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Research Article

Comparison Theorems for the Third-Order Delay Trinomial Differential Equations

B. Baculíková and J. Džurina

Department of Mathematics, Faculty of Electrical Engineering and Informatics, Technical University of Košice, Letná 9, 042 00 Košice, Slovakia

Correspondence should be addressed to J. Džurina, jozef.dzurina@tuke.sk

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The objective of this paper is to study the asymptotic properties of third-order delay trinomial differential equation $y'''(t) + p(t)y'(t) + g(t)y(\tau(t)) = 0$. Employing new comparison theorems, we can deduce the oscillatory and asymptotic behavior of the above-mentioned equation from the oscillation of a couple of the first-order differential equations. Obtained comparison principles essentially simplify the examination of the studied equations.

1. Introduction

In this paper, we are concerned with the oscillation and the asymptotic behavior of the solution of the third-order delay trinomial differential equations of the form

$$y'''(t) + p(t)y'(t) + g(t)y(\tau(t)) = 0. \tag{E}$$

In the sequel, we will assume that the following conditions are satisfied:

(i) $p(t) \geq 0$, $g(t) > 0$,

(ii) $\tau(t) \leq t$, $\lim_{t\to\infty}\tau(t) = \infty$.

By a solution of (E), we mean a function $y(t) \in C^1[T_x, \infty)$, $T_x \geq t_0$ that satisfies (E) on $[T_x, \infty)$. We consider only those solutions $y(t)$ of (E) which satisfy $\sup\{|y(t)| : t \geq T\} > 0$ for all $T \geq T_x$. We assume that (E) possesses such a solution. A solution of (E) is called oscillatory if it has arbitrarily large zeros on $[T_x, \infty)$, and otherwise it is called to be nonoscillatory. Equation (E) itself is said to be oscillatory if all its solutions are oscillatory.

Remark 1.1. All functional inequalities considered in this paper are assumed to hold eventually, that is, they are satisfied for all t large enough.

In the recent years, great attention in the oscillation theory has been devoted to the oscillatory and asymptotic properties of the third-order differential equations (see [1–20]). Various techniques appeared for the investigation of such equations. Some of them [1, 19] make use of the methods developed for the second-order equations [16, 17, 20] like the Riccati transformation and the integral averaging method and extend them to the third-order equations. Our method is based on the suitable comparison theorems.

Lazer [12] has shown that the differential equation without delay

$$y'''(t) + p(t)y'(t) + g(t)y(t) = 0 \tag{E_1}$$

has always a nonoscillatory solution satisfying the condition

$$y(t)y'(t) < 0. \tag{1.1}$$

We say that (E) has the property (P_0) if every nonoscillatory solution $y(t)$ satisfies (1.1). In [6–8, 12], the first criteria for (E_1) to have property (P_0) appeared. Those criteria have been improved in [18]. Džurina [3] has presented a set of comparison theorems that enable us to extend the results known for (E_1) to the delay equation (E). This method has been further elaborated by Parhi and Padhi [13, 14] and Džurina and Kotorová [5]. In this paper, we present a new comparison method for the studying properties of (E). We will compare (E) with a couple of the first-order delay differential equations in the sense that the oscillation of these equations yields the studied properties of (E).

2. Main Results

It will be derived that the properties of (E) are closely connected with the positive solutions of the corresponding second-order differential equation

$$v''(t) + p(t)v(t) = 0, \tag{V}$$

as the following lemma says.

Lemma 2.1. *If $v(t)$ is a positive solution of (V), then (E) can be written as the binomial equation*

$$\left(v^2(t)\left(\frac{1}{v(t)}\,y'\right)'\right)' + v(t)g(t)y(\tau(t)) = 0. \tag{E_C}$$

Proof. Straightforward computation shows that

$$\frac{1}{v(t)}\left(v^2(t)\left(\frac{1}{v(t)}\,y'(t)\right)'\right)' = y'''(t) - \frac{v''(t)}{v(t)}\,y'(t) = y'''(t) + p(t)y'(t). \tag{2.1}$$

Therefore, (E) really takes the form of (E_C). □

For our next consideration, it is desirable for (E_C) to be in a canonical form, that is, we require

$$\int^{\infty} v^{-2}(t)\mathrm{d}t = \int^{\infty} v(t)\mathrm{d}t = \infty. \tag{2.2}$$

It is clear that if $v(t)$ is a positive solution of (V), then the second integral in (2.2) is divergent. So, at first we will investigate the properties of the positive solutions of (V), and then we will be able to study the oscillation of the trinomial equation (E) with, the help of its binomial representation (E_C).

The following result (see, e.g., [4, 10] or [11]) is a consequence of Sturm's comparison theorem and guarantees the existence of a nonoscillatory solution.

Lemma 2.2. *If*

$$t^2 p(t) \leq \frac{1}{4} \quad \text{or} \quad \limsup_{t\to\infty} t^2 p(t) < \frac{1}{4}, \tag{2.3}$$

then (V) *possesses a positive solution. If*

$$\liminf_{t\to\infty} t^2 p(t) > \frac{1}{4} \quad \text{or} \quad t^2 p(t) \geq \frac{1}{4} + \varepsilon, \quad \varepsilon > 0, \tag{2.4}$$

then all solutions of (V) *are oscillatory.*

We present some properties of (V) that will be utilized later.

Lemma 2.3. *Assume that* (2.3) *is fulfilled, then* (V) *always possesses a nonoscillatory solution satisfying* (2.2).

Proof. Let $v_1(t)$ be a positive solution of (V). If $v_1(t)$ does not accomplish (2.2), then another solution of (V) is given by

$$v_2(t) = v_1(t) \int_t^{\infty} v_1^{-2}(s)\mathrm{d}s, \tag{2.5}$$

indeed, because

$$v_2'' = v_1'' \int_t^{\infty} v_1^{-2}(s)\mathrm{d}s = -p(t)v_1 \int_t^{\infty} v_1^{-2}(s)\mathrm{d}s = -p(t)v_2. \tag{2.6}$$

Moreover, $v_1(t)$ meets (2.2) by now. Really, if we denote $U(t) = \int_t^\infty v_1^{-2}(s)\mathrm{d}s$, then $\lim_{t\to\infty} U(t) = 0$. On the other hand,

$$\int_{t_0}^{\infty} v_2^{-2}(t)\mathrm{d}t = \int_{t_0}^{\infty} \frac{-U'(t)}{U^2(t)}\mathrm{d}t = \lim_{t\to\infty}\left(\frac{1}{U(t)} - \frac{1}{U(t_0)}\right) = \infty. \tag{2.7}$$

□

Picking up all the previous results, we can conclude by the following.

Corollary 2.4. *Assume that* (2.3) *is fulfilled, then the trinomial equation* (E) *can be always written in its binomial form* (E_C)*. Moreover,* (E_C) *is in the canonical form.*

In the sequel, to be sure that (V) possesses a nonoscillatory solution, we will always assume that (2.3) holds.

Now, we are ready to study the properties of (E) with the help of (E_C). Without loss of generality, we can deal only with the positive solutions of (E). Since every solution of (E) is also a solution of (E_C), we are in view of a generalization of Kiguradze's lemma (see [4] or [11]) in the following structure of the nonoscillatory solutions of (E).

Lemma 2.5. *Assume that* $v(t)$ *is a positive solution of* (V) *satisfying* (2.2)*, then every positive solution* $y(t)$ *of* (E)*is either of degree 2, that is,*

$$y > 0, \quad \frac{1}{v}y' > 0, \quad v^2\left(\frac{1}{v}y'\right)' > 0, \quad \left(v^2\left(\frac{1}{v}y'\right)'\right)' < 0, \tag{D_2}$$

or of degree 0, that is,

$$y > 0, \quad \frac{1}{v}y' < 0, \quad v^2\left(\frac{1}{v}y'\right)' > 0, \quad \left(v^2\left(\frac{1}{v}y'\right)'\right)' < 0. \tag{D_0}$$

In the sequel, we will assume that the function $v(t)$ that will be contained in our results is such solution of (V) that satisfies (2.2). If we eliminate the solutions of *degree 2* of (E), we get the studied property (P_0) of (E). The next theorem and its proof provide the details.

Theorem 2.6. *If the first-order differential equation*

$$z'(t) + v(t)g(t)\left[\int_{t_1}^{\tau(t)} v(s)\int_{t_1}^{s} v^{-2}(x)\mathrm{d}x\mathrm{d}s\right]z(\tau(t)) = 0 \tag{E_2}$$

is oscillatory, then (E) *has the property* (P_0)*.*

Proof. Assume that $y(t)$ is a positive solution of (E). It follows from Lemma 2.5 that $y(t)$ is either of *degree 2* or of *degree 0*. If $y(t)$ is of *degree 2*, then using that $z(t) = v^2(t)((1/v(t))y'(t))'$ is decreasing, we are led to

$$\begin{aligned} \frac{1}{v(t)}y'(t) &\geq \int_{t_1}^{t}\left(\frac{1}{v(u)}y'(u)\right)'\mathrm{d}u = \int_{t_1}^{t}\frac{1}{v^2(u)}\left[v^2(u)\left(\frac{1}{v(u)}y'(u)\right)'\right]\mathrm{d}u \\ &\geq z(t)\int_{t_1}^{t}\frac{1}{v^2(u)}\mathrm{d}u. \end{aligned} \tag{2.8}$$

Integrating from t_1 to t, we obtain

$$y(t) \geq \int_{t_1}^{t} z(s)v(s)\int_{t_1}^{s}\frac{1}{v^2(u)}\mathrm{d}u\,\mathrm{d}s \geq z(t)\int_{t_1}^{t} v(s)\int_{t_1}^{s}\frac{1}{v^2(u)}\mathrm{d}u\,\mathrm{d}s. \tag{2.9}$$

Obviously,

$$y(\tau(t)) \geq z(\tau(t))\int_{t_1}^{\tau(t)} v(s)\int_{t_1}^{s}\frac{1}{v^2(u)}\mathrm{d}u\,\mathrm{d}s. \tag{2.10}$$

Combining (2.10) together with (E_C), we see that

$$-z'(t) = v(t)g(t)y(\tau(t)) \geq \left[v(t)g(t)\int_{t_1}^{\tau(t)} v(s)\int_{t_1}^{s}\frac{1}{v^2(u)}\mathrm{d}u\,\mathrm{d}s\right]z(\tau(t)). \tag{2.11}$$

Or in other words, $z(t)$ is a positive solution of differential inequality

$$z'(t) + \left[v(t)g(t)\int_{t_1}^{\tau(t)} v(s)\int_{t_1}^{s}\frac{1}{v^2(u)}\mathrm{d}u\,\mathrm{d}s\right]z(\tau(t)) \leq 0. \tag{2.12}$$

Hence, by Theorem 1 in [15], we conclude that the corresponding differential equation (E_2) also has a positive solution, which contradicts to oscillation of (E_2). Therefore, $y(t)$ is of *degree 0*, and from the first two inequalities of (D_0), we conclude that (1.1) holds, which means that (E) has property (P_0). □

Applying the well-known oscillation criterion (Theorem 2.1.1 from [9]) to (E_2), we immediately get the sufficient condition for (E) to have the property (P_0).

Corollary 2.7. *Assume that*

$$\liminf_{t\to\infty}\int_{\tau(t)}^{t} v(u)g(u)\int_{t_1}^{\tau(u)} v(s)\int_{t_1}^{s} v^{-2}(x)\mathrm{d}x\mathrm{d}s\,\mathrm{d}u > \frac{1}{\mathrm{e}}, \tag{C_1}$$

then (E) *has the property* (P_0).

Remark 2.8. We note that if (E) has the property (P_0), then every positive solution $y(t)$ satisfies (D_0), and then from the first two inequalities of (D_0), we have the information only about the zero and the first derivative of $y(t)$. We have no information about the second and the third derivatives, but on the other hand, we know the sign properties of the second and the third quasiderivatives of $y(t)$.

Example 2.9. Consider the third-order trinomial equation of the form

$$y'''(t) + \frac{\alpha(1-\alpha)}{t^2}y'(t) + \frac{a}{t^3}y(\lambda t) = 0, \tag{2.13}$$

with $0 < \lambda < 1$, $0 < \alpha < 1/2$, and $a > 0$. It is easy to see that $v(t) = t^\alpha$ is the wanted solution of (V), and so (E_2) reduces to

$$z'(t) + a\left[\frac{\lambda^{2-\alpha}}{(2-\alpha)(1-2\alpha)}\frac{1}{t} + O\left(t^{-2+2\alpha}\right)\right]z(\lambda t) = 0, \tag{2.14}$$

where in the function $O(t^{-2+2\alpha})$ the terms unimportant for the oscillation of (2.14) are included. Applying the oscillation criterion from Corollary 2.7 to (2.14), we see that (2.13) has property (P_0) provided that the parameter a realizes the following condition:

$$a\frac{\lambda^{2-\alpha}}{(2-\alpha)(1-2\alpha)}\ln\left(\frac{1}{\lambda}\right) > \frac{1}{\mathrm{e}}. \tag{2.15}$$

We note that for

$$a = \left[\beta(\beta+1)(\beta+2) + \beta\alpha(1-\alpha)\right]\lambda^{\beta}, \quad \beta > 0, \tag{2.16}$$

one such solution is $y(t) = t^{-\beta}$.

Now, we turn our attention to oscillation of (E). We have known that oscillation of (E_2) brings property (P_0) of (E). If we eliminate also the case (D_0) of Lemma 2.5, we get oscillation of (E).

Theorem 2.10. *Let $\tau'(t) > 0$. Assume that there exists a function $\xi(t) \in C^1([t_0,\infty))$ such that*

$$\xi'(t) \geq 0, \quad \xi(t) > t, \quad \eta(t) = \tau(\xi(\xi(t))) < t. \tag{2.17}$$

If both the first-order delay equations (E_2) and

$$z'(t) + \left[v(t)\int_t^{\xi(t)} v^{-2}(s)\int_s^{\xi(s)} v(x)g(x)\mathrm{d}x\,\mathrm{d}s\right]z(\eta(t)) = 0 \tag{E_3}$$

are oscillatory, then (E) is oscillatory.

Proof. Assume that $y(t)$ is a positive solution of (E). It follows from Lemma 2.5 that $y(t)$ is either of *degree 2* or of *degree 0*. From Theorem 2.6, we have know that oscillation of (E_2) eliminates the solutions of *degree 2*. Consequently, $y(t)$ is of *degree 0*, which implies $y'(t) < 0$. Integration of (E_C) from t to $\xi(t)$ yields

$$v^2(t)\left(\frac{1}{v(t)}y'(t)\right)' \geq \int_t^{\xi(t)} v(x)g(x)y(\tau(x))\mathrm{d}x \geq y[\tau(\xi(t))]\int_t^{\xi(t)} v(x)g(x)\mathrm{d}x. \tag{2.18}$$

Then

$$\left(\frac{1}{v(t)}y'(t)\right)' \geq \frac{y[\tau(\xi(t))]}{v^2(t)}\int_t^{\xi(t)} v(x)g(x)\mathrm{d}x. \tag{2.19}$$

Integrating from t to $\xi(t)$ once more, we get

$$\begin{aligned} -\frac{1}{v(t)}y'(t) &\geq \int_t^{\xi(t)} \frac{y[\tau(\xi(s))]}{v^2(s)}\int_s^{\xi(s)} v(x)g(x)\mathrm{d}x\,\mathrm{d}s \\ &\geq y[\eta(t)]\int_t^{\xi(t)} \frac{1}{v^2(s)}\int_s^{\xi(s)} v(x)g(x)\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{2.20}$$

Finally, integrating from t to ∞, one gets

$$y(t) \geq \int_t^{\infty} y[\eta(u)]v(u)\int_u^{\xi(u)} \frac{1}{v^2(s)}\int_s^{\xi(s)} v(x)g(x)\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}u. \tag{2.21}$$

Let us denote the right hand side of (2.21) by $z(t)$, then $y(t) \geq z(t) > 0$, and one can easily verify that $z(t)$ is a solution of the differential inequality

$$z'(t) + \left[v(t)\int_t^{\xi(t)} v^{-2}(s)\int_s^{\xi(s)} v(x)g(x)\mathrm{d}x\,\mathrm{d}s\right]z(\eta(t)) \leq 0. \tag{2.22}$$

Then Theorem 1 in [15] shows that the corresponding differential equation (E_3) has also a positive solution. This contradiction finishes the proof. □

Applying the oscillation criterion from [9] to (E_2) and (E_3), we obtain the sufficient condition for (E) to be oscillatory.

Corollary 2.11. *Let $\tau'(t) > 0$. Assume that there exists a function $\xi(t) \in C^1([t_0,\infty))$ such that* (2.17) *holds. If, moreover,* (C_1) *is satisfied and*

$$\liminf_{t\to\infty}\int_{\eta(t)}^{t} v(u)\int_u^{\xi(u)} v^{-2}(s)\int_s^{\xi(s)} v(x)g(x)\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}u > \frac{1}{\mathrm{e}}, \tag{C_2}$$

then (E) is oscillatory.

Remark 2.12. There is an optional function $\xi(t)$ included in (E_3) and condition (C_2). There is no general rule for its choice. From the experience of the authors, we suggest to select such $\xi(t)$ for which the composite function $\xi \circ \xi$ to be "close to" the inverse function $\tau^{-1}(t)$ of $\tau(t)$. In the next example, we provide the details.

Example 2.13. We consider (2.13) again. Following Remark 2.12, we set $\xi(t) = \gamma t, 1 < \gamma < 1/\sqrt{\lambda}$, where these restrictions on γ result from (2.17). Since $v(t) = t^{\alpha}$ is a wanted solution of (V), then (E_3) reduces to

$$z'(t) + \frac{(1-\gamma^{\alpha-2})(1-\gamma^{-\alpha-1})}{(2-\alpha)(1+\alpha)} \frac{a}{t} z\left(\lambda\gamma^2 t\right) = 0. \tag{2.23}$$

Applying the oscillation criterion (C_2), we get in view of Corollary 2.11 that (2.13) is oscillatory provided that a verifies the following condition:

$$\frac{a}{(2-\alpha)(1+\alpha)} \left(1-\gamma^{\alpha-2}\right)\left(1-\gamma^{-\alpha-1}\right) \ln\left(\frac{1}{\lambda\gamma^2}\right) > \frac{1}{\mathrm{e}}. \tag{2.24}$$

Obviously, we obtain the best oscillatory result if we choose such $\gamma \in (1, 1/\sqrt{\lambda})$, for which the function

$$f(\gamma) = \left(1-\gamma^{\alpha-2}\right)\left(1-\gamma^{-\alpha-1}\right) \ln\left(\frac{1}{\lambda\gamma^2}\right) \tag{2.25}$$

attains its maximum. If we are not able to find the maximum value of $f(\gamma)$, we simply put $\gamma = (1+\sqrt{\lambda})/2\sqrt{\lambda}$, which is the middle point of the prescribed interval. In this case, (2.24) takes the form

$$\frac{a\left(1-\left(\left(1+\sqrt{\lambda}\right)/2\sqrt{\lambda}\right)^{\alpha-2}\right)\left(1-\left(\left(1+\sqrt{\lambda}\right)/2\sqrt{\lambda}\right)^{-\alpha-1}\right)\ln\left(4/\left(1+\sqrt{\lambda}\right)^2\right)}{(2-\alpha)(1+\alpha)} > \frac{1}{\mathrm{e}}. \tag{2.26}$$

Thus, it follows from Theorem 2.10 that (2.13) is oscillatory provided that (2.26) holds.

Applying MATLAB, we can draw the graph of $f(\gamma)$ with $\alpha = 0.3$, $\lambda = 0.5$ and verify that the maximum value of $f(\gamma)$ is reached for $\gamma = 1.24$. On the other hand, the middle $\gamma = 1.20$.

Therefore, Theorems 2.6 and 2.10 imply that if $\alpha = 0.3$, $\lambda = 0.5$, and

$$\begin{gathered} a > 1.1726, \quad \text{then (2.13) has the property } (P_0), \\ a > 41.3856, \quad \text{then (2.13) is oscillatory.} \end{gathered} \tag{2.27}$$

On the other hand, if we apply the middle γ, we get a bit weaker result for oscillation of (2.13), namely, $a > 43.1905$.

Remark 2.14. The oscillation of (E) is a new phenomena in the oscillation theory. The previous results [3, 5, 13] do not help to study this case, because they are based on transferring the properties of the ordinary equation (E_1) to the delay equation (E), and since (E_1) is not oscillatory, we cannot deduce oscillation of (E) from that of (E_1).

Our comparison method is based on the canonical representation (E_C) of (E). Although the condition (2.3) of Lemma 2.2 guarantees the existence of the wanted solution $v(t)$ of (V) so that canonical representation (E_C) is possible, a natural question arises; what to do if we are not able to find $v(t)$ because it is needed in the crucial (E_2) and (E_3)? In the next considerations, we crack this problem. Employing the additional condition, we revise both (E_2) and (E_3) into the form that instead of $v(t)$ requires its asymptotic representation which essentially simplifies our calculations.

We say that $v^*(t)$ is an asymptotic representation of $v(t)$ if $\lim_{t\to\infty}(v(t)/v^*(t)) = 1$. We denote this fact by $v(t) \sim v^*(t)$.

The following result is recalled from [2].

Theorem 2.15. *If*

$$\int^{\infty} sp(s)\mathrm{d}s < \infty, \tag{2.28}$$

then (V) *has a solution* $v(t)$ *with the property* $v(t) \sim 1$.

Combining Theorem 2.15 together with Corollaries 2.7 and 2.11, we get new oscillatory criterion for (E).

Theorem 2.16. *Assume that* (2.28) *holds and*

$$\liminf_{t\to\infty}\int_{\tau(t)}^{t} g(u)\frac{(\tau(u)-t_1)^2}{2}\mathrm{d}u > \frac{1}{\mathrm{e}}, \tag{C_1^*}$$

then (E) *has the property* (P_0).

If, moreover, $\tau'(t) > 0$ *and there exists a function* $\xi(t) \in C^1([t_0,\infty))$ *such that* (2.17) *holds and*

$$\liminf_{t\to\infty}\int_{\eta(t)}^{t}\int_{u}^{\xi(u)}\int_{s}^{\xi(s)} g(x)\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}u > \frac{1}{\mathrm{e}}, \tag{C_2^*}$$

then (E) *is oscillatory.*

Proof. It follows from Theorem 2.15 that for any $C \in (0,1)$, we have

$$C < v(t) < \frac{1}{C}, \tag{2.29}$$

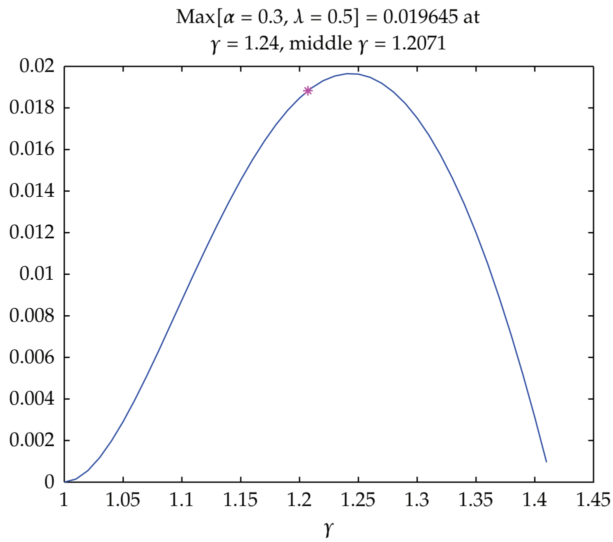

Figure 1

eventually. Moreover, (C_1^*) implies that there exists $C \in (0,1)$ such that

$$\begin{aligned}\frac{1}{\mathrm{e}} &< \liminf_{t\to\infty} C^4 \int_{\tau(t)}^{t} g(u)\frac{(\tau(u)-t_1)^2}{2}\mathrm{d}u \\ &= \liminf_{t\to\infty} \int_{\tau(t)}^{t} Cg(u) \int_{t_1}^{\tau(u)} C \int_{t_1}^{s} \frac{1}{C^{-2}}\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}u \\ &\leq \liminf_{t\to\infty} \int_{\tau(t)}^{t} v(u)g(u) \int_{t_1}^{\tau(u)} v(s) \int_{t_1}^{s} v^{-2}(x)\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}u,\end{aligned} \tag{2.30}$$

where we have used (2.29). We see that (C_1) holds and Corollary 2.7 guarantees the property (P_0) of (E).

The proof of the second part runs similarly, and so it can be omitted. $\square$

Example 2.17. Consider the third-order trinomial equation of the form

$$y'''(t) + \frac{\alpha(1-\alpha)}{t^3}y'(t) + \frac{a}{t^3}y(\lambda t) = 0, \tag{2.31}$$

with $0 < \lambda < 1$, $0 < \alpha < 1/2$, and $a > 0$. It is easy to see that (2.28) holds. Now, (C_1^*) reduces to

$$\frac{a\lambda^2}{2}\ln\left(\frac{1}{\lambda}\right) > \frac{1}{\mathrm{e}}, \tag{2.32}$$

which insures the property (P_0) of (2.23).

On the other hand, setting $\xi(t) = \gamma t$, where $1 < \gamma < 1/\sqrt{\lambda}$, the condition (C_2^*) takes the form

$$\frac{a}{2}\left(1-\frac{1}{\gamma^2}\right)\left(1-\frac{1}{\gamma}\right)\ln\left(\frac{1}{\lambda\gamma^2}\right) > \frac{1}{\mathrm{e}}. \tag{2.33}$$

If we put $\gamma = (1+\sqrt{\lambda})/2\sqrt{\lambda}$, which is the middle point of the prescribed interval, (2.33) rises to

$$\frac{a}{2}\left(1-\frac{4\lambda}{\left(1+\sqrt{\lambda}\right)^2}\right)\left(1-\frac{2\sqrt{\lambda}}{1+\sqrt{\lambda}}\right)\ln\left(\frac{4}{\left(1+\sqrt{\lambda}\right)^2}\right) > \frac{1}{\mathrm{e}}, \tag{2.34}$$

that in view of Theorem 2.16 yields the oscillation of (2.31).

3. Summary

In this paper, we have presented a new comparison principle for studying the oscillatory and asymptotic behavior of the third-order delay trinomial equation (E). Our method essentially makes use of its binomial representation (E_C), which is based on the existence of the suitable positive solution of the corresponding second-order equation (V), so that we can deduce property (P_0) or even oscillation of (E) from the oscillation of a couple of the first-order delay equations (E_2) and (E_3). Moreover, in a partial case, we can examine the studied properties of (E) without finding a positive solution of (V). Obtained comparison theorems are easily applicable.

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