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Research Article

A Note on (h, q)-Genocchi Polynomials and Numbers of Higher Order

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We investigate several arithmetic properties of (h, q)-Genocchi polynomials and numbers of higher order.

1. Introduction and Preliminaries

Recently, Kim [1] studied q-Genocchi and Euler numbers using Fermionic q-integral and introduced related applications. Kim [2] also gives the q-extensions of the Euler numbers which can be viewed as interpolating of q-analogue of Euler zeta function at negative integers and gives Bernoulli numbers at negative integers by interpolating Riemann zeta function. These numbers are very useful for number theory and mathematical physics. Kim [3, 4] studied q-Bernoulli numbers and polynomials related to Gaussian binomial coefficient and studied also some identities of q-Euler polynomials and q-stirling numbers. Kim [5] made Dedekind DC sum in the meaning of extension of Dedekind sum or Hardy sum and introduced lots of interesting results. The purpose of this paper is to investigate several arithmetic properties of (h, q)-Genocchi polynomials and numbers of higher order.

Let p be a fixed odd prime. Throughout this paper $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will, respectively, denote the ring of rational integers, the ring of p-adic rational integers, the field of p-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = p^{-1}$. When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in C$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume $|q-1|_p < p^{-1/(p-1)}$, so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. We also use the notations

$$[x]_q = \frac{1-q^x}{1-q}, \qquad [x]_{-q} = \frac{1-(-q)^x}{1+q} \tag{1.1}$$

for all $x \in \mathbb{Z}_p$ (see [5–12]). Hence, $\lim_{q \to 1}[x]_q = x$.

Let d be a fixed positive integer with $(p,d)=1$. We now set

$$X = \varprojlim_{N} \frac{\mathbb{Z}}{dp^N\mathbb{Z}}, \qquad X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a+dp\mathbb{Z}_p, \qquad a+dp^N\mathbb{Z}_p = \left\{x \in X \mid x \equiv a \left(\bmod p^N\right)\right\}, \tag{1.2}$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$. For any $N \in \mathbb{N}$, we set

$$\mu_q\left(a + dp^N\mathbb{Z}_p\right) = \frac{(-q)^a}{\left[dp^N\right]_{-q}}, \tag{1.3}$$

and this can be extended to a distribution on $\mathbb{Z}_p$.

We say that f is a uniformly differentiable function at a point $a \in \mathbb{Z}_p$ and write $f \in UD(\mathbb{Z}_p)$, if the difference quotients $F_f(x,y) = ((f(x)-f(y))/(x-y))$ have a limit $f'(a)$ as $(x,y) \to (a,a)$ (cf. [13–23]).

For $f \in UD(\mathbb{Z}_p)$, the p-adic invariant integral on $\mathbb{Z}_p$ is defined as

$$I(f) = \int_X f(x)d\mu(x) = \lim_{N\to\infty} \sum_{x=0}^{dp^N-1} f(x)(-1)^x \tag{1.4}$$

(see [14, 23]). Let $n \in N$ and $f_n(x) = f(x+n)$. From (1.4), we have

$$I(f_n) + I(f) = 2\sum_{l=0}^{n-1}(-1)^l f(l). \tag{1.5}$$

The p-adic integral has been used in many areas such as mathematics, physics, probability theory, dynamical systems, and biological models. Especially, Khrennikov [24–26] applied to many areas using ingenious technique. The Genocchi numbers G_n and polynomials $G_n(x)$ are defined by the generating functions as follows:

$$\frac{2t}{e^t+1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}, \qquad \frac{2t}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!} \tag{1.6}$$

(see [5, 7, 15]). The q-extension of Genocchi numbers are defined by

$$F_q(t) = t\sum_{m=0}^{\infty}(-1)^m q^m e^{[m]_q t} = \sum_{n=0}^{\infty} G_{n,q}\frac{t^n}{n!} \tag{1.7}$$

(see [1, 2]), and the q-extension of Genocchi polynomials is also given by

$$G_{n,q}(x)=\sum_{l=0}^{n}\binom{n}{l}q^{lx}G_{l,q}[x]_q^{n-l}. \tag{1.8}$$

In Section 2, we investigate several arithmetic properties of (h,q)-Genocchi polynomials and numbers of higher order.

2. (h,q)-Genocchi Numbers of Higher Order

Let $h,k\in N$ and $q\in C$ with $|q-1|_p<p^{-1/(p-1)}$. The (h,q)-Genocchi polynomials $G_{m,q}^{(h,k)}(x)$ of order k are defined as

$$t^k\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}e^{[x_1+\cdots x_k+x]_q t}q^{(h-1)x_1+\cdots+(h-k)x_k}d\mu_q(x_1)\cdots d\mu_q(x_k)=\sum_{m=0}^{\infty}G_{m,q}^{(h,k)}(x)\frac{t^m}{m!}, \tag{2.1}$$

where $\int_{\mathbb{Z}_p}f(x)d\mu_q(x)=\lim_{N\to\infty}(1/[p^N]_{-q})\sum_{x=0}^{p^N-1}f(x)(-q)^x$. It is easily to see that $G_{0,q}^{(h,k)}(x)=\cdots=G_{k-1,q}^{(h,k)}(x)=0$ for each $h\in\mathbb{Z}$ and $k\in\mathbb{N}$. From (2.1), we can obtain the following theorem.

Theorem 2.1. *Let $h\in\mathbb{Z}$ and $m,k\in\mathbb{N}$. Then for all $x\in\mathbb{Z}_p$,*

$$\frac{k!}{(m+k)!}G_{m+k,q}^{(h,k)}(x)=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x_1+\cdots x_k+x]_q^m q^{(h-1)x_1+\cdots+(h-k)x_k}d\mu_q(x_1)\cdots d\mu_q(x_k). \tag{2.2}$$

From Theorem 2.1, if we take $k=-m$ $(m>0)$, then

$$\frac{1}{m!\binom{m+k}{m}}G_{m+k,q}^{(-m,k)}(x)=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x_1+\cdots x_k+x]_q^m q^{-(m+1)x_1-\cdots-(m+k)x_k}d\mu_q(x_1)\cdots d\mu_q(x_k). \tag{2.3}$$

Now, we define (h,q)-Genocchi number of higher order as follows:

$$G_{m,q}^{(-m,k)}=G_{m,q}^{(-m,k)}(0). \tag{2.4}$$

From (2.4), we can derive the following theorem.

Theorem 2.2. *Let $h \in \mathbb{Z}$ and $m, k \in \mathbb{N}$. Then one has*

$$\begin{aligned}\frac{G_{m+k,q}^{(-m,k)}}{m!\binom{m+k}{m}} &= \lim_{N\to\infty}\frac{1}{[p^N]_{-q}^k}\sum_{x_1=0}^{p^N-1}\cdots\sum_{x_k=0}^{p^N-1}(-1)^{x_1+\cdots+x_k}[x_1+\cdots+x_k]_q^m q^{-x_1 m-\cdots-x_k(m+k-1)} \\ &= \frac{[2]_q^k}{(1-q)^m}\sum_{i=0}^{m}\binom{m}{i}(-1)^i\frac{1}{(1+q^{i-m})\cdots(1+q^{i-m-k+1})},\end{aligned} \tag{2.5}$$

where $\binom{m}{i} = m\cdots(m-k+1)/k!$.

Note that $\lim_{q\to1}G_{m,q}^{(-m,k)} = G_m^{(k)}$, where $G_m^{(k)}$ are the ordinary Genocchi numbers of order k defined as

$$\left(\frac{2t}{e^t+1}\right)^k = \sum_{n=0}^{\infty}G_n^{(k)}\frac{t^n}{n!}. \tag{2.6}$$

By (2.4) and (2.5), we can obtain the following theorem.

Theorem 2.3. *Let $m \in \mathbb{N}$. Then one has*

$$\frac{G_{m+1,q}^{(-m,1)}}{m+1} = \sum_{i=0}^{m}\binom{m}{i}q^{x_i}\frac{G_{i+1,q}^{(-m,1)}}{i+1}[x]_q^{m-i} = \frac{[2]_q}{(1-q)^m}\sum_{j=0}^{m}q^{ix}\binom{m}{i}\frac{(-1)^j}{(1+q^{j-m})}. \tag{2.7}$$

It is easily to check that

$$\begin{aligned}\frac{G_{n+1,q}^{(-n,1)}}{n+1} &= \int_{\mathbb{Z}_p}q^{-(n+1)t}[x+t]_q^n d\mu_q(t) \\ &= \frac{1+q}{1+q^d}[d]_q^n\sum_{x=0}^{d-1}(-1)^i q^{-ni}\int_{\mathbb{Z}_p}q^{-(n+1)dt}\left[\frac{x+i}{d}+t\right]_{q^d}^n d\mu_{q^d}(t),\end{aligned} \tag{2.8}$$

where $n, d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$. Thus we have the following theorem.

Theorem 2.4. *Let $d, n \in \mathbb{N}$ with $d \equiv 1 \pmod 2$. Then for all $x \in \mathbb{Z}_p$,*

$$\frac{G_{n+1,q}^{(-n,1)}(x)}{n+1} = \frac{1+q}{1+q^d}[d]_q^n\sum_{i=0}^{n}(-1)^i q^{-ni}G_{n+1,q}^{-n,1}\left(\frac{x+i}{d}\right). \tag{2.9}$$

We note that if we take $x = 0$, then we have

$$\frac{G_{m+1,q}^{(-m,1)}}{m+1} = \frac{1+q}{1+q^m}\sum_{k=0}^{m}\binom{m}{k}[n]_q^k\frac{G_{k+1,q^n}^{(-m,1)}}{k+1}\sum_{j=0}^{n-1}(-1)^j q^{-(m-k)j}[j]_q^{m-k}, \tag{2.10}$$

where $n \equiv 1 \pmod 2$. By (2.10), we easily see that

$$\frac{G_{m+1,q}^{(-m,1)}}{m+1} - \frac{1+q}{1+q^n}[n]_q^m \frac{G_{m+1,q}^{(-m,1)}}{m+1} = \frac{1+q}{1+q^m}\sum_{k=0}^{m-1}[n]_q^k \frac{G_{k+1,q^n}^{(-m,1)}}{k+1}\sum_{j=0}^{n-1}(-1)^j q^{-(m-k)j}[j]_q^{m-k}. \tag{2.11}$$

Note that $\lim_{q\to 1} G_{m,q}^{(-m,1)} = G_m$, where G_m are the mth Genocchi numbers defined as

$$\frac{2t}{e^t+1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}. \tag{2.12}$$

From (2.11), we can see that

$$(1-n^m)\frac{G_{m+1}}{m+1} = \sum_{k=0}^{m-1}\binom{m}{k} n^k \frac{G_{k+1}}{k+1}\sum_{j=0}^{n-1}(-1)^j j^{m-k}. \tag{2.13}$$

Let $F_q(t,x)$ be the generating function of $G_{m,q}^{(-m,1)}$ as follows:

$$F_q(t,x) = \sum_{n=0}^{\infty} \frac{G_{n+1,q}^{(-n,1)}(x)}{n+1}\frac{t^n}{n!}. \tag{2.14}$$

By (2.7) and (2.14), we see that

$$\begin{aligned} F_q(t,x) &= \sum_{k=0}^{\infty}\left((1+q)\sum_{n=0}^{\infty}(-1)^n q^{-kn}[n+x]_q^k\right)\frac{t^k}{k!} \\ &= (1+q)\sum_{n=0}^{\infty}(-1)^n \sum_{n=0}^{\infty}(-1)^n \sum_{k=0}^{\infty} q^{-kn}[n+x]_q^k \frac{t^k}{k!} \\ &= (1+q)\sum_{n=0}^{\infty}(-1)^n e^{[n+x]_q q^{-n} t}. \end{aligned} \tag{2.15}$$

By (2.14) and (2.15), we can obtain the following theorem.

Theorem 2.5. *Let $m \in \mathbb{N}$. Then for all $x \in \mathbb{Z}_p$,*

$$\frac{G_{m+1,q}^{(-m,1)}(x)}{m+1} = [2]_q \sum_{n=0}^{\infty} q^{-nm}(-1)^n [n+x]_q^m. \tag{2.16}$$

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