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Infinitely many sign-changing solutions for a Schrödinger equation

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Abstract

We study a superlinear Schrödinger equation in the whole Euclidean space $\mathbb{R}^n$. By using a suitable sign-changing critical point, we prove that the problem admits infinitely many sign-changing solutions, under weaker conditions.

Keywords: Schrödinger equation, sign-changing critical point, (w*-PS) condition

1 Introduction

In this paper, we consider the following Schrödinger equation,

$$\begin{cases} -\Delta u + V(x)u = f(x,u), x \in \mathbb{R}^N \\ u(x) \to 0, |x| \to \infty. \end{cases} \tag{1.1}$$

In order to overcome the lack of compactness of the problem, we assume that the potential $V(x)$ has a "good" behavior at infinity, in such a way the Schrödinger operator -Δ + $V(x)$ on $L^2(\mathbb{R}^N)$ has a discrete spectrum. More precisely, we suppose $(V_1) V \in L^2_{\text{loc}}(\mathbb{R}^N)$, V is bounded from below;

(V_2) There exists $r_0 > 0$ such that for any $h > 0$

$$\text{meas}(B_{r_0}(y) \cap V_h) \to 0, \quad |y| \to +\infty,$$

where meas(A) denotes the Lebesgue measure of A on $\mathbb{R}^N$, $B_{r_0}(y)$ is the ball centered at y with radius r_0 and $V^h = \{x \in \mathbb{R}^N : V(x) < h\}$.

Of course, $V(x)$ above can satisfy the condition (S_1) or $((\bar{S}_1), (\tilde{S}_1))$ in [1], so that the Schrödinger operator could have the same good properties.

We denote $\{\lambda_j\}$ to be the eigenvalues sequence of - $\Delta + V(x)$ (see Proposition 2.1 in Section 2). Set $F(x,t) = \int_0^t f(x,s)ds, \mathcal{F}(x,t) = f(x,t)t - 2F(x,t)$.

We assume the following conditions.

(f_1) $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with a subcritical growth,

$$|f(x,t)| \le c(1 + |t|^{s-1}), t \in \mathbb{R}, x \in \mathbb{R}^N,$$

where $s \in (2, 2^*)$, $f(x, t) \ge 0$ for all $(x, t) \in \mathbb{R}^N \times \mathbb{R}$ and $f(x, t) = o(|t|)$ as $|t| \to 0$.

(f_2) $\lim_{|t| \to +\infty} \frac{f(x,t)t}{|t|^2} = +\infty$ uniformly for $x \in \mathbb{R}^N$.

(f_3) There exist $\theta \geq 1$, $s \in [0, 1]$ s.t.

$$\theta \mathcal{F}(x,t) \geq \mathcal{F}(x,st), (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{1.2}$$

(f_4) $f(x, t)$ is odd in t.

Let us point out that, under our assumptions on $f(x, t)$, we can assume without loss of generality that V is strictly positive just replacing $V(x)$ with $V(x) + L$ and $f(x, u)$ with $f(x, u) + Lu$, L large enough. We shall prove the following result.

Theorem 1.1 Under assumptions (V_1), (V_2), (f_1) - (f_4), problem (1.1) has infinitely many sign-changing solutions.

Remark 1.1 In [2,3], they got sign-changing solutions for elliptic problem with Dirichlet boundary value. Those abstract results involved a Banach space of continuous functions in the Hilbert space, where the cone has a nonempty interior. This plays a crucial role. While the abstract theory in this paper only involved a Hilbert space, where the cone has an empty interior.

Remark 1.2 In [4], they showed infinitely many solutions for p-Laplace equation with Dirichlet boundary value, while we get infinitely many sign-changing solutions under similar conditions.

Remark 1.3 Equation 1.1 has been studied in [5], where they obtained the existence for sign-changing solutions in a asymptotically case.

Remark 1.4 In [1, §5.3], they also obtained infinitely many sign-changing solutions for elliptic problem with Dirichlet boundary value, under (AR) condition stronger than (f_2) and (f_3) above.

Remark 1.5 In [1, §6.4], Equation 1.1 has been studied the existence for infinitely many sign-changing solutions under conditions stronger than ours above.

2 Preliminaries

We consider the Hilbert space

$$E = \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)dx < \infty\}$$

endowed with the inner product $(u, v) = \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv)dx$ for $u, v \in E$ and norm $||u|| = (u, u)^{\frac{1}{2}}$. Clearly it is $E \lesssim H^1(\mathbb{R}^N)$. Denote $|u|_q$ to be the norm of u in $L^q(\mathbb{R}^N)$. In order to overcome the lack of compactness of the problem, the following proposition is crucial.

Proposition 2.1 [1,5] Assume $V(x)$ satisfies condition (V_1) and (V_2), or (S_1) or $(\bar{S}_1)$ and $(\tilde{S}_1)$ in [1]. Then the imbedding $E \lesssim L^q(\mathbb{R}^N)$ is continuous if $q \in [2, 2^*]$ and compact if $q \in [2, 2^*[$. Hence, the eigenvalue problem

$$-\Delta u + V(x)u = \lambda u, \quad x \in \mathbb{R}^N$$

possesses a sequence of positive eigenvalue

$$0 < \lambda_1 < \lambda_3 < \cdots < \lambda_k < \cdots \to \infty$$

with finite multiplicity for each λ_k. Moreover, the principle eigenvalue λ_1 is simple with a positive eigenfunction ϕ_1, and the eigenfunctions ϕ_k corresponding to λ_k, $k \geq 2$ are sign changing.

Let us consider the functional $J : E \to \mathbb{R}$

$$J(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x, u)dx. \tag{2.1}$$

Then $J \in C^1(E, \mathbb{R})$ and $J' = id\ (-\Delta + V)^{-1} f = id - K_J$. The critical point of J is just the weak solution of problem (1.1).

The proof if our main results will be obtained by a suitable applications of an abstract critical point theorem stated in [1]. For completeness, we recall here this theorem.

Let E be Hilbert space with norm $||u||$, and Y, M be two subspaces of E with dim $Y < \infty$, dim Y - co dim $M \geq 1$. Let G be C^1 - functional on E with $G'(u) = u - K_G(u)$ and P denote a closed convex positive cone of E. Denote $\pm D_0$ by open convex subsets of E, containing the positive cone P in its interior and $K = \{u \in E : G'(u) = 0\}$, $K[a, b] = \{u \in K : G(u) \in [a, b]\}$. Set $D = D_0 \cup (-D_0)$, $S = E \setminus D$. In applications, D contains all positive and negative critical points, and S includes all possible sign-changing critical points. Hence, nontrivial sign-changing solutions can be obtained by different choose of $\pm D_0$ and S.

Next, we assume that there is another norm $|| \cdot ||_*$ of E such that $||u||_* \leq c_*||u||$ for all $u \in E$, where $c_* > 0$ is a constant. Moreover, we assume that $||u_n - u||_* \to 0$ whenever $u_n \rightharpoonup u$ weakly in $(E, || \cdot ||)$. Write $E = M_1 \oplus M$.

Let

$$Q^*(\rho) = \{u \in M : \frac{\|u\|_*^p}{\|u\|^2} + \frac{\|u\| \ \|u\|_*}{\|u\| + D_*\|u\|_*} = \rho\}$$

where $\rho > 0$, $D_* > 0$, $p > 2$ are fixed constants. Let $Q^{**} = Q^*(\rho) \cap G^\beta \subset S$ and $\gamma = \inf_{Q^{**}} G$, where $G^\beta = \{u \in E : G(u) \leq \beta\}$, then $\beta \geq \gamma$.

Let us assume that

(A) $K_G(\pm D_0) \subset \pm D_0$;

(A_1^*) Assume that for any a, $b > 0$, there is a $c_2 = c_2(a, b) > 0$ such that $G(u) \leq a$ and $||u||_* \leq b \Rightarrow ||u|| \leq c_2$;

(A_2^*) $\lim_{u \in Y, \|u\| \to \infty} = -\infty, \quad \sup_Y G = \beta.$

In the sequel, we shall consider the following Palais-Smale condition, shortly (w^* - PS) condition.

Definition 2.1 The functional G is said to satisfy the (w^* - PS) condition if any sequence $\{u_n\}$ such that $\{G(u_n)\}$ is bounded and $G'(u_n) \to 0$, we have either $\{u_n\}$ is bounded and has a convergent subsequence or $\exists \sigma, R, \beta > 0$ s.t. for any $u \in J^{-1}([c - \sigma, c + \sigma])$ with $||u|| \geq R$, $||J'(u)||\ ||u|| \geq \beta$. If in particular, $\{G(u_n)\} \to c$, we say that (w^*-PS)$_c$ is satisfied.

The following results hold (see [1, Theorem 5.6]).

Theorem 2.1 Assume (A) and (A_1^*) and (A_2^*). If the even functional G satisfies the (w^* - PS)$_c$ condition at lever c for each $c \in [r, \beta]$, then

$$K[r - \varepsilon, \beta + \varepsilon] \cap (E \backslash P \cup (-P)) \neq \emptyset$$

for all $\varepsilon > 0$ small.

3 Proof of the main theorems

From now on, we will denote by N_k the eigenspace of λ_k. Then dim N_k <∞. We fix k and let $E_k = N_1 \oplus ... \oplus N_k$. In order to give the proof of Theorem 1.1, first we state some useful lemmas.

Lemma 3.1 $J(u) \to -\infty$, as $||u|| \to \infty$, for all $u \in E_k$.

Proof. Because dim E_k <∞, all norms in it are equivalent, then by (f_2),

$$\frac{J(u)}{||u||^2} \le \frac{1}{2} - \int_{\mathbb{R}^N} \frac{F(x,u)}{||u||^2} dx \to -\infty.$$

Consider another norm $||\cdot||_* := ||\cdot||_s$ of E, $s \in (2, 2^*)$. Then $||u||_s \le C_*||u||$ for all $u \in E$, here C_* >0 is a constant and by lemma 2.1 $||u_n - u||_* \to 0$ whenever $u_n \rightharpoonup u$ weakly in E. Write $E = E_{k-1} \oplus E_{k-1}^{\perp}$. Let

$$Q^*(\rho) = \{u \in E_{k-1}^{\perp} : \frac{||u||_s^s}{||u||^2} + \frac{||u||||u||_s}{||u|| + D_*||u||_s} = \rho\}$$

where ρ, D_* are fixed constants.

Lemma 3.2 $||u||_s \le c_1$, $\forall u \in Q^*(\rho)$, where c_1 >0 is a constant.

Proof. If $||u||_s \to \infty$, then so does $||u|| \to \infty$. Hence

$$\frac{||u||\ ||u||_s}{||u|| + D_*||u||_s} \to \infty,$$

a contradiction.

By (f_1), there exist C_F >0, $s \in (2, 2^*)$ such that

$$|F(x,u)| \le \frac{\lambda_1}{4}u^2 + C_F|u|^s, \quad x \in \mathbb{R}^N, u \in \mathbb{R}. \tag{3.1}$$

Therefore, for any a, b >0, there is a $c_2 = c_2(a, b)$ >0 such that

$$J(u) \le a, ||u||_s \le b \Rightarrow ||u|| \le c_2.$$

By lemma 3.1,

$$\lim_{u \in Y, ||u|| \to \infty} J(u) = -\infty,$$

where $Y = E_k$. Then, conditions (A_1^*) and (A_2^*) are satisfied. We define

$$\sup_Y G := \beta.$$

Let

$$Q^{**} := Q^*(\rho) \cap J^{\beta} \subset S, \quad \inf_{Q^{**}} J := \gamma.$$

Set $P = \{u \in E : u(x) \ge 0$ for a.e. $x \in \mathbb{R}^N\}$. Then, $P(-P)$ is the positive (negative) cone of E and weakly closed. By Lemma 5.4 or Lemma 6.8 [1], there is a $\delta := \delta(\beta)$ such that $dist(Q^{**}, P) = \delta(\beta)$ >0. We define

$$D(\mu_0) := \{u \in E : dist(u, P) < \mu_0\},$$

where μ_0 us determined by the following lemma.

Lemma 3.3 Under the assumptions (V_1), (V_2), and (f_1), there is a $\mu_0 \in (0, \delta)$ (may be chosen small enough) such that $K_J(\pm D(\mu_0)) \subset \pm D(\mu_0)$. Therefore, ($A$) is satisfied.

Proof. Please see Lemma 2.9 of [1] for the similar proof.

Let $D := -D(\mu_0) \cup D(\mu_0)$, $S := E \setminus D$. By Lemma 3.3, we may assume $Q^{**} \subset S$.

Lemma 3.4 Let us assume that (V_1), (V_2) and (f_2), (f_3) hold. Then, the functional J satisfies the (w*-PS) condition.

Proof. As the sequence $\{u_n\}$ such that $\{G(u_n)\}$ is bounded and $G'(u_n) \to 0$, if $\{u_n\}$ is bounded, then by Proposition 2.1 and the compact imbedding $E \lesssim L^q(\mathbb{R}^N)$, $q \in [2, 2^*[$, we have $\{u_n\}$ possesses a convergent subsequence.

Next to prove another case. If not, there exist $c \in \mathbb{R}$ and $\{u_n\} \subset E$ satisfying, as $n \to \infty$

$$J(u_n) \to c, ||u_n|| \to \infty, ||J'(u_n)||\, ||u_n|| \to 0 \tag{3.2}$$

then we have

$$\begin{aligned} &\lim_{n\to\infty} \int_{\mathbb{R}^N} (\frac{1}{2} f(x, u_n)u_n - F(x, u_n))dx \\ &= \lim_{n\to\infty} (J(u_n) - \frac{1}{2} < J'(u_n), u_n >) = c. \end{aligned} \tag{3.3}$$

Denote $v_n = \frac{u_n}{||u_n||}$, then $||v_n|| = 1$, that is $\{v_n\}$ is bounded in E. Thus, up to a subsequence, for some $v \in E$, we get

$$\begin{aligned} v_n &\rightharpoonup v \, in \, E, \\ v_n &\to v \, in \, L^p(\mathbb{R}^N), \quad for 2 \le p < 2^*, \\ v_n(x) &\to v(x) a.e. \ \ x \in \mathbb{R}^N. \end{aligned} \tag{3.4}$$

If $v \not\equiv 0$, because $||J'(u_n)||\, ||u_n|| \to 0$, as the similar proof in Lemma 6.22 of [2] or Lemma 2.2 of [4], we get a contradiction.

If $v = 0$, by condition (f_3), as the similar proof in Lemma 6.22 of [2] or Lemma 2.2 of [4], we also have

$$\int_{\mathbb{R}^N} (\frac{1}{2} f(x, u_n)u_n - F(x, u_n))dx \to \infty, \tag{3.5}$$

which contradicts (3.3).

This proves that J satisfies the (w*-PS) condition.

Remark 3.1 Our condition (f_3) here is different from (P_3) of [1, Theorem 6.14], which is used to prove the (w*-PS) condition; furthermore, it is more weaker.

Proof of Theorem 1.1. By Theorem 2.1,

$$K[r - \varepsilon, \beta + \varepsilon] \cap (E \backslash P \cup (-P)) \neq \emptyset$$

for all $\varepsilon > 0$ small. That is there exists a $u_k \in E \setminus (-P \cup P)$ (sign-changing critical point) such that

$$J'(u_k) = 0, \quad J(u_k) \in [r - 1, \beta + 1].$$

Next, we estimate the $\gamma = \inf_{Q^{**}} J$. Because of Proposition 2.1, we can adopt the similar method as in [1, p. 67]. Similar to Lemma 2.23 of [1], by choosing the constants D_* and ρ, for all $u \in Q^*(\rho)$, we may get

$$||u|| \geq \Lambda_s^* \min\{\lambda_k^{(1-\alpha)(s-2)/2}, \lambda_k^{(1-\alpha)/2}\} \min\{\rho, \rho^{1/(s-2)}\}.$$

By Lemma 2.26 of [1], for any $u \in Q^*(\rho)$, we have that

$$J(u) \geq \frac{1}{8}(\Lambda_s^*)^2 T_1 T_2,$$

where Λ_s^*, T_1, T_2 are defined in (2.49)-(2.51) in [1] with p replaced by $s \in (2, 2^*)$, $\alpha \in (0, 1)$ is a constant, and Λ_s^*, T_2 are independent of k. In particular, since $\lambda_k \to \infty$, we get

$$T_1 := \min\{\lambda_k^{(1-\alpha)(s-2)/2}, \lambda_k^{(1-\alpha)/2}\} \to \infty, as\ k \to \infty.$$

Therefore, $\gamma \to \infty$ as $k \to \infty$; hence the proof of Theorem 1.1 is finished.

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Competing interests
The author declares that they have no competing interests.

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