seen as special cases of [23, Theorem 3]. In fact, the present versions we give here are more precise than the original one. To facilitate the readers, we give the corresponding proofs here.

Lemma 2.2 *Suppose that f is a transcendental meromorphic solution of the equation*

$$f(qz)f\bigl(q^2z\bigr)\cdots f\bigl(q^mz\bigr) = \frac{a_0(z)+a_1(z)f(z)+\cdots+a_p(z)f(z)^p}{b_0(z)+b_1(z)f(z)+\cdots+b_t(z)f(z)^t}, \tag{2.1}$$

where $q \in \mathbb{C}$, $|q| > 1$ and all meromorphic coefficients are of growth $S(r,f)$. If $d = \max\{p,t\} > m$, then for sufficiently large r,

$$T(r,f) \geq K\left(\frac{d}{m}\right)^{\frac{\log r}{m\log|q|}},$$

where K (>0) is a constant. Thus, the lower order of f satisfies $\mu(f) \geq \frac{\log d-\log m}{m\log|q|}$.

Proof We will use the observation (see [10, p.249]) as

$$T\bigl(r,f\bigl(q^jz\bigr)\bigr) = T\bigl(|q|^jr,f\bigr)+O(1). \tag{2.2}$$

Noting that $|q|>1$, by (2.1), (2.2) and Valiron-Mohon'ko theorem (see [2, Theorem 2.2.5 and Corollary 2.2.7]), we have that for any given ε $(0<\varepsilon<\frac{d-m}{d+m})$,

$$\begin{aligned} d(1-\varepsilon)T(r,f) &\leq dT(r,f)+S(r,f) \leq \sum_{j=1}^{m} T\bigl(r,f\bigl(q^jz\bigr)\bigr)+S(r,f) \\ &\leq \sum_{j=1}^{m} T\bigl(|q|^jr,f\bigr)+S(r,f) \leq m(1+\varepsilon)T\bigl(|q|^mr,f\bigr), \end{aligned} \tag{2.3}$$

outside of a possible exceptional set of finite logarithmic measure. We apply [2, Lemma 1.1.1] to deal with the exceptional set here. It follows by (2.3) that for any given $\alpha>1$, there exists an $r_0>0$ such that

$$d(1-\varepsilon)T(r,f) \leq m(1+\varepsilon)T\bigl(\alpha|q|^mr,f\bigr)$$

holds for all $r \geq r_0$. Hence,

$$T\big(\alpha|q|^m r,f\big) \geq \frac{d(1-\varepsilon)}{m(1+\varepsilon)} T(r,f), \quad r \geq r_0. \tag{2.4}$$

Inductively, for any $k \in \mathbb{N}$, we have by (2.4) that

$$T\big(\big(\alpha|q|^m\big)^k r,f\big) \geq \left(\frac{d(1-\varepsilon)}{m(1+\varepsilon)}\right)^k T(r,f), \quad r \geq r_0. \tag{2.5}$$

For sufficiently large s, there exists a $k \in \mathbb{N}$ such that

$$s \in \big[\big(\alpha|q|^m\big)^k r_0, \big(\alpha|q|^m\big)^{k+1} r_0\big), \quad \textit{i.e.}\ k > \frac{\log s - \log(\alpha|q|^m r_0)}{\log(\alpha|q|^m)}. \tag{2.6}$$

We have by (2.5)-(2.6) that

$$T(s,f) \geq T\big(\big(\alpha|q|^m\big)^k r_0,f\big) \geq \left(\frac{d(1-\varepsilon)}{m(1+\varepsilon)}\right)^{\frac{\log s - \log\log(\alpha|q|^m r_0)}{\log(\alpha|q|^m)}} T(r_0,f). \tag{2.7}$$

Letting $\varepsilon \to 0$ and $\alpha \to 1$, we have by (2.7) that

$$T(s,f) \geq \left(\frac{d}{m}\right)^{\frac{\log s - \log(|q|^m r_0)}{m \log|q|}} T(r_0,f) = K\left(\frac{d}{m}\right)^{\frac{\log s}{m\log|q|}},$$

where $K = (\frac{d}{m})^{\frac{-\log(|q|^m r_0)}{m\log|q|}} T(r_0,f)$ (> 0) is a constant. Thus, we get $\mu(f) \geq \frac{\log d - \log m}{m \log|q|}$. □

Lemma 2.3 *Suppose that f is a transcendental meromorphic solution of the equation*

$$\sum_{j=1}^{m} s_j(z) f\big(q^j z\big) = \frac{a_0(z) + a_1(z)f(z) + \cdots + a_p(z)f(z)^p}{b_0(z) + b_1(z)f(z) + \cdots + b_t(z)f(z)^t}, \tag{2.8}$$

where $q \in \mathbb{C}$, $|q| > 1$ and all meromorphic coefficients are of growth $S(r,f)$. If $d = \max\{p,t\} > m$, then for sufficiently large r,

$$T(r,f) \geq K\left(\frac{d}{m}\right)^{\frac{\log r}{m\log|q|}},$$

where K (> 0) is a constant. Thus, the lower order of f satisfies $\mu(f) \geq \frac{\log d - \log m}{m\log|q|}$.

Proof Note that (2.3) holds for both (2.1) and (2.8), then the proof of Lemma 2.3 is similar to that of Lemma 2.2. □

Lemma 2.4 (See [9]) *Let f be a non-constant zero-order meromorphic function and $q \in \mathbb{C}\backslash\{0\}$. Then*

$$m\left(r, \frac{f(qz)}{f(z)}\right) = o\big(T(r,f)\big)$$

on a set of logarithmic density 1.

Lemma 2.5 (See [14]) *Suppose that f is a transcendental meromorphic solution of an equation of the form* (1.1) *with* $|q| > 1$ *and meromorphic coefficients of growth* $S(r,f)$. *Then we have that*

$$\sigma(f) = \frac{\log d}{\log |q|},$$

where $d = \deg_f R$.

3 Proofs of Theorem 1.1 and Corollary 1.1

3.1 Proof of Theorem 1.1

(i) Suppose that $f(z)$ is a transcendental meromorphic solution of order zero of (1.4). It follows by (1.4) and Lemma 2.1 that

$$\begin{aligned} nT(r,f) &= T(r,f^n) = T\bigl(r,t(z) - s(z)f(qz)f(q^2z)\cdots f(q^mz)\bigr) \\ &\leq \sum_{j=1}^{m} T\bigl(r,f(q^jz)\bigr) + T(r,t) + T(r,s) \leq m\bigl(1+o(1)\bigr)T(r,f) + O(\log r) \\ &= m\bigl(1+o(1)\bigr)T(r,f) \end{aligned} \tag{3.1}$$

on a set of lower logarithmic density 1. It is clear that (3.1) is a contradiction since $n > m$. Thus, (1.4) has no transcendental meromorphic solutions of order zero if $n > m$.

Furthermore, when $|q| > 1$, we can transform (1.4) into

$$f(qz)f(q^2z)\cdots f(q^mz) = \frac{t(z) - f^n(z)}{s(z)}. \tag{3.2}$$

By (3.2) and Lemma 2.2, we have

$$\sigma(f) \geq \mu(f) \geq \frac{\log n - \log m}{m \log |q|}.$$

(ii) Suppose that $f(z)$ is a transcendental entire solution of order zero of (1.4). By (3.2) and Lemmas 2.1, 2.4, we have

$$\begin{aligned} mT(r,f) &= mm(r,f) = m\bigl(r,f^m\bigr) \\ &\leq m\left(r,\frac{f^m}{f(qz)f(q^2z)\cdots f(q^mz)}\right) + m\bigl(r,f(qz)f(q^2z)\cdots f(q^mz)\bigr) \\ &\leq \sum_{j=1}^{m} m\left(r,\frac{f(z)}{f(q^jz)}\right) + m\left(r,\frac{t-f^n}{s}\right) \\ &\leq \sum_{j=1}^{m} o\bigl(T\bigl(r,f(q^jz)\bigr)\bigr) + m\bigl(r,f^n\bigr) + O(\log r) \\ &= n\bigl(1+o(1)\bigr)T(r,f) \end{aligned} \tag{3.3}$$

on a set of logarithmic density 1. It is clear that (3.3) is a contradiction since $n < m$. Thus, (1.4) has no transcendental meromorphic solutions of order zero if $n < m$.

3.2 Proof of Corollary 1.1

Since $n \geq 2 > 1 = m$, we immediately know by Theorem 1.1(i) that (1.5) has no transcendental meromorphic solutions of order zero. And when $|q| > 1$, if there is a transcendental meromorphic solution f of positive order of (1.5), then we have

$$\sigma(f) \geq \mu(f) \geq \frac{\log n}{\log|q|}.$$

On the other hand, we have by (1.5) and Lemma 2.5 that $\sigma(f) = \frac{\log n}{\log|q|}$. Thus, f has a regular order

$$\sigma(f) = \mu(f) = \frac{\log n}{\log|q|}.$$

4 Proofs of Theorem 1.2 and Corollaries 1.2, 1.3

4.1 Proof of Theorem 1.2

Suppose that $f(z)$ is a transcendental entire solution of order zero of (1.6). It follows by (1.6) that

$$f^{n-m}(z) = \frac{t(z)}{f^m(z)} - \frac{U_q(z,f)}{f^m(z)}. \tag{4.1}$$

Noting that $U_q(z,f)$ is a homogeneous differential-q-difference polynomial in f of degree m, by the logarithmic derivative lemma, Lemma 2.4 and (4.1), we have

$$\begin{aligned}(n-m)T(r,f) &= m\bigl(r,f^{n-m}\bigr) \leq m\left(r,\frac{t}{f^m}\right) + m\left(r,\frac{U_q(z,f)}{f^m}\right)\\ &\leq mT(r,f) + o\bigl(T(r,f)\bigr)\end{aligned} \tag{4.2}$$

on a set of logarithmic density 1. It is clear that (4.2) is a contradiction since $n > 2m$. Thus, (1.6) has no transcendental entire solution of order zero if $n > 2m$.

4.2 Proof of Corollary 1.2

Since $U_q(z,f)$ is a linear differential-q-difference polynomial in f, the degree of $U_q(z,f)$ is $m = 1$. Thus, by $n \geq 3 > 2 = 2m$, we immediately know by Theorem 1.2 that (1.6) has no transcendental entire solutions of order zero.

4.3 Proof of Corollary 1.3

The first part result of Corollary 1.3 is a special case of Corollary 1.2. The second part result of Corollary 1.3 can be proved similarly to Theorem 1.1(i) by replacing Lemma 2.2 with Lemma 2.3.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors drafted the manuscript, read and approved the final manuscript.

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