A CLASSIFICATION SCHEME FOR NONOSCILLATORY SOLUTIONS OF A HIGHER ORDER NEUTRAL DIFFERENCE EQUATION

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Nonoscillatory solutions of a nonlinear neutral type higher order difference equations are classified by means of their asymptotic behaviors. By means of the Kranoselskii's fixed point theorem, existence criteria are then provided for justification of such classification.

1. Introduction

Classification schemes for nonoscillatory solutions of nonlinear difference equations are important since further investigations of some of the qualitative behaviors of nonoscillatory solutions can then be reduced to only a number of cases. There are several studies which provide such classification schemes for difference equations, see, for example, [4–11]. In particular, in [7], a class of nonlinear neutral difference equations of the form

$$\Delta^m (x_n + c_n x_{n-k}) + f(n, x_{n-l}) = 0, \quad n = 0, 1, \ldots, \tag{1.1}$$

where m, k and l are integers such that $m \geq 2$, $k > 0$ and $l \geq 0$ is studied and classification schemes are given when $\{c_n\}$ is a nonnegative constant sequence $\{c_0\}$, and in [10], the same equation is studied with odd integer $m \geq 1$, positive integer k, integer l and $\{c_n\} = \{-1\}$.

In this paper, we continue our investigation on the possible types of nonoscillatory solutions when $\{c_n\} \subseteq (-1, 0]$ and $\lim_{n \to \infty} c_n = c_0$ (while the case $\{c_n\} \subseteq (-\infty, -1]$ will be discussed elsewhere). Besides the assumption that $\{c_n\} \subseteq (-1, 0]$, we will assume further that f is a continuous function defined on $\{0, 1, \ldots\} \times R$ such that $f = f(n, x)$ is nondecreasing in the second variable x and satisfies $x f(n, x) > 0$ for $x \neq 0$ and $n \geq 0$.

We will accomplish two things in this paper: to provide a classification scheme for the nonoscillatory solutions of (1.1) in Section 2 and establish in Section 3 sufficient and/or necessary criteria for the existence of solutions in each class. There are no overlapping

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results between our paper and [4–11], although some proofs are similar. However, the existence proofs are different in that Cheng-Patula existence theorem is applied in [7], monotone method is used in [10] while we use Krasnoselskii fixed point theorem here. We remark further that classification scheme is also provided for neutral differential equations in [2].

Before we go into details, we will need some preparatory terminologies and results. First of all, given initial x_i for $-\max\{k,l\} \le i \le 0$, we may calculate from (1.1) $x_1, x_2, x_3, \ldots$ in a recursive manner. Such a sequence $\{x_n\}$ is said to be a solution of (1.1). Among the solutions of (1.1), one is said to be nonoscillatory if it is eventually positive or eventually negative.

Given an integer a, it is convenient to set

$$N(a) = \{a, a+1, a+2, \ldots\}. \tag{1.2}$$

Given an integer $\alpha \ge 0$, the generalized factorial function $g(x) = x^{(\alpha)}$ is defined as follows

$$x^{(\alpha)} = \begin{cases} x(x-1)(x-2)\cdots(x-\alpha+1) & \alpha > 0 \\ 1 & \alpha = 0. \end{cases} \tag{1.3}$$

It is well known that $\Delta n^{(\alpha)} = \alpha n^{(\alpha-1)}$ for $\alpha > 0$ (see, e.g., [3]).

Let

$$l^\infty_{N_0} = \left\{ x = \{x_n\}_{n \ge N_0} : \sup_{n \ge N_0} \frac{|x_n|}{r_n} < \infty \right\}, \tag{1.4}$$

where $N_0 > 0$ is an integer and $\{r_n\}_{n \ge N_0}$ is a positive sequence with a uniform positive lower bound. When endowed with the usual linear structure and the norm $\|x\| = \sup_{n \ge N_0}(|x_n|/r_n)$, $(l^\infty_{N_0}, \|\cdot\|)$ is a Banach space. A set $B \subseteq l^\infty_{N_0}$ is said to be uniformly Cauchy if for any $\varepsilon > 0$ there exists an integer $M \ge N_0$ such that

$$\left| \frac{x_i}{r_i} - \frac{x_j}{r_j} \right| < \varepsilon \quad i, j > M \tag{1.5}$$

for all $x = \{x_n\} \in B$.

Lemma 1.1. *A bounded and uniformly Cauchy subset $B \subseteq l^\infty_{N_0}$ is relatively compact.*

Proof. By assumption, we know that for any such $\varepsilon > 0$, there exists an integer $M \ge N_0 > 0$ such that for any $x \in B$, we have

$$\left| \frac{x_i}{r_i} - \frac{x_j}{r_j} \right| < \frac{\varepsilon}{3}, \quad i, j \ge M. \tag{1.6}$$

Let $\Gamma > 0$ be a bound for B. That is $\|x\| \le \Gamma$ for all $x \in B$. Choose integers M_n, $n = N_0, N_0 + 1, \ldots, M$, and numbers $y_n^{(1)} < y_n^{(2)} < \cdots < y_n^{(M_n)}$ such that $y_n^{(1)} = -r_n\Gamma$, $y_n^{(M_n)} = r_n\Gamma$ and

$$\left| \frac{y_n^{(j+1)}}{r_n} - \frac{y_n^{(j)}}{r_n} \right| < \frac{\varepsilon}{3}, \quad 1 \le j \le M_n - 1. \tag{1.7}$$

Now define a sequence $\{v_k\}_{k \geqslant N_0}$ as follows. Let v_{N_0} be one of the values $\{y_{N_0}^{(1)}, \ldots, y_{N_0}^{(M_{N_0})}\}$, v_{N_0+1} be one of the values $\{y_{N_0+1}^{(1)}, \ldots, y_{N_0+1}^{(M_{N_0+1})}\}$. In general, for $N_0 \le k \le M$, let v_k equal one of the values $\{y_k^{(1)}, \ldots, y_k^{(M_k)}\}$. For $k > M$, let $v_k = (r_k/r_M)v_M$. It is clear that the sequence $\{v_k\}_{k \geqslant N_0}$ belongs to $l_{N_0}^{\infty}$. Let L be the set of all possible sequences $\{v_k\}_{k \ge N_0}$ defined as above. Note that L has $M_{N_0}M_{N_0+1} \cdots M_M$ elements.

We assert that L is a finite ε-net for B. It is sufficient to show that for any $x = \{x_k\}_{k \geqslant N_0} \in B$, L contains a sequence $v = \{v_k\}_{k \geqslant N_0}$ such that

$$\|x - v\| = \sup_{n \ge N_0} \frac{|x_n - v_n|}{r_n} < \varepsilon. \tag{1.8}$$

Indeed, by definition of L, we can choose a sequence $\{v_k\}_{k \geqslant N_0}$ in L such that

$$\left| \frac{x_k}{r_k} - \frac{v_k}{r_k} \right| < \frac{\varepsilon}{3}, \quad N_0 \le k \le M. \tag{1.9}$$

Furthermore, by (1.6), (1.9), and the definition of $v = \{v_k\}_{k \geqslant N_0}$, for $k > M$, we have

$$\left| \frac{x_k}{r_k} - \frac{v_k}{r_k} \right| = \left| \frac{x_k}{r_k} - \frac{v_M}{r_M} \right| \le \left| \frac{x_k}{r_k} - \frac{x_M}{r_M} \right| + \left| \frac{x_M}{r_M} - \frac{v_M}{r_M} \right| \le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}. \tag{1.10}$$

From (1.9) and (1.10), we see that (1.8) holds. The proof is complete. □

LEMMA 1.2. *Suppose that* $\lim_{n\to\infty} c_n = c_0$ *with* $c_0 \in (-1, 0]$ *and the sequence* $\{x_n/n^{(i)}\}$ *is eventually positive or eventually negative, where i is a nonnegative integer. Suppose further that* $z_n = x_n + c_n x_{n-k}$ *and* $\lim_{n\to\infty}(z_n/n^{(i)}) = b$. *Then* $\lim_{n\to\infty}(x_n/n^{(i)}) = b/(1 + c_0)$.

Proof. Without loss of generality, we assume that $x_n/n^{(i)} > 0$ for any positive integer n. In case b is finite, we assert that $\{x_n/n^{(i)}\}$ is bounded. Otherwise, there would exist a sequence $\{n_\lambda\}$ of integers with $n_\lambda \to \infty$ for $\lambda \to \infty$ such that

$$\lim_{\lambda\to\infty} \frac{x_{n_\lambda}}{n_\lambda^{(i)}} = \infty, \quad x_n \le x_{n_\lambda},\ 0 < n \le n_\lambda. \tag{1.11}$$

On the other hand, we have

$$\frac{z_{n_\lambda}}{n_\lambda^{(i)}} = \frac{x_{n_\lambda}}{n_\lambda^{(i)}} + c_{n_\lambda}\frac{x_{n_\lambda - k}}{n_\lambda^{(i)}} \ge (1 + c_{n_\lambda})\frac{x_{n_\lambda}}{n_\lambda^{(i)}} \longrightarrow \infty \tag{1.12}$$

as $\lambda \to \infty$. This is contrary to the fact that b is finite.

Let $\limsup_{n\to\infty}(x_n/n^{(i)}) = Q$ and $\liminf_{n\to\infty}(x_n/n^{(i)}) = q$. Then $0 \le q \le Q < \infty$. Moreover, there exist $\{n_\lambda\}$ and $\{\overline{n}_\lambda\}$ such that $\lim_{\lambda\to\infty} n_\lambda = \infty$, $\lim_{\lambda\to\infty} \overline{n}_\lambda = \infty$, $\lim_{\lambda\to\infty}(x_{n_\lambda}/n_\lambda{}^{(i)}) = Q$ and $\lim_{\lambda\to\infty}(x_{\overline{n}_\lambda}/\overline{n}_\lambda{}^{(i)}) = q$. Note that

$$\begin{aligned} b &= \lim_{\lambda\to\infty} \frac{z_{n_\lambda}}{n_\lambda{}^{(i)}} = \lim_{\lambda\to\infty}\left(\frac{x_{n_\lambda}}{n_\lambda{}^{(i)}} + c_{n_\lambda}\frac{x_{n_\lambda-k}}{n_\lambda{}^{(i)}}\right) \\ &\ge \lim_{\lambda\to\infty} \frac{x_{n_\lambda}}{n_\lambda{}^{(i)}} + \liminf_{\lambda\to\infty} c_{n_\lambda}\frac{x_{n_\lambda-k}}{(n_\lambda-k)^{(i)}}\frac{(n_\lambda-k)^{(i)}}{n_\lambda{}^{(i)}} \ge Q + c_0 Q, \\ b &= \lim_{\lambda\to\infty} \frac{z_{\overline{n}_\lambda}}{\overline{n}_\lambda{}^{(i)}} = \lim_{\lambda\to\infty}\left(\frac{x_{\overline{n}_\lambda}}{\overline{n}_\lambda{}^{(i)}} + c_{\overline{n}_\lambda}\frac{x_{\overline{n}_\lambda-k}}{\overline{n}_\lambda{}^{(i)}}\right) \\ &\le \lim_{\lambda\to\infty} \frac{x_{\overline{n}_\lambda}}{\overline{n}_\lambda{}^{(i)}} + \limsup_{\lambda\to\infty} c_{\overline{n}_\lambda}\frac{x_{\overline{n}_\lambda-k}}{(\overline{n}_\lambda-k)^{(i)}}\frac{(\overline{n}_\lambda-k)^{(i)}}{\overline{n}_\lambda{}^{(i)}} \le q + c_0 q, \end{aligned} \tag{1.13}$$

we have $(1+c_0)q \ge (1+c_0)Q$. It follows that $q \ge Q$. Hence $q = Q$ and it implies that $\lim_{n\to\infty}(x_n/n^{(i)})$ exists. In view of $z_n = x_n + c_n x_{n-k}$ and $\lim_{n\to\infty}(z_n/n^{(i)}) = b$, we have

$$\lim_{n\to\infty} \frac{x_n}{n^{(i)}} = \frac{b}{1+c_0}. \tag{1.14}$$

In case b is infinite, then $b = \infty$ or $b = -\infty$. We assert that $b = -\infty$ cannot hold. In fact, for given c_1 with $-c_0 < c_1 < 1$, there exists a large integer N_0 such that $-c_n \le c_1$ for $n \ge N_0$. Hence, if $b = -\infty$, then $z_n = x_n + c_n x_{n-k} < 0$ for $n \ge N$ and

$$x_n < -c_n x_{n-k} \le c_1 x_{n-k}, \quad n \ge N, \tag{1.15}$$

where $N \ge N_0$ is some positive integer. It implies that

$$0 < x_{N+\lambda k} < c_1 x_{N+(\lambda-1)k} < \cdots < c_1^\lambda x_N. \tag{1.16}$$

So that $\lim_{\lambda\to\infty} x_{N+\lambda k} = 0$. Thus

$$\lim_{\lambda\to\infty} z_{N+\lambda k} = 0 \tag{1.17}$$

which implies that $b = -\infty$ is impossible.

Now, for arbitrary $M > 0$, there exists a sufficiently large integer N such that

$$\frac{z_n}{n^{(i)}} = \frac{x_n}{n^{(i)}} + c_n\frac{x_{n-k}}{n^{(i)}} \ge M, \quad n \ge N. \tag{1.18}$$

It follows that

$$\frac{x_n}{n^{(i)}} \ge M, \quad n \ge N. \tag{1.19}$$

That is $\lim_{n\to\infty}(x_n/n^{(i)}) = \infty$. The proof is complete. □

The following two propositions are respectively in [1, Theorems 1.7.9 and 1.7.11].

Lemma 1.3. *Suppose that the sequence* $\{x_n\}$ *and* $\{y_n\}$ *satisfy the following conditions,*

(i) $y_n > 0$ *and* $\Delta y_n > 0$ *for all large integers n and* $\lim_{n\to\infty} y_n = \infty$, *and*

(ii) $\lim_{n\to\infty}(\Delta x_n/\Delta y_n) = b$.

Then $\lim_{n\to\infty}(x_n/y_n) = \lim_{n\to\infty}(\Delta x_n/\Delta y_n) = b$, *where b can be finite or infinite.*

Lemma 1.4. *Let* $u = u(n)$ *be a sequence defined for* $n \in N(a)$, $u(n) > 0$ *with* $\Delta^m u(n)$ *of constant sign on* $N(a)$ *and not identically zero. Then, there exists an integer* m^*, $0 \le m^* \le m$ *with* $m+m^*$ *odd for* $\Delta^m u(n) \le 0$ *or,* $m+m^*$ *even for* $\Delta^m u(n) \ge 0$ *and such that*

$$\begin{gathered} m^* \le m-1 \text{ implies } (-1)^{m^*+i}\Delta^i u(n) > 0 \quad \forall n \in N(a),\ m^*+1 \le i \le m, \\ m^* \ge 1 \text{ implies } \Delta^i u(n) > 0 \quad \forall \text{ large } n \in N(a),\ 1 \le i \le m^*. \end{gathered} \tag{1.20}$$

Remark 1.5. If $u(n) < 0$ in Lemma 1.4, then there exists m^*, $0 \le m^* \le m$ with $m+m^*$ odd for $\Delta^m u(n) \ge 0$ or, $m+m^*$ even for $\Delta^m u(n) \le 0$ and such that

$$\begin{gathered} m^* \le m-1 \text{ implies } (-1)^{m^*+i}\Delta^i u(n) < 0 \quad \forall n \in N(a),\ m^*+1 \le i \le m, \\ m^* \ge 1 \text{ implies } \Delta^i u(n) < 0 \quad \forall \text{ large } n \in N(a),\ 1 \le i \le m^*. \end{gathered} \tag{1.21}$$

Lemma 1.6 (Kranoselskii's fixed point theorem). *Suppose B is a Banach space and* Ω *is a bounded, convex and closed subset of B. Let* $U, S : \Omega \to B$ *satisfy the following conditions.*

(i) $Ux + Sy \in \Omega$ *for any* $x, y \in \Omega$,

(ii) *U is a contraction mapping, and*

(iii) *S is completely continuous.*

Then $U + S$ *has a fixed point in* Ω.

2. Classifications of nonoscillatory solutions

In the following discussions, we assume throughout that

$$\lim_{n\to\infty} c_n = c_0 \in (-1,0]. \tag{2.1}$$

We set

$$z_n = x_n + c_n x_{n-k} \tag{2.2}$$

whenever it is defined. Equation (1.1) can now be written as

$$\Delta^m z_n = -f(n, x_{n-l}). \tag{2.3}$$

We will propose a classification scheme for the nonoscillatory solutions of (1.1). For this purpose, we first note that if $x = \{x_n\}$ is an eventually negative solution of (1.1), then $y = \{y_n\}$ defined by $y_n = -x_n$ will satisfy

$$\Delta^m(y_n + c y_{n-k}) + \tilde{f}(n, y_{n-l}) = 0, \tag{2.4}$$

where

$$\tilde{f}(n,u) = -f(n,-u), \quad n \in N(0),\ u \in R \tag{2.5}$$

has the same properties satisfied by f, that is, $\tilde{f}$ is a continuous function defined on $\{0,1,\ldots\}\times R$ such that $\tilde{f}=\tilde{f}(n,u)$ is nondecreasing in the second variable u and satisfies $u\tilde{f}(n,u)>0$ for $u\neq 0$ and $n\geq 0$. Therefore, we may restrict our attention to the set S^+ of all eventually positive solutions of (1.1). Motivated by the classification scheme in [2], we make use of the following notations for classifying our eventually positive solutions:

$$A_k(\alpha,\beta)=\left\{\{x_n\}\in S^+:\lim_{n\to\infty}\frac{x_n}{n^{(k-1)}}=\alpha,\lim_{n\to\infty}\frac{x_n}{n^{(k)}}=\beta\right\},\quad k\geq 1,$$
$$A_0(\alpha)=\left\{\{x_n\}\in S^+:\lim_{n\to\infty}x_n=\alpha\right\}. \tag{2.6}$$

Theorem 2.1. (a) *Suppose that m is even. If $x=\{x_n\}$ is an eventually positive solution of (1.1), then either $x\in A_0(0)$ or there are some $j\in\{1,2,\ldots,m/2\}$ and $a>0$ such that x belongs to $A_{2j-1}(\infty,a)$, $A_{2j-1}(\infty,0)$ or $A_{2j-1}(a,0)$.* (b) *Suppose that m is odd. If $x=\{x_n\}$ is an eventually positive solution of (1.1), then either x belongs to $A_0(\alpha)$ for some $\alpha\geq 0$, or there are $j\in\{1,2,\ldots,(m-1)/2\}$ and $a>0$ such that x belongs to $A_{2j}(\infty,a)$, $A_{2j}(\infty,0)$ or $A_{2j}(a,0)$.*

Proof. Let m is even and $x=\{x_n\}$ be an eventually positive solution of (1.1). Then, in view of (2.3), there exists some integer $N>0$ such that $\Delta^m z_n<0$ for $n\geq N$. Therefore, z_n is eventually of fixed sign. For the sake of simplicity, we may assume that $\{z_n\}$ is of fixed sign for $n\geq N$.

First of all, suppose $z_n<0$ for $n\geq N$. By the same reasoning as in the proof of Lemma 1.2, we may show that

$$\lim_{\lambda\to\infty}z_{N+\lambda k}=0. \tag{2.7}$$

On the other hand, in view of Lemma 1.4, there exists some even m^* with $0\leq m^*\leq m$ such that eventually $\Delta^i z_n<0$ for $0\leq i\leq m^*$ and $(-1)^{m^*+i}\Delta^i z_n<0$ for $m^*+1\leq i\leq m$. There are now two cases to consider.

Case 1 ($m^*=0$). Then we have eventually

$$z_n<0,\qquad \Delta z_n>0. \tag{2.8}$$

By (2.8), we can set

$$\lim_{n\to\infty}z_n=L_0\leq 0. \tag{2.9}$$

In view of (2.7), we find that $\lim_{n\to\infty}z_n=0$. By Lemma 1.2, we have $\lim_{n\to\infty}x_n=0$. Hence x belongs to $A_0(0)$.

Case 2 ($m^*\geq 2$). Then we have eventually

$$z_n<0,\qquad \Delta z_n<0. \tag{2.10}$$

It implies $\lim_{n\to+\infty}z_n<0$ which is contrary to (2.7). Hence $m^*\geq 2$ does not hold.

Now we suppose $z_n > 0$ for $n \geq N$. Similar to the proof in [7, Theorem 1], we may see that x belongs to $A_{2j-1}(\infty, a)$, $A_{2j-1}(\infty, 0)$ or $A_{2j-1}(a, 0)$ for some $j \in \{1, 2, \ldots, m/2\}$ and $a > 0$.

When m is odd, the proof is similar to those above and hence is skipped. The proof is complete. □

3. Existence criteria

Eventually positive (and by analog eventually negative) solutions of (1.1) have been classified according to Theorem 2.1. We now justify our classification schemes by finding existence criteria for each type of solutions.

Theorem 3.1. *Suppose that m is even. If (1.1) has a solution in $A_{2j-1}(\infty, a)$ for some $j \in \{1, 2, \ldots, m/2\}$ and $a > 0$, then there exists some $K > 0$ such that*

$$\sum_{i=0}^{\infty} \frac{(i+m-2j)^{(m-2j)}}{(m-2j)!} f\left(i, K(i-l)^{(2j-1)}\right) < \infty. \tag{3.1}$$

The converse is also true.

Proof. First of all, we remark that

$$\sum_{i_\lambda=n}^{\infty} \sum_{i_{\lambda-1}=i_\lambda}^{\infty} \cdots \sum_{i_2=i_3}^{\infty} \sum_{i_1=i_2}^{\infty} f(i_1, x_{i_1-l}) = \sum_{i=n}^{\infty} \frac{(i-n+\lambda-1)^{(\lambda-1)}}{(\lambda-1)!} f(i, x_{i-l}). \tag{3.2}$$

Let $x = \{x_n\}$ be an eventually positive solution of (1.1) in $A_{2j-1}(\infty, a)$. Then we may suppose that there exists an integer $N_0 > 0$ such that $x_n > 0$ and $x_{n-l} > 0$ for $n > N_0$. In view of (2.3), we have $\Delta^m z_n < 0$ for $n > N_0$. Thereby $\{\Delta^i z_n\}$ is eventually monotonic for $i = 0, 1, 2, \ldots, m-1$. Since $\lim_{n\to\infty}(x_n/n^{(2j-1)}) = a > 0$, there exists some integer $N_1 > N_0$ such that

$$\frac{1}{2} a n^{(2j-1)} \leq x_n \leq \frac{3}{2} a n^{(2j-1)}, \quad n \geq N_1. \tag{3.3}$$

Note that $\lim_{n\to\infty}(z_n/n^{(2j-1)}) = (1+c_0)a$ implies

$$\lim_{n\to\infty} \Delta^{2j-1} z_n = (1+c_0) a (2j-1)!. \tag{3.4}$$

By (3.4) and the monotonicity of $\Delta^i z_n$, we have

$$\lim_{n\to\infty} \Delta^i z_n = 0, \quad i = 2j, 2j+1, \ldots, m-1. \tag{3.5}$$

Summing (2.3) $m-2j$ times from n to N_1 and invoking (3.5) in each time, we obtain

$$\begin{aligned} \Delta^{2j} z_n &= -\sum_{i_{m-2j}=n}^{\infty} \cdots \sum_{i_2=i_3}^{\infty} \sum_{i_1=i_2}^{\infty} f(i_1, x_{i_1-l}) \\ &= -\sum_{i=n}^{\infty} \frac{(i-n+m-2j-1)^{(m-2j-1)}}{(m-2j-1)!} f(i, x_{i-l}), \quad n \geq N_1. \end{aligned} \tag{3.6}$$

Summing the above equation again from N_1 to n, we obtain

$$\Delta^{2j-1}z_{n+1} = \Delta^{2j-1}z_{N_1} - \sum_{i_2=N_1}^{n}\sum_{i_1=i_2}^{\infty}\frac{(i_1-i_2+m-2j-1)^{(m-2j-1)}}{(m-2j-1)!}f(i_1,x_{i_1-l}). \tag{3.7}$$

By (3.4), the above equation implies that

$$\sum_{i_2=n}^{\infty}\sum_{i_1=i_2}^{\infty}\frac{(i_1-i_2+m-2j-1)^{(m-2j-1)}}{(m-2j-1)!}f(i_1,x_{i_1-l}) < \infty, \quad n \geq N_1. \tag{3.8}$$

That is,

$$\sum_{i=n}^{\infty}\frac{(i-n+m-2j)^{(m-2j)}}{(m-2j)!}f(i,x_{i-l}) < \infty, \quad n \geq N_1. \tag{3.9}$$

Let $K = a/2$. In view of (3.3), (3.9) and the monotonicity of $f(n,x)$ in x, we see that (3.1) holds.

Conversely, suppose (3.1) holds for some $K > 0$. Set $R_n = n^{(2j-1)}$. In view of (3.2), we have

$$\begin{aligned}\sum_{i_{m-2j+1}=n}^{\infty}\sum_{i_{m-2j}=i_{m-2j+1}}^{\infty}\cdots\sum_{i_2=i_3}^{\infty}\sum_{i_1=i_2}^{\infty}f\left(i_1,K(i_1-l)^{(2j-1)}\right)\\ = \sum_{i=n}^{\infty}\frac{(i-n+m-2j)^{(m-2j)}}{(m-2j)!}f\left(i,K(i-l)^{(2j-1)}\right).\end{aligned} \tag{3.10}$$

Note that (2.1), there are two cases to consider.

In case $-1 < c_0 < 0$, take c_1 so that $-c_0 < c_1 < (1-4c_0)/5 < 1$. Then $(1-5c_1)/(4c_0) < 1$. Note that $\lim_{n\to\infty}(|c_n|R_n/R_{n-k-l}) = |c_0|$, $\lim_{n\to\infty}(R_{n-k}/R_n) = 1$ and (3.1) holds. Thus there exists an integer $N > k+l$ such that when $n \geq N$, we have

$$\frac{|c_n|R_n}{R_{n-k-l}} \leq c_1, \tag{3.11}$$

$$-c_n \leq c_1, \tag{3.12}$$

$$\frac{R_{n-k}}{R_n} \geq \frac{1-5c_1}{4c_n}, \tag{3.13}$$

$$\sum_{i_{m-2j+1}=N}^{\infty}\sum_{i_{m-2j}=i_{m-2j+1}}^{\infty}\cdots\sum_{i_1=i_2}^{\infty}f\left(i_1,K(i_1-l)^{(2j-1)}\right) < \frac{(1-c_1)K}{8}. \tag{3.14}$$

Take $N_0 = N-k-l$, $r_n = R_n^2$ and define the Banach space $l_{N_0}^{\infty}$ as in (1.4). Let

$$\Omega = \left\{x \in l_{N_0}^{\infty} : \frac{1}{2}KR_n \leq x_n \leq KR_n\right\}. \tag{3.15}$$

Then it is obvious that Ω is a bounded, convex and closed subset of $l_{N_0}^\infty$, and for any $x \in \Omega$ and $n \geq N_0 + l$, we have

$$f(n, x_{n-l}) \leq f\left(n, K(n-l)^{(2j-1)}\right). \tag{3.16}$$

Define operators U and S on Ω as follows:

$$(Ux)_n = \begin{cases} -\frac{3}{4}c_1 K R_n - \frac{c_N x_{N-k}}{R_N} R_n & N_0 \leq n < N \\ -\frac{3}{4}c_1 K R_n - c_n x_{n-k} & n \geq N, \end{cases}$$
$$(Sx)_n = \begin{cases} \frac{3}{4}K R_n & N_0 \leq n < N \\ \frac{3}{4}K R_n + F(n) & n \geq N, \end{cases} \tag{3.17}$$

where

$$F(n) = \sum_{i_m=N}^{n-1} \cdots \sum_{i_{m-2j+3}=N}^{i_{m-2j+4}-1} \sum_{i_{m-2j+2}=N}^{i_{m-2j+3}-1} \sum_{i_{m-2j+1}=i_{m-2j+2}}^{\infty} \sum_{i_{m-2j}=i_{m-2j+1}}^{\infty} \cdots \sum_{i_1=i_2}^{\infty} f(i_1, x_{i_1-l}). \tag{3.18}$$

In view of (3.16) and (3.14),we have

$$F(n) \leq \sum_{i_m=N}^{n-1} \cdots \sum_{i_{m-2j+3}=N}^{i_{m-2j+4}-1} \sum_{i_{m-2j+2}=N}^{i_{m-2j+3}-1} \frac{(1-c_1)K}{8} = \frac{(1-c_1)K(n-N)^{(2j-1)}}{8(2j-1)!} \leq \frac{(1-c_1)K}{8} R_n \tag{3.19}$$

for $n \geq N$.

Next, we will show that the operators U and S satisfy the conditions of Kranoselskii's fixed point theorem.

First, we claim that $Ux + Sy \in \Omega$ for any $x, y \in \Omega$. Indeed, for $N_0 \leq n < N$, in view of (3.13) and (3.12), we have

$$(Ux)_n + (Sy)_n = \left(\frac{3}{4}(1-c_1)K - c_N \frac{x_{N-k}}{R_N}\right) R_n \geq \left(\frac{3}{4}(1-c_1)K - c_N K \frac{R_{N-k}}{2R_N}\right) R_n \geq \frac{1}{2} K R_n,$$
$$(Ux)_n + (Sy)_n \leq \left(\frac{3}{4}(1-c_1)K - c_N K \frac{R_{N-k}}{R_N}\right) R_n \leq \left(\frac{3}{4}(1-c_1) + c_1\right) K R_n \leq K R_n. \tag{3.20}$$

When $n \geq N$, invoking (3.13) again, we have

$$(Ux)_n + (Sy)_n \geq \frac{3}{4}(1-c_1)K R_n - c_n x_{n-k} \geq \frac{3}{4}(1-c_1)K R_n - c_n \frac{1}{2} K \frac{R_{n-k}}{R_n} R_n \geq \frac{1}{2} K R_n \tag{3.21}$$

and, in view of (3.19) and (3.12), we have

$$\begin{aligned}(Ux)_n+(Sy)_n &\le \frac{3}{4}(1-c_1)KR_n - c_n x_{n-k} + \frac{(1-c_1)K}{8}R_n \\ &\le \frac{3}{4}(1-c_1)KR_n - c_n KR_{n-k} + \frac{(1-c_1)K}{8}R_n \le KR_n.\end{aligned} \tag{3.22}$$

That is, $Ux+Sy\in\Omega$ for any $x,y\in\Omega$.

Let $x,y\in\Omega$. In view of (3.11), we have

$$\begin{aligned}\frac{1}{R_n^2}\left|(Ux)_n-(Uy)_n\right| &= \frac{|c_N|\,|x_{N-k}-y_{N-k}|}{R_N R_n} \\ &= \frac{|x_{N-k}-y_{N-k}|}{R_{N-k}^2}\frac{|c_N|R_{N-k}^2}{R_N R_n} \le c_1 \sup_{n\ge N_0}\frac{|x_n-y_n|}{R_n^2}\end{aligned} \tag{3.23}$$

for $N_0\le n<N$. And, for $n\ge N$, we have

$$\frac{1}{R_n^2}\left|(Ux)_n-(Uy)_n\right| \le |c_n| \sup_{n\ge N_0}\frac{|x_n-y_n|}{R_n^2}. \tag{3.24}$$

Therefore, we have

$$\|Ux-Uy\|\le c_1\|x-y\| \tag{3.25}$$

for any $x,y\in\Omega$. Hence, U is a contraction mapping.

Next, we will prove that S is a completely continuous mapping. Indeed, it is obvious that $(Sx)_n\ge (K/2)R_n$ for $n\ge N_0$ and $(Sx)_n\le KR_n$ for $N_0\le n<N$. When $n\ge N$, by means of (3.19), we have

$$(Sx)_n\le \frac{3}{4}KR_n+\frac{(1-c_1)K}{8}R_n\le KR_n. \tag{3.26}$$

That is, the operator S maps Ω into Ω.

Now we consider the continuity of S. Let $x^{(\lambda)}\in\Omega$ and $\|x^{(\lambda)}-x\|\to 0$ when $\lambda\to\infty$, we assert that $Sx^{(\lambda)}$ converges to Sx by $\|\cdot\|$. Indeed, $\|x^{(\lambda)}-x\|\to 0$ implies that $x\in\Omega$ and $|x_n^{(\lambda)}-x_n|\to 0$ when $\lambda\to\infty$ for any integer $n\ge N_0$. Thereby, we have

$$\left|f\left(n,x_{n-l}^{(\lambda)}\right)-f(n,x_{n-l})\right|\longrightarrow 0,\quad \lambda\longrightarrow\infty \tag{3.27}$$

for any integer $n\ge N_0+l$. By definition of S, we have

$$\left|\left(Sx^{(\lambda)}\right)_n-(Sx)_n\right|=0 \tag{3.28}$$

for $N_0\le n<N$ and

$$\left|\left(Sx^{(\lambda)}\right)_n-(Sx)_n\right|\le H_\lambda(n) \tag{3.29}$$

for $n \geq N$, where

$$H_\lambda(n) = \sum_{i_m=N}^{n-1} \cdots \sum_{i_{m-2j+3}=N}^{i_{m-2j+4}-1} \sum_{i_{m-2j+2}=N}^{i_{m-2j+3}-1} \sum_{i_{m-2j+1}=i_{m-2j+2}}^{\infty} \sum_{i_{m-2j}=i_{m-2j+1}}^{\infty} \cdots \sum_{i_1=i_2}^{\infty} \left| f\left(i_1, x_{i_1-l}^{(\lambda)}\right) - f\left(i_1, x_{i_1-l}\right) \right|. \tag{3.30}$$

In view of (3.16), we have

$$\left| f\left(i_1, x_{i_1-l}^{(\lambda)}\right) - f\left(i_1, x_{i_1-l}\right) \right| \leq 2f\left(i_1, K(i_1-l)^{(2j-1)}\right), \quad n \geq N_0 + l. \tag{3.31}$$

Thus

$$\begin{aligned} H_\lambda(n) &\leq \sum_{i_m=N}^{n-1} \cdots \sum_{i_{m-2j+3}=N}^{i_{m-2j+4}-1} \sum_{i_{m-2j+2}=N}^{i_{m-2j+3}-1} \sum_{i_{m-2j+1}=N}^{\infty} \sum_{i_{m-2j}=i_{m-2j+1}}^{\infty} \cdots \sum_{i_1=i_2}^{\infty} \left| f\left(i_1, x_{i_1-l}^{(\lambda)}\right) - f\left(i_1, x_{i_1-l}\right) \right| \\ &\leq R_n \sum_{i_{m-2j+1}=N}^{\infty} \sum_{i_{m-2j}=i_{m-2j+1}}^{\infty} \cdots \sum_{i_1=i_2}^{\infty} \left| f\left(i_1, x_{i_1-l}^{(\lambda)}\right) - f\left(i_1, x_{i_1-l}\right) \right|. \end{aligned} \tag{3.32}$$

To sum up, we have

$$\begin{aligned} \left\| \left(Sx^{(\lambda)}\right)_n - (Sx)_n \right\| &\leq \sup_{n \geq N_0} \frac{1}{R_n} \sum_{i_{m-2j+1}=N}^{\infty} \sum_{i_{m-2j}=i_{m-2j+1}}^{\infty} \cdots \sum_{i_1=i_2}^{\infty} \left| f\left(i_1, x_{i_1-l}^{(\lambda)}\right) - f\left(i_1, x_{i_1-l}\right) \right| \\ &= \sup_{n \geq N_0} \frac{1}{R_n} \sum_{i=N}^{\infty} \frac{(i-N+m-2j)^{(m-2j)}}{(m-2j)} \left| f\left(i, x_{i-l}^{(\lambda)}\right) - f\left(i, x_{i-l}\right) \right|. \end{aligned} \tag{3.33}$$

In view of (3.27) and (3.31), the Lebesque's dominated theorem [3] then implies $\lim_{\lambda\to\infty} \|(Sx^\lambda) - (Sx)\| = 0$. This means S is continuous.

Finally, we prove that $S\Omega$ is relatively compact. We assert that $S\Omega$ is uniformly Cauchy. Indeed, for any $\varepsilon > 0$, there exists $N_1 > N$ such that $1/R_n < \varepsilon/3K$ for $n \geq N_1$. For any $x \in \Omega$ and $i_1, i_2 \geq N_1$, in view of (3.19), we have that

$$\left| \frac{(Sx)_{i_1}}{R_{i_1}^2} - \frac{(Sx)_{i_2}}{R_{i_2}^2} \right| \leq \frac{(Sx)_{i_1}}{R_{i_1}^2} + \frac{(Sx)_{i_2}}{R_{i_2}^2} \leq \frac{3K}{4}\left(R_{i_1}^{-1} + R_{i_2}^{-1}\right) + \sum_{j=1}^{2} \frac{(1-c_1)K}{8R_{i_j}} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{12} < \varepsilon. \tag{3.34}$$

By Lemma 1.1, $S\Omega$ is relatively compact.

To sum up, we have proved that S is a completely continuous mapping.

By the Kranoselskii's fixed point theorem, there then exists $x = \{x_n\} \in \Omega$ such that $(Ux)_n + (Sx)_n = x_n$. Therefore, we have

$$x_n = \frac{3}{4}(1-c_1)KR_n - c_n x_{n-k} + F(n), \quad n \geq N. \tag{3.35}$$

It is easy to verify that x_n satisfy (1.1). Furthermore, we have

$$\begin{aligned}\Delta^{2j-1}F(n) &= \sum_{i_{m-2j+1}=n}^{\infty} \sum_{i_{m-2j}=i_{m-2j+1}}^{\infty} \cdots \sum_{i_1=i_2}^{\infty} f(i_1, x_{i_1-l}) \\ &\leq \sum_{i_{m-2j+1}=n}^{\infty} \sum_{i_{m-2j}=i_{m-2j+1}}^{\infty} \cdots \sum_{i_1=i_2}^{\infty} f\left(i_1, K(i_1-l)^{(2j-1)}\right).\end{aligned} \tag{3.36}$$

In view of (3.1) and (3.10), we have

$$\lim_{n\to\infty} \Delta^{2j-1}F(n) = 0, \tag{3.37}$$

so that

$$\lim_{n\to\infty} \frac{F(n)}{n^{(2j-1)}} = 0. \tag{3.38}$$

Now we turn to (3.35) and obtain

$$\lim_{n\to\infty} \frac{z_n}{n^{(2j-1)}} = \frac{3}{4}(1-c_1)K. \tag{3.39}$$

By Lemma 1.2, we have

$$\lim_{n\to\infty} \frac{x_n}{n^{(2j-1)}} = \frac{3(1-c_1)K}{4(1+c_0)}, \tag{3.40}$$

which infers that $\lim_{n\to\infty}(x_n/n^{(2j-2)}) = \infty$. In summary, (1.1) has a solution in $A_{2j-1}(\infty, a)$ when $-1 < c_0 < 0$.

In case $c_0 = 0$, take c_1 so that $0 < c_1 \leq 1/3$. Then, there exists an integer $N > k+l$ such that when $n \geq N$, (3.11) to (3.14) hold. Take operators U and S to be the same operators as above. Then we may prove in similar manners that (1.1) has a solution in $A_{2j-1}(\infty, a)$. The proof is complete. □

A similar theorem holds when m is odd, the proof is similar to that of Theorem 3.1 and hence is skipped.

Theorem 3.2. *Suppose m is odd. If (1.1) has a solution in $A_{2j}(\infty, a)$ for some $j \in \{1, 2, \ldots, (m-1)/2\}$ and $a > 0$, then there exists some $K > 0$ such that*

$$\sum_{i=0}^{\infty} \frac{(i+m-2j-1)^{(m-2j-1)}}{(m-2j-1)!} f\left(i, K(i-l)^{(2j)}\right) < \infty. \tag{3.41}$$

The converse also holds.

Theorem 3.3. *Suppose that m is even. If (1.1) has an eventually positive solution in* $A_{2j-1}(a, 0)$ *for some* $j \in \{1,\dots,m/2\}$ *and* $a > 0$, *then there is some* $K > 0$ *such that*

$$\sum_{i=0}^{\infty} \frac{(i+m-2j+1)^{(m-2j+1)}}{(m-2j+1)!} f\left(i, K(i-l)^{(2j-2)}\right) < \infty. \tag{3.42}$$

The converse is also true.

The proof is similar to that of Theorem 3.1 by taking $R_n = n^{(2j-2)}$.

Theorem 3.4. *Suppose m is odd. If (1.1) has a solution in* $A_{2j}(a,0)$ *for some* $j \in \{1,2,\dots,(m-1)/2\}$ *and* $a > 0$, *then there is some* $K > 0$ *such that*

$$\sum_{i=0}^{\infty} \frac{(i+m-2j)^{(m-2j)}}{(m-2j)!} f\left(i, K(i-l)^{(2j-1)}\right) < \infty. \tag{3.43}$$

The converse also holds.

Theorem 3.5. *Suppose that m is even. If (1.1) has a solution in* $A_{2j-1}(\infty,0)$ *for some* $j \in \{1,2,\dots,m/2\}$, *then*

$$\sum_{i=0}^{\infty} \frac{(i+m-2j)^{(m-2j)}}{(m-2j)!} f\left(i, (i-l)^{(2j-2)}\right) < \infty, \tag{3.44}$$

$$\sum_{i=0}^{\infty} \frac{(i+m-2j+1)^{(m-2j+1)}}{(m-2j+1)!} f\left(i, (i-l)^{(2j-1)}\right) = \infty. \tag{3.45}$$

Conversely, if there is some $j \in \{1,2,\dots,m/2\}$ *such that*

$$\sum_{i=0}^{\infty} \frac{(i+m-2j)^{(m-2j)}}{(m-2j)!} f\left(i, (i-l)^{(2j-1)}\right) < \infty, \tag{3.46}$$

$$\sum_{i=0}^{\infty} \frac{(i+m-2j+1)^{(m-2j+1)}}{(m-2j+1)!} f\left(i, (i-l)^{(2j-2)}\right) = \infty, \tag{3.47}$$

then (1.1) has a solution in $A_{2j-1}(\infty,0)$.

Proof. Let $x = \{x_n\}$ be an eventually positive solution of (1.1) in $A_{2j-1}(\infty,0)$. Note that $\lim_{n\to\infty}(x_n/n^{(2j-1)}) = 0$, $\lim_{n\to\infty}(x_n/n^{(2j-2)}) = \infty$ and (2.3) holds. Therefore there exists an integer $N_0 > 0$ such that

$$\Delta^m z_n < 0, \quad n \ge N_0, \tag{3.48}$$

$$x_n \le n^{(2j-1)}, \quad n \ge N_0, \tag{3.49}$$

$$x_n \ge n^{(2j-2)}, \quad n \ge N_0. \tag{3.50}$$

In view of (3.48), $\{\Delta^i z_n\}$ is eventually monotonic for $i=0,1,2,\dots,m-1$. Since $\lim_{n\to\infty}(z_n/n^{(2j-1)})=0$ and $\lim_{n\to\infty}(z_n/n^{(2j-2)})=\infty$, we have

$$\lim_{n\to\infty}\Delta^{2j-1}z_n=0, \tag{3.51}$$

$$\lim_{n\to\infty}\Delta^{2j-2}z_n=\infty. \tag{3.52}$$

In view of (3.51) and the monotonicity of $\{\Delta^i z_n\}$, we see that

$$\lim_{n\to\infty}\Delta^i z_n=0,\quad i=2j,2j+1,\dots,m-1. \tag{3.53}$$

Now summing (2.3) $m-2j+1$ times from N_0+l to n and invoking (3.53), we have

$$\Delta^{2j-1}z_{n+1}=\Delta^{2j-1}z_{N_0+l}-\sum_{i_2=N_0+l}^{n}\sum_{i_1=i_2}^{\infty}\frac{(i_1-i_2+m-2j-1)^{(m-2j-1)}}{(m-2j-1)!}f(i_1,x_{i_1-l}). \tag{3.54}$$

Noticing (3.50) and (3.51), we see that (3.44) holds.

By taking limits on both sides of (3.54) as $n\to\infty$ and then replacing N_0+l by n, we see from (3.51) that

$$\Delta^{2j-1}z_n=\sum_{i_1=n}^{\infty}\frac{(i_1-n+m-2j)^{(m-2j)}}{(m-2j)!}f(i_1,x_{i_1-l}),\quad n\ge N_0. \tag{3.55}$$

Summing (3.55) from N_0+l to n, we have

$$\Delta^{2j-2}z_{n+1}-\Delta^{2j-2}z_{N_0+l}=\sum_{i_2=N_0+l}^{n}\sum_{i_1=i_2}^{\infty}\frac{(i_1-n+m-2j)^{(m-2j)}}{(m-2j)!}f(i_1,x_{i_1-l}). \tag{3.56}$$

Invoking (3.49) and (3.52), we see that (3.45) holds.

Conversely, we demonstrate the sufficiency. Suppose that $-1<c_0<0$. Set $L_n=n^{(2j-2)}$ and $R_n=n^{(2j-1)}$. Take c_1 so that $-c_0<c_1<(1-4c_0)/5<1$. Similar to the proof of Theorem 3.1, there exists an integer $N>k+l$ such that when $n\ge N$, we have

$$\begin{gathered}2L_n\le R_n,\qquad -c_n\le c_1,\qquad \frac{|c|R_n}{R_{n-k-l}}\le c_1,\\ \frac{L_{n-k}}{L_n}\ge\frac{1-5c_1}{4c_n},\qquad \sum_{i_{m-2j+1}=N}^{\infty}\sum_{i_{m-2j}=i_{m-2j+1}}^{\infty}\cdots\sum_{i_1=i_2}^{\infty}f\Big(i_1,(i_1-l)^{(2j-1)}\Big)<\frac{1-c_1}{8}.\end{gathered} \tag{3.57}$$

Take $N_0=N-k-l$, $r_n=R_n^2$ and define the Banach space $l_{N_0}^{\infty}$ as in (1.4). Let

$$\Omega=\{x\in l_{N_0}^{\infty}:L_n\le x_n\le R_n\}. \tag{3.58}$$

Define two operators on Ω as follows:

$$(Ux)_n = \begin{cases} \frac{1}{2}L_n & N_0 \le n < N \\ -\frac{3}{2}c_1 L_n - c_n x_{n-k} & n \ge N, \end{cases}$$
$$(Sx)_n = \begin{cases} \frac{1}{2}L_n & N_0 \le n < N \\ \frac{3}{2}L_n + F(n) & n \ge N, \end{cases} \tag{3.59}$$

where

$$F(n) = \sum_{i_m=N}^{n-1} \cdots \sum_{i_{m-2j+3}=N}^{i_{m-2j+4}-1} \sum_{i_{m-2j+2}=N}^{i_{m-2j+3}-1} \sum_{i_{m-2j+1}=i_{m-2j+2}}^{\infty} \sum_{i_{m-2j}=i_{m-2j+1}}^{\infty} \cdots \sum_{i_1=i_2}^{\infty} f(i_1, x_{i_1-l}). \tag{3.60}$$

Analogous to the discussions in Theorem 3.1, there exists $x = \{x_n\} \in \Omega$ such that

$$x_n = \frac{3}{2}(1-c_1)L_n - c_n x_{n-k} + F(n), \quad n \ge N. \tag{3.61}$$

From (3.61) and (3.46), we see that

$$\lim_{n\to\infty} \frac{z_n}{n^{(2j-1)}} = 0. \tag{3.62}$$

By Lemma 1.2, we have

$$\lim_{n\to\infty} \frac{x_n}{n^{(2j-1)}} = 0. \tag{3.63}$$

Note that (3.47) implies

$$\lim_{n\to\infty} \Delta^{2j-2} F(n) = \infty. \tag{3.64}$$

Hence, in view of (3.61) and Lemma 1.2, we see that

$$\lim_{n\to\infty} \frac{x_n}{n^{(2j-2)}} = \infty. \tag{3.65}$$

This means (1.1) has an eventually positive solution in $A_{2j-1}(\infty, 0)$ when $-1 < c_0 < 0$.

When $c_0 = 0$, we take c_1 so that $0 < c_1 \le 1/3$ and the rest of proof is the same as the above and is thus skipped. The proof is complete. □

A result similar to Theorem 3.5 is the following.

Theorem 3.6. *Suppose m is odd. If (1.1) has a solution in $A_{2j}(\infty,0)$ for some $j \in \{1,2,\ldots,(m-1)/2\}$, then*

$$\sum_{i=0}^{\infty} \frac{(i+m-2j-1)^{(m-2j-1)}}{(m-2j-1)!} f\left(i,(i-l)^{(2j-1)}\right) < \infty, \qquad \sum_{i=0}^{\infty} \frac{(i+m-2j)^{(m-2j)}}{(m-2j)!} f\left(i,(i-l)^{(2j)}\right) = \infty. \tag{3.66}$$

Conversely, if

$$\sum_{i=0}^{\infty} \frac{(i+m-2j-1)^{(m-2j-1)}}{(m-2j-1)!} f\left(i,(i-l)^{(2j)}\right) < \infty, \qquad \sum_{i=0}^{\infty} \frac{(i+m-2j)^{(m-2j)}}{(m-2j)!} f\left(i,(i-l)^{(2j-1)}\right) = \infty, \tag{3.67}$$

then (1.1) has a solution in $A_{2j}(\infty,0)$.

Theorem 3.7. *Suppose m is even and $c_0 < 0$. If there exist constants $\alpha > 0$, c_1 with $0 < c_1 < -c_0$ and integer $M > k+l$ such that*

$$c_1 e^{\alpha k} > 1 \tag{3.68}$$

as well as

$$\sum_{i=n}^{\infty} \frac{(i-n+m-1)^{(m-1)}}{(m-1)!} f\left(i,\frac{1}{i-l}\right) \le \left(c_1 e^{\alpha k}-1\right)e^{-\alpha n}, \quad n \ge M, \tag{3.69}$$

then (1.1) has a solution in $A_0(0)$.

Proof. First note that there exists integer $N > M$ such that

$$e^{-\alpha n} < \frac{1}{n}, \quad n \ge N-k-l, \qquad -c_n \ge c_1, \quad n \ge N,$$
$$\frac{-c_n}{n-k} \le \frac{1}{n}, \quad n \ge N,$$
$$\sum_{i=n}^{\infty} \frac{(i-n+m-1)^{(m-1)}}{(m-1)!} f\left(i,\frac{1}{i-l}\right) \le \left(c_1 e^{\alpha k}-1\right)e^{-\alpha n}, \quad n \ge N. \tag{3.70}$$

Take $N_0 = N-k-l$, $r_n = 1$ and define the Banach space $l^\infty_{N_0}$ as in (1.4). Let

$$\Omega = \left\{x \in l^\infty_{N_0} : e^{-\alpha n} \le x_n \le \frac{1}{n}\right\}. \tag{3.71}$$

Define two operators on Ω as follows:

$$(Ux)_n = 0 \quad \text{for } n \ge N_0, \qquad (Sx)_n = \begin{cases} \dfrac{1}{n} & N_0 \le n < N \\ -c_n x_{n-k} - F(n) & n \ge N, \end{cases} \tag{3.72}$$

where

$$F(n) = \sum_{i_m=n}^{\infty} \sum_{i_{m-1}=i_m}^{\infty} \cdots \sum_{i_2=i_3}^{\infty} \sum_{i_1=i_2}^{\infty} f\left(i_1, x_{i_1-l}\right). \tag{3.73}$$

By arguments similar to those in the proof of Theorem 3.1, we may prove that there exists $x = \{x_n\} \in \Omega$ such that

$$x_n = -c_n x_{n-k} + F(n),\ n \geq N. \tag{3.74}$$

In view of the definition of Ω, we see that x is a solution of (1.1) in $A_0(0)$. The proof is complete. □

A variant of Theorem 3.7 is the following and its proof is omitted.

THEOREM 3.8. *Suppose m is odd and $c_0 < 0$. If there exist constants $\alpha > 0$, c_1, c_2 with $0 < c_1 < -c_0 < c_2 < 1$ and integer $M > l + k$ such that*

$$c_1 e^{\alpha k} > 1 \tag{3.75}$$

as well as

$$\sum_{i=n}^{\infty} \frac{(i-n+m-1)^{(m-1)}}{(m-1)!} f\left(i, \frac{1}{i-l}\right) \leq \frac{1}{n} - \frac{c_2}{n-k}, \quad n \geq M, \tag{3.76}$$

then (1.1) has a solution in $A_0(0)$.

THEOREM 3.9. *Suppose that m is odd. If (1.1) has a solution in $A_0(a)$ for some $a > 0$, then there exists some $K > 0$ such that*

$$\sum_{i=0}^{\infty} \frac{(i+m-1)^{(m-1)}}{(m-1)!} f(i,K) < \infty. \tag{3.77}$$

The converse also holds.

The proof is similar to that of Theorem 3.1 and is skipped.

Example 3.10. Consider the equation

$$\Delta^2\left(x_n - \frac{3}{4}x_{n-1}\right) + \frac{1}{16}x_{n-1} = 0, \tag{3.78}$$

here $f(n,x) = (1/16)x$.

It is clear that

$$\sum_{i=0}^{\infty} f\left(i, K(i-1)\right) = \infty, \qquad \sum_{i=0}^{\infty}(i+1) f(i,K) = \infty \tag{3.79}$$

for any $K > 0$ and

$$\sum_{i=0}^{\infty}(i+1) f(i, i-1) = \infty. \tag{3.80}$$

Hence by Theorems 3.1 and 3.3, an eventually positive solution of (3.78) cannot be in $A_1(\infty,a)$ nor in $A_1(a,0)$. In addition, by Theorem 3.5, an eventually positive solution of (3.78) cannot be in $A_1(\infty,0)$. However, by Theorem 2.1, (3.78) has a solution in $A_0(0)$ if it has some eventually positive solution. Indeed, $\{x_n\} = \{1/2^n\}$ satisfies (3.78) and $\lim_{n\to\infty} x_n = 0$.

Consider another equation

$$\Delta^3\left(x_n - \frac{1}{5}x_{n-2}\right) + \frac{1}{80}x_{n-1} = 0, \tag{3.81}$$

here $f(n,x) = (1/80)x$. Then, it is easy to see that

$$\sum_{i=0}^{\infty} f\left(i, K(i-1)^{(2)}\right) = \infty, \qquad \sum_{i=0}^{\infty}(i+1)f\left(i,K(i-1)\right) = \infty, \tag{3.82}$$

as well as

$$\sum_{i=0}^{\infty} \frac{(i+2)^{(2)}}{2} f(i,K) = \infty \tag{3.83}$$

for any $K > 0$, and

$$\sum_{i=0}^{\infty} f(i,i-1) = \infty. \tag{3.84}$$

By the similar reasons to the above, (3.81) has a solution in $A_0(0)$ if it has some eventually positive solution. Indeed, $\{x_n\} = \{1/2^n\}$ is such a solution of (3.81).

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