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Research Article

Existence of Periodic Solutions for p-Laplacian Equations on Time Scales

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We systematically explore the periodicity of Liénard type p-Laplacian equations on time scales. Sufficient criteria are established for the existence of periodic solutions for such equations, which generalize many known results for differential equations when the time scale is chosen as the set of the real numbers. The main method is based on the Mawhin's continuation theorem.

1. Introduction

In the past decades, periodic problems involving the scalar p-Laplacian were studied by many authors, especially for the second-order and three-order p-Laplacian differential equation, see [1–8] and the references therein. Of the aforementioned works, Lu in [1] investigated the existence of periodic solutions for a p-Laplacian Liénard differential equation with a deviating argument

$$(\varphi_p(y'(t)))' + f(y(t))y'(t) + h(y(t)) + g(y(t-\tau(t))) = e(t), \tag{1.1}$$

by Mawhin's continuation theorem of coincidence degree theory [3]. The author obtained a new result for the existence of periodic solutions and investigated the relation between the existence of periodic solutions and the deviating argument $\tau(t)$. Cheung and Ren [4] studied the existence of T-periodic solutions for a p-Laplacian Liénard equation with a deviating argument

$$\left(\varphi_p\left(x'(t)\right)\right)' + f(x(t))x'(t) + g(x(t-\tau(t))) = e(t), \tag{1.2}$$

by Mawhin's continuation theorem. Two results for the existence of periodic solutions were obtained. Such equations are derived from many fields, such as fluid mechanics and elastic mechanics.

The theory of time scales has recently received a lot of attention since it has a tremendous potential for applications. For example, it can be used to describe the behavior of populations with hibernation periods. The theory of time scales was initiated by Hilger [9] in his Ph.D. thesis in 1990 in order to unify continuous and discrete analysis. By choosing the time scale to be the set of real numbers, the result on dynamic equations yields a result concerning a corresponding ordinary differential equation, while choosing the time scale as the set of integers, the same result leads to a result for a corresponding difference equation. Later, Bohner and Peterson systematically explore the theory of time scales and obtain many perfect results in [10] and [11]. Many examples are considered by the authors in these books.

But the research of periodic solutions on time scales has not got much attention, see [12–16]. The methods usually used to explore the existence of periodic solutions on time scales are many fixed point theory, upper and lower solutions, Masseras theorem, and so on. For example, Kaufmann and Raffoul in [12] use a fixed point theorem due to Krasnosel'ski to show that the nonlinear neutral dynamic system with delay

$$x^{\Delta}(t) = -a(t)x^{\sigma}(t) + c(t)x^{\Delta}(t-k) + q(t, x(t), x(t-k)), \quad t \in \mathbb{T}, \tag{1.3}$$

has a periodic solution. Using the contraction mapping principle the authors show that the periodic solution is unique under a slightly more stringent inequality.

The Mawhin's continuation theorem has been extensively applied to explore the existence problem in ordinary differential (difference) equations but rarely applied to dynamic equations on general time scales. In [13], Bohner et al. introduce the Mawhin's continuation theorem to explore the existence of periodic solutions in predator-prey and competition dynamic systems, where the authors established some suitable sufficient criteria by defining some operators on time scales.

In [14], Li and Zhang have studied the periodic solutions for a periodic mutualism model

$$\begin{aligned} x^{\Delta}(t) &= r_1(t)\left[\frac{k_1(t) + \alpha_1(t)\exp\{y(t-\tau_2(t, y(t)))\}}{1 + \exp\{y(t-\tau_2(t, y(t)))\}} - \exp\{x(t-\sigma_1(t, x(t)))\}\right], \\ y^{\Delta}(t) &= r_2(t)\left[\frac{k_2(t) + \alpha_2(t)\exp\{x(t-\tau_1(t, y(t)))\}}{1 + \exp\{x(t-\tau_1(t, x(t)))\}} - \exp\{y(t-\sigma_2(t, y(t)))\}\right] \end{aligned} \tag{1.4}$$

on a time scale $\mathbb{T}$ by employing Mawhin's continuation theorem, and have obtained three sufficient criteria.

Combining Brouwer's fixed point theorem with Horn's fixed point theorem, two classes of one-order linear dynamic equations on time scales

$$\begin{gathered} x^{\Delta}(t) = a(t)x(t) + h(t), \\ x^{\Delta}(t) = f(t,x), \text{ with the initial condition } x(t_0) = x_0, \end{gathered} \tag{1.5}$$

are considered in [15] by Liu and Li. The authors presented some interesting properties of the exponential function on time scales and obtain a sufficient and necessary condition that guarantees the existence of the periodic solutions of the equation $x^{\Delta}(t) = a(t)x(t) + h(t)$.

In [16], Bohner et al. consider the system

$$x^{\Delta}(t) = G\left(t, \exp\{x(g_1(t))\}, \exp\{x(g_2(t))\}, \ldots, \exp\{x(g_n(t))\}, \int_{-\infty}^{t} c(t,s)\exp\{x(s)\}\Delta s\right), \tag{1.6}$$

easily verifiable sufficient criteria are established for the existence of periodic solutions of this class of nonautonomous scalar dynamic equations on time scales, the approach that authors used in this paper is based on Mawhin's continuation theorem.

In this paper, we consider the existence of periodic solutions for p-Laplacian equations on a time scales $\mathbb{T}$

$$\left(\varphi_p\left(x^{\Delta}(t)\right)\right)^{\Delta} + f(x(t))x^{\Delta}(t) + g(x(t)) = e(t), \quad t \in \mathbb{T}, \tag{1.7}$$

where $p > 2$ is a constant, $\varphi_p(s) = |s|^{p-2}s$, $f, g \in C(R,R)$, $e \in C(\mathbb{T}, R)$, and e is a function with periodic $\omega > 0$. $\mathbb{T}$ is a periodic time scale which has the subspace topology inherited from the standard topology on R. Sufficient criteria are established for the existence of periodic solutions for such equations, which generalize many known results for differential equations when the time scales are chosen as the set of the real numbers. The main method is based on the Mawhin's continuation theorem.

If $\mathbb{T} = R$, (1.7) reduces to the differential equation

$$(\varphi_p(x'(t)))' + f(x(t))x'(t) + g(x(t)) = e(t). \tag{1.8}$$

We will use Mawhin's continuation theorem to study (1.7).

2. Preliminaries

In this section, we briefly give some basic definitions and lemmas on time scales which are used in what follows. Let $\mathbb{T}$ be a time scale (a nonempty closed subset of R). The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to R^{+}$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \qquad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \qquad \mu(t) = \sigma(t) - t. \tag{2.1}$$

We say that a point $t \in \mathbb{T}$ is left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$. If $t < sup\,\mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense. A point $t \in \mathbb{T}$ is called left-scattered if $\rho(t) < t$, while right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then we set $\mathbb{T}^k = \mathbb{T} \setminus \{m\}$, otherwise set $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then set $\mathbb{T}_k = \mathbb{T} \setminus \{m\}$, otherwise set $\mathbb{T}_k = \mathbb{T}$.

A function $f : \mathbb{T} \to R$ is right-dense continuous (rd-continuous) provided that it is continuous at right-dense point in $\mathbb{T}$ and its left side limits exist at left-dense points in $\mathbb{T}$. If f is continuous at each right-dense point and each left-dense point, then f is said to be continuous function on $\mathbb{T}$.

Definition 2.1 (see [10]). Assume $f : \mathbb{T} \to R$ is a function and let $t \in \mathbb{T}^k$. We define $f^{\Delta}(t)$ to be the number (if it exists) with the property that for a given $\varepsilon > 0$, there exists a neighborhood U of t such that

$$\left|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]\right| < \varepsilon|\sigma(t) - s|, \quad \text{for all } s \in U. \tag{2.2}$$

We call $f^{\Delta}(t)$ the delta derivative of f at t.

If f is continuous, then f is right-dense continuous, and if f is delta differentiable at t, then f is continuous at t.

Let f be right-dense continuous. If $F^{\Delta}(t) = f(t)$, for all $t \in \mathbb{T}$, then we define the delta integral by

$$\int_a^t f(s)\Delta s = F(t) - F(a), \quad \text{for } t, a \in \mathbb{T}. \tag{2.3}$$

Definition 2.2 (see [12]). We say that a time scale $\mathbb{T}$ is periodic if there is $p > 0$ such that if $t \in \mathbb{T}$, then $t \pm p \in \mathbb{T}$. For $\mathbb{T} \neq R$, the smallest positive p is called the period of the time scale.

Definition 2.3 (see [12]). Let $\mathbb{T} \neq R$ be a periodic time scale with period p. We say that the function $f : \mathbb{T} \to R$ is periodic with period ω if there exists a natural number n such that $\omega = np$, $f(t + \omega) = f(t)$ for all $t \in \mathbb{T}$, and ω is the smallest number such that $f(t + \omega) = f(t)$. If $\mathbb{T} = R$, we say that f is periodic with period $\omega > 0$ if ω is the smallest positive number such that $f(t + \omega) = f(t)$ for all $t \in \mathbb{T}$.

Lemma 2.4 (see [10]). *If $a, b \in \mathbb{T}$, $\alpha, \beta \in R$, and $f, g \in C(\mathbb{T}, R)$, then*

(*A*1) $\int_a^b [\alpha f(t) + \beta g(t)]\Delta t = \alpha \int_a^b f(t)\Delta t + \beta \int_a^b g(t)\Delta t$;

(*A*2) *if $f(t) \geq 0$ for all $a \leq t < b$, then $\int_a^b f(t)\Delta t \geq 0$;*

(*A*3) *if $|f(t)| \leq g(t)$ on $[a, b) := \{t \in \mathbb{T} : a \leq t < b\}$, then $|\int_a^b f(t)\Delta t| \leq \int_a^b g(t)\Delta t$.*

Lemma 2.5 (Hölder's inequality [11]). *Let $a,\ b \in \mathbb{T}$. For rd-continuous functions $f,\ g : [a,b] \to R$, one has*

$$\int_a^b |f(t)g(t)|\Delta t \le \left(\int_a^b |f(t)|^p \Delta t\right)^{1/p} \left(\int_a^b |g(t)|^q \Delta t\right)^{1/q}, \tag{2.4}$$

where $p > 1$ and $q = p/(p-1)$.

For convenience, we denote

$$\kappa = \min\{[0,\infty) \cap \mathbb{T}\}, \quad I_\omega = [\kappa, \kappa+\omega] \cap \mathbb{T}, \quad \overline{g} = \frac{1}{\omega}\int_{I_\omega} g(s)\Delta s = \frac{1}{\omega}\int_\kappa^{\kappa+\omega} g(s)\Delta s, \tag{2.5}$$

where $g \in C(\mathbb{T}, R)$ is an ω-periodic real function, that is, $g(t+\omega) = g(t)$ for all $t \in \mathbb{T}$.

Next, let us recall the continuation theorem in coincidence degree theory. To do so, we introduce the following notations.

Let X, Y be real Banach spaces, $L : \operatorname{Dom} L \subset X \to Y$ a linear mapping, $N : X \to Y$ a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim\operatorname{Ker} L = \operatorname{codim}\operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in Y. If L is a Fredholm mapping of index zero and there exist continuous projections $P : X \to X,\ Q : Y \to Y$ such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I-Q)$, then it follows that $L|_{\operatorname{Dom} L \cap \operatorname{Ker} P} : (I-P)X \to \operatorname{Im} L$ is invertible. We denote the inverse of that map by K_P. If Ω is an open bounded subset of X, the mapping N will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N : \overline{\Omega} \to X$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$.

Lemma 2.6 (continuation theorem). *Suppose that X and Y are two Banach spaces, and $L : \operatorname{Dom} L \subset X \to Y$ is a Fredholm operator of index 0. Furthermore, let $\Omega \subset X$ be an open bounded set and $N : \overline{\Omega} \to Y$ L-compact on $\overline{\Omega}$. If*

- $(B1)$ $Lx \ne \lambda Nx$, *for all* $x \in \partial\Omega \cap \operatorname{Dom} L, \lambda \in (0,1)$,
- $(B2)$ $Nx \notin \operatorname{Im} L$, *for all* $x \in \partial\Omega \cap \operatorname{Ker} L$,
- $(B3)$ $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \ne 0$, *where* $J : \operatorname{Im} Q \to \operatorname{Ker} L$ *is an isomorphism,*

then the equation $Lx = Nx$ has at least one solution in $\overline{\Omega} \cap \operatorname{Dom} L$.

Lemma 2.7 (see [13]). *Let $t_1, t_2 \in I_\omega$ and $t \in \mathbb{T}$. If $g : \mathbb{T} \to R$ is ω-periodic, then*

$$g(t) \le g(t_1) + \int_\kappa^{\kappa+\omega} \left|g^\Delta(s)\right|\Delta s, \qquad g(t) \ge g(t_2) - \int_\kappa^{\kappa+\omega} \left|g^\Delta(s)\right|\Delta s. \tag{2.6}$$

In order to use Mawhin's continuation theorem to study the existence of ω-periodic solutions for (1.7), we consider the following system:

$$\begin{aligned} x_1^\Delta(t) &= \varphi_q(x_2(t)) = |x_2(t)|^{q-2}x_2(t), \\ x_2^\Delta(t) &= -f(x_1(t))\varphi_q(x_2(t)) - g(x_1(t)) + e(t), \end{aligned} \tag{2.7}$$

where $1 < q < 2$ is a constant with $1/p + 1/q = 1$. Clearly, if $x(t) = (x_1(t), x_2(t))^{\top}$ is an ω-periodic solution to (2.7), then $x_1(t)$ must be an ω-periodic solution to (1.7). Thus, in order to prove that (1.7) has an ω-periodic solution, it suffices to show that (2.7) has an ω-periodic solution.

Now, we set $\Psi^{\omega} = \{(u, v) \in C(\mathbb{T}, R^2) : u(t+\omega) = u(t),\ v(t+\omega) = v(t),\ \text{for all}\ \ t \in \mathbb{T}\}$ with the norm $\|(u, v)\| = \max_{t\in I_{\omega}}|u(t)| + \max_{t\in I_{\omega}}|v(t)|,\ \text{for}\ (u, v) \in \Psi^{\omega}$. It is easy to show that Ψ^{ω} is a Banach space when it is endowed with the above norm $\|\cdot\|$.

Let

$$\begin{aligned}
\Psi_0^{\omega} &= \{(u, v) \in \Psi^{\omega} : \overline{u} = 0, \overline{v} = 0\}, \\
\Psi_c^{\omega} &= \left\{(u, v) \in \Psi^{\omega} : (u(t), v(t)) \equiv (h_1, h_2) \in R^2,\ \text{for}\ t \in \mathbb{T}\right\}.
\end{aligned} \tag{2.8}$$

Then it is easy to show that Ψ_0^{ω} and Ψ_c^{ω} are both closed linear subspaces of Ψ^{ω}. We claim that $\Psi^{\omega} = \Psi_0^{\omega} \oplus \Psi_c^{\omega}$, and $\dim \Psi_c^{\omega} = 2$. Since for any $(u, v) \in \Psi_0^{\omega} \cap \Psi_c^{\omega}$, we have $(u(t), v(t)) = (h_1, h_2) \in R^2$, and

$$\overline{u} = \frac{1}{\omega}\int_{\kappa}^{\kappa+\omega} u(s)\Delta s = h_1 = 0, \qquad \overline{v} = \frac{1}{\omega}\int_{\kappa}^{\kappa+\omega} v(s)\Delta s = h_2 = 0, \tag{2.9}$$

so we obtain $(u, v) = (h_1, h_2) = (0, 0)$.

Take $X = Y = \Psi^{\omega}$. Define

$$L : \operatorname{Dom} L = \left\{x = (x_1, x_2)^{\top} \in C^1\left(\mathbb{T}, R^2\right) : x(t+\omega) = x(t), x^{\Delta}(t+\omega) = x^{\Delta}(t)\right\} \subset X \longrightarrow Y, \tag{2.10}$$

by

$$Lx(t) = x^{\Delta}(t) = \begin{pmatrix} x_1^{\Delta}(t) \\ x_2^{\Delta}(t) \end{pmatrix}, \tag{2.11}$$

and $N : X \to Y$, by

$$Nx(t) = \begin{pmatrix} \varphi_q(x_2(t)) \\ -f(x_1(t))\varphi_q(x_2(t)) - g(x_1(t)) + e(t) \end{pmatrix}. \tag{2.12}$$

Define the operator $P : X \to X$ and $Q : Y \to Y$ by

$$Px = P\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \end{pmatrix}, \quad Qy = Q\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \overline{y}_1 \\ \overline{y}_2 \end{pmatrix}, \quad x \in X,\ y \in Y. \tag{2.13}$$

It is easy to see that (2.7) can be converted to the abstract equation $Lx = Nx$.

Then $\operatorname{Ker} L = \Psi_c^{\omega}$, $\operatorname{Im} L = \Psi_0^{\omega}$, and $\dim\operatorname{Ker} L = 2 = \operatorname{codim}\operatorname{Im} L$. Since Ψ_0^{ω} is closed in Ψ^{ω}, it follows that L is a Fredholm mapping of index zero. It is not difficult to show that P and Q are continuous projections such that $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q)$. Furthermore, the generalized inverse (to L_P) $K_P : \operatorname{Im} L \to \operatorname{Ker} P \cap \operatorname{Dom} L$ exists and is given by

$$K_P\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} X_1 - \overline{X_1} \\ X_2 - \overline{X_2} \end{pmatrix}, \quad \text{where } X_i(t) = \int_{\kappa}^{t} x_i(s)\Delta s,\ i = 1, 2. \tag{2.14}$$

Since for every $x \in \operatorname{Ker} P \cap \operatorname{Dom} L$, we have

$$\begin{aligned}
K_P L x(t) = K_P\begin{pmatrix} x_1^{\Delta}(t) \\ x_2^{\Delta}(t) \end{pmatrix} &= \begin{pmatrix} \displaystyle\int_{\kappa}^{t} x_1^{\Delta}(s)\Delta s - \frac{1}{\omega}\int_{\kappa}^{\kappa+\omega}\int_{\kappa}^{t} x_1^{\Delta}(s)\Delta s\Delta t \\ \displaystyle\int_{\kappa}^{t} x_2^{\Delta}(s)\Delta s - \frac{1}{\omega}\int_{\kappa}^{\kappa+\omega}\int_{\kappa}^{t} x_2^{\Delta}(s)\Delta s\Delta t \end{pmatrix} \\
&= \begin{pmatrix} x_1(t) - x_1(\kappa) - \displaystyle\frac{1}{\omega}\int_{\kappa}^{\kappa+\omega}(x_1(t) - x_1(\kappa))\Delta t \\ x_2(t) - x_2(\kappa) - \displaystyle\frac{1}{\omega}\int_{\kappa}^{\kappa+\omega}(x_2(t) - x_2(\kappa))\Delta t \end{pmatrix} \\
&= \begin{pmatrix} x_1(t) - \displaystyle\frac{1}{\omega}\int_{\kappa}^{\kappa+\omega} x_1(t)\Delta t \\ x_2(t) - \displaystyle\frac{1}{\omega}\int_{\kappa}^{\kappa+\omega} x_2(t)\Delta t \end{pmatrix},
\end{aligned} \tag{2.15}$$

from the definition of P and the condition that $x \in \operatorname{Ker} P \cap \operatorname{Dom} L$, then $(1/\omega)\int_{\kappa}^{\kappa+\omega} x_1(t)\Delta t = (1/\omega)\int_{\kappa}^{\kappa+\omega} x_2(t)\Delta t = 0$. Thus, we get $K_P L x(t) = x(t)$. Similarly, we can prove that $LK_P x(t) = x(t)$, for every $x(t) \in \operatorname{Im} L$. So the operator K_P is well defined. Thus,

$$QN\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \displaystyle\frac{1}{\omega}\int_{\kappa}^{\kappa+\omega} \varphi_q(x_2(s))\Delta s \\ \displaystyle\frac{1}{\omega}\int_{\kappa}^{\kappa+\omega} \left[-f(x_1(s))\varphi_q(x_2(s)) - g(x_1(s)) + e(s)\right]\Delta s \end{pmatrix}. \tag{2.16}$$

Denote $Nx_1 = N_1,\ Nx_2 = N_2$. We have

$$K_P(I-Q)N\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$
$$=\begin{pmatrix}\displaystyle\int_\kappa^t\left[N_1(s)-\frac{1}{\omega}\int_\kappa^{\kappa+\omega}N_1(r)\Delta r\right]\Delta s-\frac{1}{\omega}\int_\kappa^{\kappa+\omega}\int_\kappa^t\left[N_1(s)-\frac{1}{\omega}\int_\kappa^{\kappa+\omega}N_1(r)\Delta r\right]\Delta s\Delta t\\ \displaystyle\int_\kappa^t\left[N_2(s)-\frac{1}{\omega}\int_\kappa^{\kappa+\omega}N_2(r)\Delta r\right]\Delta s-\frac{1}{\omega}\int_\kappa^{\kappa+\omega}\int_\kappa^t\left[N_2(s)-\frac{1}{\omega}\int_\kappa^{\kappa+\omega}N_2(r)\Delta r\right]\Delta s\Delta t\end{pmatrix}. \tag{2.17}$$

Clearly, QN and $K_P(I-Q)N$ are continuous. Since X is a Banach space, it is easy to show that $\overline{K_P(I-Q)N(\overline{\Omega})}$ is a compact for any open bounded set $\Omega\subset X$. Moreover, $QN(\overline{\Omega})$ is bounded. Thus, N is L-compact on $\overline{\Omega}$.

3. Main Results

In this section, we present our main results.

Theorem 3.1. *Suppose that there exist positive constants d_1 and d_2 such that the following conditions hold:*

(i) $u(\sigma(t))u^\Delta(t)f(u(t))<0, |u(\sigma(t))|>d_1,\ t\in\mathbb{T}$,

(ii) $u(\sigma(t))(g(u(t))-e(t))<0, |u(\sigma(t))|>d_2,\ t\in\mathbb{T}$,

then (1.7) *has at least one ω-periodic solution.*

Proof. Consider the equation $Lx=\lambda Nx,\ \lambda\in(0,1)$, where L and N are defined by the second section. Let $\Omega_1=\{x\in X: Lx=\lambda Nx,\ \lambda\in(0,1)\}$.

If $x=\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix}\in\Omega_1$, then we have

$$\begin{aligned}x_1^\Delta(t)&=\lambda\varphi_q(x_2(t)),\\ x_2^\Delta(t)&=-f(x_1(t))x_1^\Delta(t)-\lambda g(x_1(t))+\lambda e(t).\end{aligned}\tag{3.1}$$

From the first equation of (3.1), we obtain $x_2(t)=\varphi_p((1/\lambda)x_1^\Delta(t))$, and then by substituting it into the second equation of (3.1), we get

$$\left[\varphi_p\left(\frac{1}{\lambda}x_1^\Delta(t)\right)\right]^\Delta=-f(x_1(t))x_1^\Delta(t)-\lambda g(x_1(t))+\lambda e(t). \tag{3.2}$$

Integrating both sides of (3.2) from κ to $\kappa+\omega$, noting that $x_1(\kappa) = x_1(\kappa+\omega)$, $x_1^{\Delta}(\kappa) = x_1^{\Delta}(\kappa+\omega)$, and applying Lemma 2.4, we have

$$\int_{\kappa}^{\kappa+\omega} f(x_1(t))x_1^{\Delta}(t)\Delta t = -\int_{\kappa}^{\kappa+\omega} \left[g(x_1(t)) - e(t)\right]\Delta t, \tag{3.3}$$

that is,

$$\int_{\kappa}^{\kappa+\omega} \left[f(x_1(t))x_1^{\Delta}(t) + g(x_1(t)) - e(t)\right]\Delta t = 0. \tag{3.4}$$

There must exist $\xi \in I_\omega$ such that

$$f(x_1(\xi))x_1^{\Delta}(\xi) + g(x_1(\xi)) - e(\xi) \geq 0. \tag{3.5}$$

From conditions (i) and (ii), when $x(\sigma(\xi)) > \max\{d_1, d_2\}$, we have $f(x_1(\xi))x_1^{\Delta}(\xi) < 0$, and $g(x_1(\xi)) - e(\xi) < 0$, which contradicts to (3.5). Consequently $x(\sigma(\xi)) \leq \max\{d_1, d_2\}$. Similarly, there must exist $\eta \in I_\omega$ such that

$$f(x_1(\eta))x_1^{\Delta}(\eta) + g(x_1(\eta)) - e(\eta) \leq 0. \tag{3.6}$$

Then we have $x(\sigma(\eta)) \geq -\max\{d_1, d_2\}$. Applying Lemma 2.7, we get

$$-\max\{d_1, d_2\} - \int_{\kappa}^{\kappa+\omega} \left|x_1^{\Delta}(s)\right|\Delta s \leq x_1(t) \leq \max\{d_1, d_2\} + \int_{\kappa}^{\kappa+\omega} \left|x_1^{\Delta}(s)\right|\Delta s. \tag{3.7}$$

Let $d = \max\{d_1, d_2\}$. Then (3.7) equals to the following inequality:

$$|x_1(t)| \leq d + \int_{\kappa}^{\kappa+\omega} \left|x_1^{\Delta}(s)\right|\Delta s. \tag{3.8}$$

Let $E_1 = \{t \in I_\omega : |x_1(t)| \leq d\}$, $E_2 = \{t \in I_\omega : |x_1(t)| > d\}$.

Consider the second equation of (3.1) and (3.8), then we have

$$\begin{aligned}
\int_{\kappa}^{\kappa+\omega} x_1^{\Delta}(t)x_2(t)\Delta t &= -\int_{\kappa}^{\kappa+\omega} x_1(\sigma(t))x_2^{\Delta}(t)\Delta t \\
&= \int_{\kappa}^{\kappa+\omega} f(x_1(t))x_1^{\Delta}(t)x_1(\sigma(t))\Delta t + \lambda\int_{\kappa}^{\kappa+\omega} x_1(\sigma(t))\left[g(x_1(t)) - e(t)\right]\Delta t \\
&\le \int_{\kappa}^{\kappa+\omega} \left|f(x_1(t))\right|\left|x_1^{\Delta}(t)\right||x_1(\sigma(t))|\Delta t + \lambda\int_{E_1} x_1(\sigma(t))\left[g(x_1(t)) - e(t)\right]\Delta t \\
&\quad + \lambda\int_{E_2} x_1(\sigma(t))\left[g(x_1(t)) - e(t)\right]\Delta t \\
&\le \sup_{t\in I_\omega}\left|f(x_1(t))\right|\left(d + \int_{\kappa}^{\kappa+\omega}\left|x_1^{\Delta}(t)\right|\Delta t\right)\int_{\kappa}^{\kappa+\omega}\left|x_1^{\Delta}(t)\right|\Delta t \\
&\quad + \lambda\int_{E_1} x_1(\sigma(t))\left[g(x_1(t)) - e(t)\right]\Delta t \\
&\le \sup_{t\in I_\omega}\left|f(x_1(t))\right|\left(\int_{\kappa}^{\kappa+\omega}\left|x_1^{\Delta}(t)\right|\Delta t\right)^2 + d\sup_{t\in I_\omega}\left|f(x_1(t))\right|\int_{\kappa}^{\kappa+\omega}\left|x_1^{\Delta}(t)\right|\Delta t \\
&\quad + \lambda\int_{E_1} x_1(\sigma(t))\left[g(x_1(t)) - e(t)\right]\Delta t.
\end{aligned} \tag{3.9}$$

Applying Lemma 2.5, we obtain that

$$\begin{aligned}
\frac{1}{\lambda^{p-1}}\int_{\kappa}^{\kappa+\omega}\left|x_1^{\Delta}(t)\right|^p\Delta t &\le \omega\sup_{t\in I_\omega}\left|f(x_1(t))\right|\int_{\kappa}^{\kappa+\omega}\left|x_1^{\Delta}(t)\right|^2\Delta t + d\sup_{t\in I_\omega}\left|f(x_1(t))\right|\int_{\kappa}^{\kappa+\omega}\left|x_1^{\Delta}(t)\right|\Delta t \\
&\quad + \lambda\left(d + \int_{\kappa}^{\kappa+\omega}\left|x_1^{\Delta}(t)\right|\Delta t\right)\int_{\kappa}^{\kappa+\omega}\left|g(x_1(t)) - e(t)\right|\Delta t \\
&\le Q_1\int_{\kappa}^{\kappa+\omega}\left|x_1^{\Delta}(t)\right|^2\Delta t + Q_2\int_{\kappa}^{\kappa+\omega}\left|x_1^{\Delta}(t)\right|\Delta t \\
&\quad + \lambda d\omega\sup_{t\in I_\omega}\left|g(x_1(t)) - e(t)\right| \\
&\le Q_1\int_{\kappa}^{\kappa+\omega}\left|x_1^{\Delta}(t)\right|^2\Delta t + Q_2\int_{\kappa}^{\kappa+\omega}\left|x_1^{\Delta}(t)\right|\Delta t + Q_3,
\end{aligned} \tag{3.10}$$

where

$$Q_1 = \omega \sup_{t\in I_\omega}\left|f(x_1(t))\right|, \qquad Q_2 = d\sup_{t\in I_\omega}\left|f(x_1(t))\right| + \lambda\omega \sup_{t\in I_\omega}\left|g(x_1(t)) - e(t)\right|,$$
$$Q_3 = \lambda d\omega \sup_{t\in I_\omega}\left|g(x_1(t)) - e(t)\right|. \tag{3.11}$$

That is,

$$\frac{1}{\lambda^{p-1}}\int_\kappa^{\kappa+\omega}\left|x_1^\Delta(t)\right|^p \Delta t \le Q_1\int_\kappa^{\kappa+\omega}\left|x_1^\Delta(t)\right|^2 \Delta t + Q_2\int_\kappa^{\kappa+\omega}\left|x_1^\Delta(t)\right|\Delta t + Q_3. \tag{3.12}$$

Thus,

$$\int_\kappa^{\kappa+\omega}\left|x_1^\Delta(t)\right|^p \Delta t \le \lambda^{p-1}Q_1\omega^{(p-2)/p}\left(\int_\kappa^{\kappa+\omega}\left|x_1^\Delta(t)\right|^p \Delta t\right)^{2/p}$$
$$+ \lambda^{p-1}Q_2\omega^{(p-1)/p}\left(\int_\kappa^{\kappa+\omega}\left|x_1^\Delta(t)\right|^p \Delta t\right)^{1/p} + \lambda^{p-1}Q_3. \tag{3.13}$$

Since $p > 2$, then we obtain that there exists a positive constant M_1 such that

$$\left|x_1^\Delta(t)\right| \le M_1. \tag{3.14}$$

Therefore,

$$|x_1(t)| \le d + M_1\omega := M_2, \qquad |x_2(t)| \le \frac{M_2{}^{p-1}}{\lambda^{p-1}} := M_3. \tag{3.15}$$

Let $\Omega_2 = \{x : x \in \operatorname{Ker} L,\ QNx = 0\}$. If $x \in \Omega_2$, then $x \in R^2$ is a constant vector with

$$|x_2(t)|^{q-2}x_2(t) = 0,$$
$$\frac{1}{\omega}\int_\kappa^{\kappa+\omega}\left[f(x_1(t))x_1^\Delta(t) + g(x_1(t)) - e(t)\right]\Delta t = 0. \tag{3.16}$$

From the second equation of (3.16) we get

$$\int_\kappa^{\kappa+\omega} f(x_1(t))x_1^\Delta(t)\Delta t = -\int_\kappa^{\kappa+\omega}\left[g(x_1(t)) - e(t)\right]\Delta t, \tag{3.17}$$

that is,

$$\int_\kappa^{\kappa+\omega}\left[f(x_1(t))x_1^\Delta(t) + g(x_1(t)) - e(t)\right]\Delta t = 0. \tag{3.18}$$

By assumptions (i) and (ii), we see that $|x_1(t)| \leq M_2$ and $x_2(t) = 0$, which implies $\Omega_2 \subset \Omega_1$.

Now, we set $\Omega = \{x : x = (x_1, x_2)^\top,\ |x_1| < M_2 + 1, |x_2| < M_3 + 1\}$. Then $\Omega_1 \subset \Omega$. Thus from (3.8) and (3.14), we see that conditions *(B1)* and *(B2)* of Lemma 2.6 are satisfied. The remainder is verifying condition *(B3)* of Lemma 2.6. In order to do it, let

$$J : \operatorname{Im} Q \longrightarrow \operatorname{Ker} L, \qquad J(x_1, x_2) = (x_1, x_2). \tag{3.19}$$

Set

$$\Delta_0 = \left\{x = (x_1, x_2)^\top \in R^2 : |x_1| < M_2 + 1,\ x_2 = 0\right\}. \tag{3.20}$$

It is easy to see that the equation $QN(x_1, x_2)^\top(t) = (0, 0)^\top$, that is,

$$\begin{gathered} \varphi_q(x_2(t)) = 0, \\ \frac{1}{\omega}\int_\kappa^{\kappa+\omega} \left[f(x_1(t))\varphi_q(x_2(t)) + g(x_1(t)) - e(t)\right]\Delta t = 0, \end{gathered} \tag{3.21}$$

has no solution in $\overline{(\Omega \cap \operatorname{Ker} L) \setminus \Delta_0}$. So $\deg\{JQN,\ \Omega \cap \operatorname{Ker}, 0\} = \deg\{JQN, \Delta_0,\ 0\}$.

Let

$$QN_0 x(t) = \begin{pmatrix} 0 \\ \dfrac{1}{\omega}\displaystyle\int_\kappa^{\kappa+\omega} \left[g(x_1(t)) - e(t)\right]\Delta t \end{pmatrix}. \tag{3.22}$$

If $x \in \partial\Delta_0$, then we get

$$\|JQN_0 x - JQNx\| = \max_{x_2=0,\ |x_1|=M_2+1} \left|\varphi_q(x_2)\right| + \frac{1}{\omega}\max_{x_2=0,\ |x_1|=M_2+1}\left|\int_\kappa^{\kappa+\omega} f(x_1(t))\varphi_q(x_2)\Delta t\right| = 0, \tag{3.23}$$

so we have

$$\deg\{JQN, \Delta_0, 0\} = \deg\{JQN_0, \Delta_0, 0\} \neq 0. \tag{3.24}$$

Then we see that

$$\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} = \deg\{JQN_0, \Delta_0, 0\} \neq 0, \tag{3.25}$$

so the condition *(B3)* of Lemma 2.6 is satisfied, the proof is complete. □

When $\int_\kappa^{\kappa+\omega} e(t)\Delta t = 0,\ g(x(t)) = \beta(t)x(t)$, where $\beta(t) = \beta(t + T),\ t \in [0, T]$, we have the following result.

Corollary 3.2. *Suppose that the following conditions hold:*

(i) $\beta(t) > 0$, *for all* $t \in I_\omega$;

(ii) $u(t)u^{\Delta}(t)f(u(t)) > 0, |u| > d$,

then (1.7) *has at least one* ω*-periodic solution.*

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