Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2010, Article ID 289340, 8 pages
doi:10.1155/2010/289340

Research Article

On the Oscillation of Second-Order Neutral Delay Differential Equations

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Received 8 October 2009; Accepted 10 January 2010

Academic Editor: Toka Diagana

Some new oscillation criteria for the second-order neutral delay differential equation $(r(t)z'(t))' + q(t)x(\sigma(t)) = 0$, $t \geq t_0$ are established, where $\int_{t_0}^{\infty}(1/r(t))dt = \infty$, $z(t) = x(t) + p(t)x(\tau(t))$, $0 \leq p(t) \leq p_0 < \infty$, $q(t) > 0$. These oscillation criteria extend and improve some known results. An example is considered to illustrate the main results.

1. Introduction

Neutral differential equations find numerous applications in natural science and technology. For instance, they are frequently used for the study of distributed networks containing lossless transmission lines; see Hale [1]. In recent years, many studies have been made on the oscillatory behavior of solutions of neutral delay differential equations, and we refer to the recent papers [2–23] and the references cited therein.

This paper is concerned with the oscillatory behavior of the second-order neutral delay differential equation

$$\left(r(t)z'(t)\right)' + q(t)x(\sigma(t)) = 0, \quad t \geq t_0, \tag{1.1}$$

where $z(t) = x(t) + p(t)x(\tau(t))$.

In what follows we assume that

(I_1) $p, q \in C([t_0, \infty), R)$, $0 \le p(t) \le p_0 < \infty$, $q(t) > 0$,

(I_2) $r \in C([t_0, \infty), R)$, $r(t) > 0$, $\int_{t_0}^{\infty}(1/r(t))\mathrm{d}t = \infty$,

(I_3) $\tau, \sigma \in C([t_0, \infty), R)$, $\tau(t) \le t$, $\sigma(t) \le t$, $\tau'(t) = \tau_0 > 0$, $\sigma'(t) > 0$, $\lim_{t\to\infty}\tau(t) = \lim_{t\to\infty}\sigma(t) = \infty$, $\tau(\sigma(t)) = \sigma(\tau(t))$, where τ_0 is a constant.

Some known results are established for (1.1) under the condition $0 \le p(t) < 1$. Grammatikopoulos et al. [6] obtained that if $0 \le p(t) \le 1$, $q(t) \ge 0$ and, $\int_{t_0}^{\infty} q(s)[1-p(s-\sigma)]\mathrm{d}s = \infty$, then the second-order neutral delay differential equation

$$\left[y(t) + p(t)y(t-\tau)\right]'' + q(t)y(t-\sigma) = 0 \tag{1.2}$$

oscillates. In [13], by employing Riccati technique and averaging functions method, Ruan established some general oscillation criteria for second-order neutral delay differential equation

$$\left[a(t)\left(x(t) + p(t)x(t-\tau)\right)'\right]' + q(t)f(x(t-\sigma)) = 0. \tag{1.3}$$

Xu and Meng [18] as well as Zhuang and Li [23] studied the oscillation of the second-order neutral delay differential equation

$$\left[r(t)\left(y(t) + p(t)y(\tau(t))\right)'\right]' + \sum_{i=1}^{n} q_i(t) f_i\left(y(\sigma_i(t))\right) = 0. \tag{1.4}$$

Motivated by [11], we will further the investigation and offer some more general new oscillation criteria for (1.1), by employing a class of function Y, operator T, and the Riccati technique and averaging technique.

Following [11], we say that a function $\phi = \phi(t, s, l)$ belongs to the function class Y, denoted by $\phi \in Y$ if $\phi \in C(E, R)$, where $E = \{(t, s, l) : t_0 \le l \le s \le t < \infty\}$, which satisfies $\phi(t, t, l) = 0$, $\phi(t, l, l) = 0$, and $\phi(t, s, l) > 0$, for $l < s < t$, and has the partial derivative $\partial\phi/\partial s$ on E such that $\partial\phi/\partial s$ is locally integrable with respect to s in E. By choosing the special function ϕ, it is possible to derive several oscillation criteria for a wide range of differential equations.

Define the operator $T[\cdot; l, t]$ by

$$T[g; l, t] = \int_l^t \phi(t, s, l) g(s)\mathrm{d}s, \tag{1.5}$$

for $t \ge s \ge l \ge t_0$ and $g \in C^1[t_0, \infty)$. The function $\varphi = \varphi(t, s, l)$ is defined by

$$\frac{\partial\phi(t, s, l)}{\partial s} = \varphi(t, s, l)\phi(t, s, l). \tag{1.6}$$

It is easy to see that $T[\cdot;l,t]$ is a linear operator and that it satisfies

$$T\left[g';l,t\right] = -T\left[g\varphi;l,t\right], \quad \text{for } g(s) \in C^1[t_0,\infty). \tag{1.7}$$

2. Main Results

In this section, we give some new oscillation criteria for (1.1). We start with the following oscillation criteria.

Theorem 2.1. *If*

$$\int_{t_0}^{\infty} Q(t)dt = \infty, \tag{2.1}$$

where $Q(t) := \min\{q(t), q(\tau(t))\}$, *then* (1.1) *oscillates.*

Proof. Let x be a nonoscillatory solution of (1.1). Then there exists $t_1 \geq t_0$ such that $x(t) \neq 0$, for all $t \geq t_1$. Without loss of generality, we assume that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$, for all $t \geq t_1$. From (1.1), we have

$$\left(r(t)z'(t)\right)' = -q(t)x(\sigma(t)) < 0, \quad t \geq t_1. \tag{2.2}$$

Therefore $r(t)z'(t)$ is a decreasing function. We claim that $z'(t) > 0$ for $t \geq t_1$. Otherwise, there exists $t_2 \geq t_1$ such that $z'(t_2) < 0$. Then from (2.2) we obtain

$$r(t)z'(t) \leq r(t_2)z'(t_2), \quad t \geq t_2, \tag{2.3}$$

and hence,

$$z(t) \leq z(t_2) - \left[-r(t_2)z'(t_2)\right]\int_{t_2}^{t} \frac{ds}{r(s)}. \tag{2.4}$$

Taking $t \to \infty$, we get $z(t) \to -\infty$, $t \to \infty$. This contradiction proves that $z'(t) > 0$ for $t \geq t_1$. Using definition of $z(t)$ and applying (1.1), we get for sufficiently large t

$$\left(r(t)z'(t)\right)' + q(t)x(\sigma(t)) + p_0 q(\tau(t))x(\sigma(\tau(t))) + \frac{p_0}{\tau'(t)}\left(r(\tau(t))z'(\tau(t))\right)' = 0, \tag{2.5}$$

and thus,

$$\left(r(t)z'(t)\right)' + Q(t)z(\sigma(t)) + \frac{p_0}{\tau'(t)}\left(r(\tau(t))z'(\tau(t))\right)' \leq 0. \tag{2.6}$$

Integrating (2.6) from t_3 $(\geq t_1)$ to t, we obtain

$$\int_{t_3}^{t}\left(r(s)z'(s)\right)'\mathrm{d}s+\int_{t_3}^{t}Q(s)z(\sigma(s))\mathrm{d}s+p_0\int_{t_3}^{t}\frac{1}{\tau'(s)}\left(r(\tau(s))z'(\tau(s))\right)'\mathrm{d}s\leq 0. \tag{2.7}$$

Noting that $\tau'(t)=\tau_0>0$, we have

$$\begin{aligned}\int_{t_3}^{t}Q(s)z(\sigma(s))\mathrm{d}s&\leq-\int_{t_3}^{t}\left(r(s)z'(s)\right)'\mathrm{d}s-p_0\int_{t_3}^{t}\frac{1}{(\tau'(s))^2}\left(r(\tau(s))z'(\tau(s))\right)'\mathrm{d}(\tau(s))\\&=-\int_{t_3}^{t}\left(r(s)z'(s)\right)'\mathrm{d}s-\frac{p_0}{\tau_0^2}\int_{\tau(t_3)}^{\tau(t)}\left(r(u)z'(u)\right)'\mathrm{d}u\\&=r(t_3)z'(t_3)-r(t)z'(t)+\frac{p_0}{\tau_0^2}r(\tau(t_3))z'(\tau(t_3))-\frac{p_0}{\tau_0^2}r(\tau(t))z'(\tau(t)).\end{aligned} \tag{2.8}$$

Since $z'(t)>0$ for $t\geq t_1$, we can find a constant $c>0$ such that $z(\sigma(t))\geq c$ for $t\geq t_3\geq t_1$. Then from (2.8) and the fact that $r(t)z'(t)$ is eventually decreasing, we have

$$\int_{t_3}^{\infty}Q(t)\mathrm{d}t<\infty, \tag{2.9}$$

which is a contradiction to (2.1). This completes the proof. □

Theorem 2.2. *Assume that $\sigma(t)\leq\tau(t)$, and there exist functions $\phi\in Y$ and $k\in C^1([t_0,\infty),R^+)$ such that*

$$\limsup_{t\to\infty}T\left[k(s)Q(s)-\frac{\left(1+(p_0/\tau_0)\right)\left(\varphi+(k'(s)/k(s))\right)^2}{4}\frac{r(\sigma(s))k(s)}{\sigma'(s)};l,t\right]>0, \tag{2.10}$$

where $Q(t)$ is defined as in Theorem 2.1, the operator T is defined by (1.5), *and $\varphi=\varphi(t,s,l)$ is defined by* (1.6). *Then every solution x of* (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of (1.1). Then there exists $t_1\geq t_0$ such that $x(t)\neq 0$ for all $t\geq t_1$. Without loss of generality, we assume that $x(t)>0$, $x(\tau(t))>0$, and $x(\sigma(t))>0$, for all $t\geq t_1$. Define

$$\omega(t)=k(t)\frac{r(t)z'(t)}{z(\sigma(t))},\quad t\geq t_1. \tag{2.11}$$

Then $w(t)>0$ and

$$\omega'(t)=k'(t)\frac{r(t)z'(t)}{z(\sigma(t))}+k(t)\frac{\left(r(t)z'(t)\right)'z(\sigma(t))-r(t)z'(t)z'(\sigma(t))\sigma'(t)}{z^2(\sigma(t))}. \tag{2.12}$$

By (2.2) and the fact $z'(t) > 0$, we get

$$\frac{z'(\sigma(t))}{z'(t)} \geq \frac{r(t)}{r(\sigma(t))}. \tag{2.13}$$

From (2.11), (2.12), and (2.13), we have

$$\omega'(t) \leq k(t)\frac{(r(t)z'(t))'}{z(\sigma(t))} + \frac{k'(t)}{k(t)}\omega(t) - \frac{\sigma'(t)}{r(\sigma(t))k(t)}\omega^2(t). \tag{2.14}$$

Similarly, define

$$\nu(t) = k(t)\frac{r(\tau(t))z'(\tau(t))}{z(\sigma(t))}, \quad t \geq t_1. \tag{2.15}$$

Then $\nu(t) > 0$ and

$$\nu'(t) = k'(t)\frac{r(\tau(t))z'(\tau(t))}{z(\sigma(t))} + k(t)\frac{(r(\tau(t))z'(\tau(t)))'z(\sigma(t)) - r(\tau(t))z'(\tau(t))z'(\sigma(t))\sigma'(t)}{z^2(\sigma(t))}. \tag{2.16}$$

By (2.2) and the facting $z'(t) > 0$, noting that $\sigma(t) \leq \tau(t)$, we get

$$\frac{z'(\sigma(t))}{z'(\tau(t))} \geq \frac{r(\tau(t))}{r(\sigma(t))}. \tag{2.17}$$

From (2.15), (2.16), and (2.17), we have

$$\nu'(t) \leq k(t)\frac{(r(\tau(t))z'(\tau(t)))'}{z(\sigma(t))} + \frac{k'(t)}{k(t)}\nu(t) - \frac{\sigma'(t)}{r(\sigma(t))k(t)}\nu^2(t). \tag{2.18}$$

Therefore, from (2.14) and (2.18), we get

$$\begin{aligned}\omega'(t) + \frac{p_0}{\tau_0}\nu'(t) \leq{}& k(t)\frac{(r(t)z'(t))'}{z(\sigma(t))} + \frac{p_0}{\tau_0}k(t)\frac{(r(\tau(t))z'(\tau(t)))'}{z(\sigma(t))}\\ &+ \frac{k'(t)}{k(t)}\omega(t) - \frac{\sigma'(t)}{r(\sigma(t))k(t)}\omega^2(t) + \frac{p_0}{\tau_0}\frac{k'(t)}{k(t)}\nu(t) - \frac{p_0}{\tau_0}\frac{\sigma'(t)}{r(\sigma(t))k(t)}\nu^2(t).\end{aligned} \tag{2.19}$$

From (2.6), we obtain

$$\begin{aligned}\omega'(t) + \frac{p_0}{\tau_0}\nu'(t) \leq{}& -k(t)Q(t) + \frac{k'(t)}{k(t)}\omega(t) - \frac{\sigma'(t)}{r(\sigma(t))k(t)}\omega^2(t)\\ &+ \frac{p_0}{\tau_0}\frac{k'(t)}{k(t)}\nu(t) - \frac{p_0}{\tau_0}\frac{\sigma'(t)}{r(\sigma(t))k(t)}\nu^2(t).\end{aligned} \tag{2.20}$$

Applying $T[\cdot;l,t]$ to (2.20), we get

$$T\left[\omega'(s)+\frac{p_0}{\tau_0}\nu'(s);l,t\right] \le T\left[-k(s)Q(s)+\frac{k'(s)}{k(s)}\omega(s)-\frac{\sigma'(s)}{r(\sigma(s))k(s)}\omega^2(s)+\frac{p_0}{\tau_0}\frac{k'(s)}{k(s)}\nu(s)-\frac{p_0}{\tau_0}\frac{\sigma'(s)}{r(\sigma(s))k(s)}\nu^2(s);l,t\right]. \tag{2.21}$$

By (1.7) and the above inequality, we obtain

$$T[k(s)Q(s);l,t] \le T\left[\left(\varphi+\frac{k'(s)}{k(s)}\right)\omega(s)-\frac{\sigma'(s)}{r(\sigma(s))k(s)}\omega^2(s)+\frac{p_0}{\tau_0}\left(\varphi+\frac{k'(s)}{k(s)}\right)\nu(s)-\frac{p_0}{\tau_0}\frac{\sigma'(s)}{r(\sigma(s))k(s)}\nu^2(s);l,t\right]. \tag{2.22}$$

Hence, from (2.22) we have

$$T[k(s)Q(s);l,t] \le T\left[\left(\frac{(\varphi+(k'(s)/k(s)))^2}{4}+\frac{(p_0/\tau_0)(\varphi+(k'(s)/k(s)))^2}{4}\right)\frac{r(\sigma(s))k(s)}{\sigma'(s)};l,t\right], \tag{2.23}$$

that is,

$$T\left[k(s)Q(s)-\frac{(1+(p_0/\tau_0))(\varphi+(k'(s)/k(s)))^2}{4}\frac{r(\sigma(s))k(s)}{\sigma'(s)};l,t\right] \le 0. \tag{2.24}$$

Taking the super limit in the above inequality, we get

$$\limsup_{t\to\infty} T\left[k(s)Q(s)-\frac{(1+(p_0/\tau_0))(\varphi+(k'(s)/k(s)))^2}{4}\frac{r(\sigma(s))k(s)}{\sigma'(s)};l,t\right] \le 0, \tag{2.25}$$

which contradicts (2.10). This completes the proof. □

Remark 2.3. With the different choice of k and ϕ, Theorem 2.2 can be stated with different conditions for oscillation of (1.1). For example, if we choose $\phi(t,s,l)=\rho(s)(t-s)^{\sigma}(s-l)^{\mu}$ for $\sigma>1/2$, $\mu>1/2$, $\rho\in C^1([t_0,\infty),(0,\infty))$, then

$$\varphi(t,s,l)=\frac{\rho'(s)}{\rho(s)}+\frac{\mu t-(\sigma+\mu)s+\sigma l}{(t-s)(s-l)}. \tag{2.26}$$

By Theorem 2.2 we can obtain the oscillation criterion for (1.1), the details are left to the reader.

For an application, we give the following example to illustrate the main results.

Example 2.4. Consider the following equation:

$$(x(t) + 2x(t - \pi))'' + x(t - \pi) = 0, \quad t \geq t_0. \tag{2.27}$$

Let $r(t) = 1$, $p(t) = 2$, $q(t) = 1$, and $\tau(t) = \sigma(t) = t - \pi$, then by Theorem 2.1 every solution of (2.27) oscillates; for example, $x(t) = \sin t$ is an oscillatory solution of (2.27).

Remark 2.5. The recent results cannot be applied in (2.27) since $p(t) = 2 > 1$; so our results are new ones.

Acknowledgments

This research is supported by the Natural Science Foundation of China (60774004, 60904024), China Postdoctoral Science Foundation Funded Project (20080441126, 200902564), Shandong Postdoctoral Funded Project (200802018) and the Natural Scientific Foundation of Shandong Province (Y2008A28, ZR2009AL003), also supported by University of Jinan Research Funds for Doctors (XBS0843).

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