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Some identities of Frobenius-Euler polynomials arising from umbral calculus

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Abstract
In this paper, we study some interesting identities of Frobenius-Euler polynomials arising from umbral calculus.

1 Introduction

Let $\mathbf{C}$ be the complex number field, and let $\mathbf{F}$ be the set of all formal power series in the variable t over $\mathbf{C}$ with

$$\mathbf{F} = \left\{ f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \,\middle|\, a_k \in \mathbf{C} \right\}.$$

We use notation $\mathbb{P} = \mathbf{C}[x]$ and $\mathbb{P}^*$ denotes the vector space of all linear functional on $\mathbb{P}$.

Also, $\langle L|p(x)\rangle$ denotes the action of the linear functional L on the polynomial $p(x)$, and we remind that the vector space operations on $\mathbb{P}^*$ is defined by

$$\begin{aligned} &\langle L + M|p(x)\rangle = \langle L|p(x)\rangle + \langle M|p(x)\rangle, \\ &\langle cL|p(x)\rangle = c\langle L|p(x)\rangle \quad \text{(see [1])}, \end{aligned}$$

where c is any constant in $\mathbf{C}$.

The formal power series

$$f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \in \mathbf{F} \quad \text{(see [1, 2])}, \tag{1}$$

defines a linear functional on $\mathbb{P}$ by setting

$$\langle f(t)|x^n\rangle = a_n, \quad \text{for all } n \geq 0. \tag{2}$$

In particular,

$$\langle t^k|x^n\rangle = n!\delta_{n,k}, \tag{3}$$

where $\delta_{n,k}$ is the Kronecker symbol. If $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^k\rangle}{k!} t^k$, then we get $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$ and so as linear functionals $L = f_L(t)$ (see [1, 2]).

In addition, the map $L \longmapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathbf{F}$ (see [1, 2]). Henceforth, $\mathbf{F}$ will denote both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathbf{F}$ will be thought of as both a formal power series and a linear functional. We shall call $\mathbf{F}$ the umbral algebra (see [1, 2]).

Let us give an example. For y in $\mathbf{C}$ the evaluation functional is defined to be the power series e^{yt}. From (2), we have $\langle e^{yt}|x^n\rangle = y^n$ and so $\langle e^{yt}|p(x)\rangle = p(y)$ (see [1, 2]). Notice that for all $f(t)$ in $\mathbf{F}$,

$$f(t) = \sum_{k=0}^{\infty} \frac{\langle f(t)|x^t\rangle}{k!} t^k \tag{4}$$

and for all polynomial $p(x)$

$$p(x) = \sum_{k\geq 0} \frac{\langle t^k|p(x)\rangle}{k!} x^k \quad \text{(see [1, 2])}. \tag{5}$$

For $f_1(t), f_2(t), \dots, f_m(t) \in \mathbf{F}$, we have

$$\begin{aligned} &\bigl\langle f_1(t) f_2(t) \cdots f_m(t) | x^n \bigr\rangle \\ &\quad = \sum \binom{n}{i_1, \dots, i_m} \bigl\langle f_1(t)|x^{i_1}\bigr\rangle \cdots \bigl\langle f_n(t)|x^{i_m}\bigr\rangle, \end{aligned}$$

where the sum is over all nonnegative integers $i_1, i_2, \dots, i_m$ such that $i_1 + \cdots + i_m = n$ (see [1, 2]). The order $o(f(t))$ of the power series $f(t) \neq 0$ is the smallest integer k for which a_k does not vanish. We define $o(f(t)) = \infty$ if $f(t) = 0$. We see that $o(f(t)g(t)) = o(f(t)) + o(g(t))$ and $o(f(t) + g(t)) \geq \min\{o(f(t)), o(g(t))\}$. The series $f(t)$ has a multiplicative inverse, denoted by $f(t)^{-1}$ or $\frac{1}{f(t)}$, if and only if $o(f(t)) = 0$. Such series is called an invertible series. A series $f(t)$ for which $o(f(t)) = 1$ is called a delta series (see [1, 2]). For $f(t), g(t) \in \mathbf{F}$, we have $\langle f(t)g(t)|p(x)\rangle = \langle f(t)|g(t)p(x)\rangle$.

A delta series $f(t)$ has a compositional inverse $\bar{f}(t)$ such that $f(\bar{f}(t)) = \bar{f}(f(t)) = t$.

For $f(t), g(t) \in \mathbf{F}$, we have $\langle f(t)g(t)|p(x)\rangle = \langle f(t)|g(t)p(x)\rangle$.

From (5), we have

$$p^{(k)}(x) = \frac{d^k p(x)}{dx^k} = \sum_{l=k}^{\infty} \frac{\langle t^l|p(x)\rangle}{l!} l(l-1)\cdots(l-k+1)x^{l-k}.$$

Thus, we see that

$$p^{(k)}(0) = \bigl\langle t^k|p(x)\bigr\rangle = \bigl\langle 1|p^{(k)}(x)\bigr\rangle. \tag{6}$$

By (6), we get

$$t^k p(x) = p^{(k)}(x) = \frac{d^k(p(x))}{dx^k} \quad \text{(see [1, 2])}. \tag{7}$$

By (7), we have

$$e^{yt} p(x) = p(x+y) \quad \text{(see [1, 2])}. \tag{8}$$

Let $S_n(x)$ be a polynomial with $\deg S_n(x) = n$.

Let $f(t)$ be a delta series, and let $g(t)$ be an invertible series. Then there exists a unique sequence $S_n(x)$ of polynomials such that $\langle g(t)f(t)^k | S_n(x)\rangle = n!\delta_{n,k}$ for all $n, k \geq 0$. The sequence $S_n(x)$ is called the Sheffer sequence for $(g(t), f(t))$ or that $S_n(t)$ is Sheffer for $(g(t), f(t))$.

The Sheffer sequence for $(1, f(t))$ is called the associated sequence for $f(t)$ or $S_n(x)$ is associated to $f(t)$. The Sheffer sequence for $(g(t), t)$ is called the Appell sequence for $g(t)$ or $S_n(x)$ is Appell for $g(t)$ (see [1, 2]). The umbral calculus is the study of umbral algebra and the modern classical umbral calculus can be described as a systemic study of the class of Sheffer sequences. Let $p(x) \in \mathbb{P}$. Then we have

$$\left\langle \frac{e^{yt}-1}{t} \middle| p(x) \right\rangle = \int_0^y p(u)\,du, \tag{9}$$

$$\bigl\langle f(t) | xp(x) \bigr\rangle = \bigl\langle \partial_t f(t) | p(x) \bigr\rangle = \bigl\langle f'(t) | p(x) \bigr\rangle, \tag{10}$$

and

$$\bigl\langle e^{yt} - 1 | p(x) \bigr\rangle = p(y) - p(0) \quad \text{(see [1, 2])}. \tag{11}$$

Let $S_n(x)$ be Sheffer for $(g(t), f(t))$. Then

$$h(t) = \sum_{k=0}^{\infty} \frac{\langle h(t) | S_k(x)\rangle}{k!} g(t)f(t)^k, \quad h(t) \in \mathbf{F}, \tag{12}$$

$$p(x) = \sum_{k\geq 0} \frac{\langle g(t)f(t)^k | p(x)\rangle}{k!} S_k(x), \quad p(x) \in \mathbb{P}, \tag{13}$$

$$\frac{1}{g(\bar{f}(t))} e^{y\bar{f}(t)} = \sum_{k=0}^{\infty} \frac{S_k(y)}{k!} t^k, \quad \text{for all } y \in \mathbf{C}, \tag{14}$$

$$f(t)S_n(x) = nS_{n-1}(x). \tag{15}$$

For λ $(\neq 1) \in \mathbf{C}$, we recall that the Frobenius-Euler polynomials are defined by the generating function to be

$$\frac{1-\lambda}{e^t - \lambda} e^{xt} = e^{H(x|\lambda)t} = \sum_{n=0}^{\infty} H_n(x|\lambda) \frac{t^n}{n!}, \tag{16}$$

with the usual convention about replacing $H^n(x|\lambda)$ by $H_n(x|\lambda)$ (see [3]). In the special case, $x = 0$, $H_n(0|\lambda) = H_n(\lambda)$ are called the nth Frobenius-Euler numbers. By (16), we get

$$H_n(x|\lambda) = \bigl(H(\lambda) + x\bigr)^n = \sum_{l=0}^{n} \binom{n}{l} H_{n-l}^{(\lambda)} x^l, \tag{17}$$

and

$$\bigl(H(\lambda) + 1\bigr)^n - \lambda H_n(\lambda) = (1-\lambda)\delta_{0,n} \quad \text{(see [1, 4–13])}. \tag{18}$$

From (17), we note that the leading coefficient of $H_n(x|\lambda)$ is $H_0(\lambda) = 1$. So, $H_n(x|\lambda)$ is a monic polynomial of degree n with coefficients in $\mathbf{Q}(\lambda)$.

In this paper, we derive some new identities of Frobenius-Euler polynomials arising from umbral calculus.

2 Applications of umbral calculus to Frobenius-Euler polynomials

Let $S_n(x)$ be an Appell sequence for $g(t)$. From (14), we have

$$\frac{1}{g(t)}x^n = S_n(x) \quad \text{if and only if} \quad x^n = g(t)S_n(x) \quad (n \geq 0). \tag{19}$$

For $\lambda\ (\neq 1) \in \mathbf{C}$, let us take $g_\lambda(t) = \frac{e^t-\lambda}{1-\lambda} \in \mathbf{F}$.

Then we see that $g_\lambda(t)$ is an invertible series.

From (16), we have

$$\sum_{k=0}^{\infty} \frac{H_k(x|\lambda)}{k!}t^k = \frac{1}{g_\lambda(t)}e^{xt}. \tag{20}$$

By (20), we get

$$\frac{1}{g_\lambda(t)}x^n = H_n(x|\lambda) \quad \bigl(\lambda\ (\neq 1) \in \mathbf{C}, n \geq 0\bigr), \tag{21}$$

and by (17), we get

$$tH_n(x|\lambda) = H_n'(x|\lambda) = nH_{n-1}(x|\lambda). \tag{22}$$

Therefore, by (21) and (22), we obtain the following proposition.

Proposition 1 *For $\lambda\ (\neq 1) \in \mathbf{C}$, $n \geq 0$, we see that $H_n(x|\lambda)$ is the Appell sequence for $g_\lambda(t) = \frac{e^t-\lambda}{1-\lambda}$.*

From (20), we have

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H_k(x|\lambda)}{k!}kt^{k-1} &= \frac{xg_\lambda(t)e^{xt} - g_\lambda'(t)e^{xt}}{g_\lambda(t)^2} \\ &= \sum_{k=0}^{\infty}\left\{x\frac{1}{g_\lambda(t)}x^k - \frac{g_\lambda'(t)}{g_\lambda(t)}\frac{1}{g_\lambda(t)}x^k\right\}\frac{t^k}{k!}.\end{aligned} \tag{23}$$

By (21) and (23), we get

$$H_{k+1}(x|\lambda) = xH_k(x|\lambda) - \frac{g_\lambda'(t)}{g_\lambda(t)}H_k(x|\lambda). \tag{24}$$

Therefore, by (24) we obtain the following theorem.

Theorem 2 *Let $g_\lambda(t) = \frac{e^t-\lambda}{1-\lambda} \in \mathbf{F}$. Then we have*

$$H_{k+1}(x|\lambda) = \left(x - \frac{g_\lambda'(t)}{g_\lambda(t)}\right)H_k(x|\lambda) \quad (k \geq 0).$$

From (16), we have

$$\sum_{n=0}^{\infty}\bigl(H_n(x+1|\lambda)-\lambda H_n(x|\lambda)\bigr)\frac{t^n}{n!}=\frac{1-\lambda}{e^t-\lambda}e^{(x+1)t}-\lambda\frac{1-\lambda}{e^t-\lambda}e^{xt}=(1-\lambda)e^{xt}. \tag{25}$$

By (25), we get

$$H_n(x+1|\lambda)-\lambda H_n(x|\lambda)=(1-\lambda)x^n. \tag{26}$$

From Theorem 2, we can derive the following equation (27):

$$g_\lambda(t)H_{k+1}(x|\lambda)=\bigl(g_\lambda(t)x-g'_\lambda(t)\bigr)H_k(x|\lambda). \tag{27}$$

By (27), we get

$$\left(\frac{e^t-\lambda}{1-\lambda}\right)H_{k+1}(x|\lambda)=\frac{e^t-\lambda}{1-\lambda}xH_k(x|\lambda)-\frac{e^t}{1-\lambda}H_k(x|\lambda). \tag{28}$$

From (8) and (28), we have

$$\begin{aligned}H_{k+1}(x+1|\lambda)-\lambda H_{k+1}(x|\lambda)&=(x+1)H_k(x+1|\lambda)-\lambda xH_k(x|\lambda)-H_k(x+1|\lambda)\\&=xH_k(x+1|\lambda)-\lambda xH_k(x|\lambda).\end{aligned}$$

Therefore, by (26), we obtain the following theorem.

Theorem 3 *For $k\geq 0$, we have*

$$H_{k+1}(x+1|\lambda)=\lambda H_{k+1}(x|\lambda)+(1-\lambda)x^{k+1}.$$

From (16), (17), and (18), we note that

$$\begin{aligned}\int_x^{x+y}H_n(u|\lambda)\,du&=\frac{1}{n+1}\bigl\{H_{n+1}(x+y|\lambda)-H_{n+1}(x|\lambda)\bigr\}\\&=\frac{1}{n+1}\sum_{k=1}^{\infty}\binom{n+1}{k}H_{n+1-k}(x|\lambda)y^k\\&=\sum_{k=1}^{\infty}\frac{n(n-1)\cdots(n-k+2)}{k!}H_{n+1-k}(x|\lambda)y^k\\&=\sum_{k=1}^{\infty}\frac{y^k}{k!}t^{k-1}H_n(x|\lambda)\\&=\frac{1}{t}\left(\sum_{k=0}^{\infty}\frac{y^k}{k!}t^k-1\right)H_n(x|\lambda)\\&=\frac{e^{yt}-1}{t}H_n(x|\lambda).\end{aligned} \tag{29}$$

Therefore, by (29), we obtain the following theorem.

Theorem 4 *For* $\lambda\ (\neq 1) \in \mathbf{C}$, $n \geq 0$, *we have*

$$\int_x^{x+y} H_n(u|\lambda)\,du = \frac{e^{yt}-1}{t} H_n(x|\lambda).$$

By (15) and Proposition 1, we get

$$t\left\{\frac{1}{n+1} H_{n+1}(x|\lambda)\right\} = H_n(x|\lambda). \tag{30}$$

From (30), we can derive equation (31):

$$\begin{aligned}\left\langle e^{yt}-1 \middle| \frac{H_{n+1}(x|\lambda)}{n+1}\right\rangle &= \left\langle \frac{e^{yt}-1}{t} \middle| t\left\{\frac{H_{n+1}(x|\lambda)}{n+1}\right\}\right\rangle \\ &= \left\langle \frac{e^{yt}-1}{t} \middle| H_n(x|\lambda)\right\rangle. \end{aligned}\tag{31}$$

By (11) and (31), we get

$$\begin{aligned}\left\langle \frac{e^{yt}-1}{t} \middle| H_n(x|\lambda)\right\rangle &= \left\langle e^{yt}-1 \middle| \frac{H_{n+1}(x|\lambda)}{n+1}\right\rangle \\ &= \frac{1}{n+1}\left\{H_{n+1}(y|\lambda) - H_{n+1}(\lambda)\right\} = \int_0^y H_n(u|\lambda)\,du. \end{aligned}\tag{32}$$

Therefore, by (32), we obtain the following corollary.

Corollary 5 *For* $n \geq 0$, *we have*

$$\left\langle \frac{e^{yt}-1}{t} \middle| H_n(x|\lambda)\right\rangle = \int_0^y H_n(u|\lambda)\,du.$$

Let $\mathbb{P}(\lambda) = \{p(x) \in \mathbf{Q}(\lambda)[x] \,|\, \deg p(x) \leq n\}$ be a vector space over $\mathbf{Q}(\lambda)$.
For $p(x) \in \mathbb{P}_n(\lambda)$, let us take

$$p(x) = \sum_{k=0}^{n} b_k H_k(x|\lambda). \tag{33}$$

By Proposition 1, $H_n(x|\lambda)$ is an Appell sequence for $g_\lambda(t) = \frac{e^t-\lambda}{1-\lambda}$ where $\lambda\ (\neq 1) \in \mathbf{C}$. Thus, we have

$$\left\langle \frac{e^t-\lambda}{1-\lambda} t^k \middle| H_n(x|\lambda)\right\rangle = n!\delta_{n,k}. \tag{34}$$

From (33) and (34), we can derive

$$\begin{aligned}\left\langle \frac{e^t-\lambda}{1-\lambda} t^k \middle| p(x)\right\rangle &= \sum_{l=0}^{n} b_l \left\langle \frac{e^t-\lambda}{1-\lambda} t^k \middle| H_l(x|\lambda)\right\rangle \\ &= \sum_{l=0}^{n} b_l l! \delta_{l,k} = k! b_k. \end{aligned}\tag{35}$$

Thus, by (35), we get

$$\begin{aligned} b_k &= \frac{1}{k!}\left\langle \frac{e^t-\lambda}{1-\lambda}t^k \middle| p(x)\right\rangle \\ &= \frac{1}{k!(1-\lambda)}\left\langle (e^t-\lambda)t^k | p(x)\right\rangle \\ &= \frac{1}{k!(1-\lambda)}\left\langle e^t-\lambda | p^{(k)}(x)\right\rangle. \end{aligned} \tag{36}$$

From (11) and (36), we have

$$b_k = \frac{1}{k!(1-\lambda)}\left\{p^{(k)}(1)-\lambda p^{(k)}(0)\right\}, \tag{37}$$

where $p^{(k)}(x) = \frac{d^k p(x)}{dx^k}$.

Therefore, by (37), we obtain the following theorem.

Theorem 6 *For $p(x) \in \mathbb{P}_n(\lambda)$, let us assume that $p(x) = \sum_{k=0}^n b_k H_k(x|\lambda)$. Then we have*

$$b_k = \frac{1}{k!(1-\lambda)}\left\{p^{(k)}(1)-\lambda p^{(k)}(0)\right\},$$

where $p^{(k)}(1) = \frac{d^k p(x)}{dx^k}|_{x=1}$.

The higher-order Frobenius-Euler polynomials are defined by

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)^r e^{xt} = \sum_{n=0}^{\infty} H_n^{(r)}(x|\lambda)\frac{t^n}{n!}, \tag{38}$$

where $\lambda\ (\neq 1) \in \mathbf{C}$ and $r \in \mathbf{N}$ (see [4, 11]).

In the special case, $x = 0$, $H_n^{(r)}(0|\lambda) = H_n^{(r)}(\lambda)$ are called the nth Frobenius-Euler numbers of order r. From (38), we have

$$\begin{aligned} H_n^{(r)}(x) &= \sum_{l=0}^{n}\binom{n}{l}H_{n-l}^{(r)}(\lambda)x^l \\ &= \sum_{n_1+\cdots+n_r=n}\binom{n}{n_1,\ldots,n_r}H_{n_1}(x|\lambda)\cdots H_{n_r}(x|\lambda). \end{aligned} \tag{39}$$

Note that $H_n^{(r)}(x|\lambda)$ is a monic polynomial of degree n with coefficients in $\mathbf{Q}(\lambda)$.

For $r \in \mathbf{N}$, $\lambda\ (\neq 1) \in \mathbf{C}$, let $g_\lambda^r(t) = (\frac{e^t-\lambda}{1-\lambda})^r$. Then we easily see that $g_\lambda^r(t)$ is an invertible series.

From (38) and (39), we have

$$\frac{1}{g_\lambda^r(t)}e^{xt} = \sum_{n=0}^{\infty} H_n^{(r)}(x|\lambda)\frac{t^n}{n!}, \tag{40}$$

and

$$tH_n^{(r)}(x|\lambda) = nH_{n-1}^{(r)}(x|\lambda). \tag{41}$$

By (40), we get

$$\frac{1}{g_\lambda^r(t)} x^n = H_n^{(r)}(x|\lambda) \quad (n \in \mathbf{Z}_+, r \in \mathbf{N}). \tag{42}$$

Therefore, by (41) and (42), we obtain the following proposition.

Proposition 7 *For $n \in \mathbf{Z}_+$, $H_n^{(r)}(x|\lambda)$ is an Appell sequence for*

$$g_\lambda^r(t) = \left(\frac{e^t - \lambda}{1 - \lambda}\right)^r.$$

Moreover,

$$\frac{1}{g_\lambda^r(t)} x^n = H_n^{(r)}(x|\lambda) \quad \textit{and} \quad tH_n^{(r)}(x|\lambda) = nH_{n-1}^{(r)}(x|\lambda).$$

Remark Note that

$$\left\langle \frac{1-\lambda}{e^t - \lambda} \middle| x^n \right\rangle = H_n(\lambda). \tag{43}$$

From (43), we have

$$\left\langle \left(\frac{1-\lambda}{e^t - \lambda}\right)^r \middle| x^n \right\rangle = \sum_{n=n_1+\cdots+n_r} \binom{n}{n_1, \ldots, n_r} \left\langle \frac{1-\lambda}{e^t - \lambda} \middle| x^{n_1} \right\rangle \cdots \left\langle \frac{1-\lambda}{e^t - \lambda} \middle| x^{n_r} \right\rangle, \tag{44}$$

$$\left\langle \left(\frac{1-\lambda}{e^t - \lambda}\right)^r \middle| x^n \right\rangle = H_n^{(r)}(\lambda). \tag{45}$$

By (43), (44), and (45), we get

$$\sum_{n=i_1+\cdots+i_r} \binom{n}{i_1, \ldots, i_r} H_{i_1}(\lambda) \cdots H_{i_r}(\lambda) = H_n^{(r)}(\lambda).$$

Let us take $p(x) \in \mathbb{P}_n(\lambda)$ with

$$p(x) = \sum_{k=0}^{n} C_k^{(r)} H_k^{(r)}(x|\lambda). \tag{46}$$

From the definition of Appell sequences, we have

$$\left\langle \left(\frac{e^t - \lambda}{1 - \lambda}\right)^r \middle| H_n^{(r)}(x|\lambda) \right\rangle = n!\delta_{n,k}. \tag{47}$$

By (46) and (47), we get

$$\begin{aligned} \left\langle \left(\frac{e^t - \lambda}{1 - \lambda}\right)^r t^k \middle| p(x) \right\rangle &= \sum_{l=0}^{n} C_l^{(r)} \left\langle \left(\frac{e^t - \lambda}{1 - \lambda}\right)^r t^k \middle| H_l(x|\lambda) \right\rangle \\ &= \sum_{l=0}^{n} C_l^{(r)} l!\delta_{l,k} = k!C_k^{(r)}. \end{aligned} \tag{48}$$

Thus, from (48), we have

$$\begin{aligned}
C_k^{(r)} &= \frac{1}{k!}\left\langle \left(\frac{e^t-\lambda}{1-\lambda}\right)^r t^k \middle| p(x) \right\rangle \\
&= \frac{1}{k!(1-\lambda)^r}\left\langle \left(e^t-\lambda\right)^r t^k | p(x) \right\rangle \\
&= \frac{1}{k!(1-\lambda)^r}\sum_{l=0}^{r}\binom{r}{l}(-\lambda)^{r-l}\left\langle e^{lt} | p^{(k)}(x)\right\rangle \\
&= \frac{1}{k!(1-\lambda)^r}\sum_{l=0}^{r}\binom{r}{l}(-\lambda)^{r-l}p^{(k)}(l).
\end{aligned} \tag{49}$$

Therefore, by (46) and (49), we obtain the following theorem.

Theorem 8 *For $p(x) \in \mathbb{P}_n(\lambda)$, let*

$$p(x) = \sum_{k=0}^{n} C_k^{(r)} H_k^{(r)}(x|\lambda).$$

Then we have

$$C_k^{(r)} = \frac{1}{k!(1-\lambda)^r}\sum_{l=0}^{r}\binom{r}{l}(-\lambda)^{r-l}p^{(k)}(l),$$

where $r \in \mathbf{N}$ and $p^{(k)}(l) = \frac{d^k p(x)}{dx^k}|_{x=l}$.

Remark Let $S_n(x)$ be a Sheffer sequence for $(g(t), f(t))$. Then Sheffer identity is given by

$$S_n(x+y) = \sum_{k=0}^{n}\binom{n}{k}P_k(y)S_{n-k}(x) = \sum_{k=0}^{n}\binom{n}{k}P_k(x)S_{n-k}(y), \tag{50}$$

where $P_k(y) = g(t)S_k(y)$ is associated to $f(t)$ (see [1, 2]).

From (21), Proposition 1, and (50), we have

$$\begin{aligned}
H_n(x+y|\lambda) &= \sum_{k=0}^{n}\binom{n}{k}P_k(y)S_{n-k}(x) \\
&= \sum_{k=0}^{n}\binom{n}{k}H_{n-k}(y|\lambda)x^k.
\end{aligned}$$

By Proposition 7 and (50), we get

$$H_n^{(r)}(x+y|\lambda) = \sum_{k=0}^{n}\binom{n}{k}H_{n-k}^{(r)}(y|\lambda)x^k.$$

Let α $(\neq 0) \in \mathbf{C}$. Then we have

$$H_n(\alpha x|\lambda) = \alpha^n \frac{g_\lambda(t)}{g_\lambda(\frac{t}{\alpha})}H_n(x|\lambda).$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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