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Research Article

Solutions of 2*n*th-Order Boundary Value Problem for Difference Equation via Variational Method

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The variational method and critical point theory are employed to investigate the existence of solutions for $2n$th-order difference equation $\Delta^n(p_{k-n}\Delta^n y_{k-n}) + (-1)^{n+1} f(k, y_k) = 0$ for $k \in [1, N]$ with boundary value condition $y_{1-n} = y_{2-n} = \cdots = y_0 = 0,\ y_{N+1} = \cdots = y_{N+n} = 0$ by constructing a functional, which transforms the existence of solutions of the boundary value problem (BVP) to the existence of critical points for the functional. Some criteria for the existence of at least one solution and two solutions are established which is the generalization for BVP of the even-order difference equations.

1. Introduction

Difference equations have been applied as models in vast areas such as finance insurance, biological populations, disease control, genetic study, physical field, and computer application technology. Because of their importance, many literature deals with its existence and uniqueness problems. For example, see [1–10].

We notice that the existing results are usually obtained by various analytical techniques, for example, the conical shell fixed point theorem [1, 6], Banach contraction map method [7], Leray-Schauder fixed point theorem [2, 10], and the upper and lower solution method [3]. It seems that the variational technique combining with the critical point theory [11] developed in the recent decades is one of the effective ways to study the boundary value problems of difference equations. However because the variational method requires a "symmetrical" functional, it is hard for the odd-order difference equations to create a functional satisfying the "symmetrical" property. Therefore, the even-order difference equations have been investigated in most references.

Let $a, b, N > 1$, $n \geq 1, k$ be integers, and $a < b$, $[a, b] := \{a, a+1, \ldots, b\}$ be a discrete interval in $\mathbb{Z}$. Inspired by [5, 8], in this paper, we try to investigate the following $2n$th-order boundary value problem (BVP) of difference equation via variational method combining with some traditional analytical skills:

$$\Delta^n(p_{k-n}\Delta^n y_{k-n}) + (-1)^{n+1} f(k, y_k) = 0 \quad k \in [1, N], \tag{1.1}$$

$$y_{1-n} = y_{2-n} = \cdots = y_0 = 0, \qquad y_{N+1} = \cdots = y_{N+n} = 0, \tag{1.2}$$

where $\Delta^n y_k = \Delta^{n-1} y_{k+1} - \Delta^{n-1} y_k$ ($n \geqslant 1$) is the forward difference operator; $p_k \in \mathbb{R}$ for $k \in [1-n, N]$ and $f \in C([1,N] \times \mathbb{R}, \mathbb{R})$. A variational functional for BVP (1.1)-(1.2) is constructed which transforms the existence of solutions of the boundary value problem (BVP) to the existence of critical points of this functional. In order to prove the existence criteria of critical points of the functional, some lemmas are given in Section 2. Two criteria for the existence of at least one solution and two solutions for BVP (1.1)-(1.2) are established in Section 3 which is the generalization for BVP of the even-order difference equations. The existence results obtained in this paper are not found in the references, to the best of our knowledge.

For convenience, we will use the following notations in the following sections:

$$F(k,u) = \int_0^u f(k,s)ds, \qquad \overline{p} = \max_{k \in [1-n,N]} |p_k|, \qquad \underline{p} = \min_{k \in [1-n,N]} |p_k|. \tag{1.3}$$

2. Variational Structure and Preliminaries

We need two lemmas from [12] or [11].

Lemma 2.1. *Let H be a real reflexive Banach space with a norm $\|\cdot\|$, and let ϕ be a weakly lower (upper) semicontinuous functional, such that*

$$\lim_{\|x\| \to \infty} \phi(x) = +\infty \qquad \left(\text{or} \lim_{\|x\| \to \infty} \phi(x) = -\infty\right), \tag{2.1}$$

then there exists $x_0 \in H$ such that

$$\phi(x_0) = \inf_{x \in H} \phi(x_0) \qquad \left(\text{or } \phi(x_0) = \sup_{x \in H} \phi(x_0)\right). \tag{2.2}$$

Furthermore, if ϕ has bounded linear Gâteaux derivative, then $\phi'(x_0) = 0$.

Lemma 2.2 (mountain-pass lemma). *Let H be a real Banach space, and let $\phi : H \to \mathbb{R}$ be continuously differential, satisfying the P-S condition. Assume that $x_0, x_1 \in H$ and Ω is an open neighborhood of x_0, but $x_1 \notin \Omega$. If $\max\{\phi(x_0), \phi(x_1)\} < \inf_{x \in \partial\Omega} \phi(x)$, then $c = \inf_{h \in \Gamma} \max_{t \in [0,1]} \phi(h(t))$ is the critical value of ϕ, where*

$$\Gamma = \{h \mid h : [0,1] \longrightarrow H,\ h \text{ is continuous},\ h(0) = x_0,\ h(1) = x_1\}. \tag{2.3}$$

This means that there exists $x_2 \in H$, s.t. $\phi'(x_2) = 0$, $\phi(x_2) = c$.

The following lemma will be used in the proof of Lemma 2.4.

Lemma 2.3. *If $A_{m\times m}$ is a symmetric and positive-defined real matrix, $B_{m\times n}$ is a real matrix, B^T is the transposed matrix of B. Then B^TAB is positive defined if and only if* $\operatorname{rank} B = n$.

Proof. Since A is positive defined, then

$$\begin{aligned} &B^TAB \text{ is positive-defined} \Longleftrightarrow \forall x \neq 0,\ x^TB^TABx > 0 \\ &\Longleftrightarrow \forall x \neq 0,\ (Bx)^TA(Bx) > 0 \Longleftrightarrow \forall x \neq 0,\ Bx \neq 0 \Longleftrightarrow \operatorname{rank} B = n. \end{aligned} \tag{2.4}$$

□

Let H be a Hilbert space defined by

$$H \triangleq \{y : [1-n, N+n] \longrightarrow \mathbb{R} \mid y_{1-n} = y_{2-n} = \cdots = y_0 = 0,\ y_{N+1} = \cdots = y_{N+n} = 0\} \tag{2.5}$$

with the norm

$$\|y\| = \sqrt{\sum_{k=1}^{N} y_k^2}, \quad y \in H. \tag{2.6}$$

Hence H is an N-dimensional Hilbert space. For any $q > 1$, let $\|y\|_q = (\sum_{k=1}^{N} y_k^2)^{1/q}$, then one can show that there exist constants $q_1, q_2 > 0$, s.t. $q_1\|y\| \leqslant \|y\|_q \leqslant q_2\|y\|$; that is, $\|\cdot\|_q$ is an equivalent norm of $\|\cdot\|$ (see [9, page 68]).

Lemma 2.4. *There is*

$$\lambda\|x\|^2 \leqslant \sum_{k=1-n}^{N} (\Delta^n x_k)^2 \leqslant 4^n\|x\|^2, \quad x \in H,\ \textit{where } \lambda \textit{ is a postive constant}. \tag{2.7}$$

Proof. Since $x \in H$, $\Delta^{n-j}x_{N+1} = \Delta^{n-j}x_{j-n} = 0$, $j = 1, 2, \ldots, n$. By using the inequality $(a-b)^2 \leqslant 2(a^2+b^2)$, $a, b \in \mathbb{R}$, we have

$$\begin{aligned} \sum_{k=1-n}^{N} (\Delta^n x_k)^2 &= \sum_{k=1-n}^{N} (\Delta^{n-1}x_{k+1} - \Delta^{n-1}x_k)^2 \leq 2\sum_{k=1-n}^{N}\left[\left(\Delta^{n-1}x_{k+1}\right)^2 + (\Delta^{n-1}x_k)^2\right] \\ &= 2\left[\sum_{k=2-n}^{N+1}\left(\Delta^{n-1}x_k\right)^2 + \sum_{k=2-n}^{N}\left(\Delta^{n-1}x_k\right)^2\right] = 4\sum_{k=2-n}^{N}(\Delta^{n-1}x_k)^2 \\ &\leq 4\times 2\left[\sum_{k=2-n}^{N}\left(\Delta^{n-2}x_{k+1}\right)^2 + \sum_{k=2-n}^{N}(\Delta^{n-2}x_k)^2\right] \\ &= 4\times 2\left[\sum_{k=3-n}^{N+1}\left(\Delta^{n-2}x_k\right)^2 + \sum_{k=2-n}^{N}\left(\Delta^{n-2}x_k\right)^2\right] = 4^2\sum_{k=3-n}^{N}(\Delta^{n-2}x_k)^2. \end{aligned} \tag{2.8}$$

Repeating the above process, we obtain

$$\sum_{k=1-n}^{N} (\Delta^n x_k)^2 \leqslant 4^n \sum_{k=1}^{N} (x_k)^2 = 4^n \|x\|^2. \tag{2.9}$$

On the other hand, define $b_k \triangleq \Delta^n x_k = \sum_{j=0}^{n} (-1)^j C_n^j x_{k+n-j}$, $k \in [1-n, N]$, where C_n^j is the combination number, then we can rewrite $\{b_k\}_{1-n}^{N}$ in a vector form, that is, $b = Bx$, where $b = (b_{1-n}, b_{2-n}, \ldots, b_N)^T$, $x = (x_1, x_2, \ldots, x_N)^T$, and

$$B = \begin{pmatrix} 1 & & & & & & & \\ c_1 & 1 & & & & & & \\ \vdots & \vdots & \ddots & & & & & \\ c_n & c_{n-1} & \cdots & 1 & & & & \\ & c_n & \cdots & c_1 & 1 & & & \\ & & \ddots & \vdots & \vdots & \ddots & & \\ & & & c_n & c_{n-1} & \cdots & 1 & \\ & & & & c_n & \cdots & c_1 & 1 \\ & & & & & \ddots & \vdots & \vdots \\ & & & & & & c_n & c_{n-1} \\ & & & & & & & c_n \end{pmatrix}_{(N+n)\times N}, \tag{2.10}$$

where $c_i = (-1)^i C_n^i$. Hence $\operatorname{rank} B = N$. Note that

$$\sum_{k=1-n}^{N} (\Delta^n x_k)^2 = b^T b = (Bx)^T Bx = x^T B^T Bx, \tag{2.11}$$

and by Lemma 2.3 with $A_{m\times m} = I_{(N+n)\times(N+n)}$, we know that $B^T B$ is positive defined. Hence all eigenvalues of $B^T B$ are real and positive. Let λ be the minimal eigenvalue of these N eigenvalues, then $\lambda > 0$. Therefore $x^T B^T Bx \geqslant \lambda x^T x$, that is, $\sum_{1-n}^{N} (\Delta^n x_k)^2 \geqslant \lambda \|x\|^2$. □

However, how to find the λ in Lemma 2.4 is a skillful and challenging task. The following lemma from [13] offers some help for the estimation of λ.

Lemma 2.5 (Brualdi [13]). *If $A = [a_{ij}] \in M_N$ is weak irreducible, then each eigenvalue is contained in the set*

$$\bigcup_{\gamma \in C(A)} \left\{ z \in \mathbb{C} : \prod_{P_i \in \gamma} |z - a_{ii}| \leqslant \prod_{P_i \in \gamma} R_i' \right\}. \tag{2.12}$$

In Lemma 2.5, $\mathbb{C}$ is the complex number set, and the denotations $C(A)$, γ, P_i, R_i' can be found in [13]. Since B^TB is positive defined, all eigenvalues are positive real numbers. Therefore, by Lemma 2.5, let

$$\mathbb{B} = \bigcup_{\gamma \in C(B^TB)} \left\{ z \in \mathbb{R} : \prod_{P_i \in \gamma} |z - b_{ii}| \leqslant \prod_{P_i \in \gamma} R_i' \right\}, \tag{2.13}$$

where $B^TB = [b_{ij}]$. $\mathbb{B}$ is a subset of $\mathbb{R}$ and can be calculated directly from B^TB. Define $\overline{\lambda} = \max\{0, \min\{\mathbb{B}\}\}$. If $\overline{\lambda} > 0$, we can use this $\overline{\lambda}$ as λ in Lemma 2.4. If $\overline{\lambda} = 0$, then one needs to calculate the eigenvalues directly.

Define the functional ϕ on H by

$$\phi(y) \triangleq \sum_{k=1-n}^{N} \left[\frac{1}{2} p_k (\Delta^n y_k)^2 - F(k, y_k) \right]. \tag{2.14}$$

Then ϕ is C^1 with

$$(\phi'(y), x) = \sum_{k=1-n}^{N} [p_k(\Delta^n y_k)(\Delta^n x_k) - x_k f(k, y_k)], \tag{2.15}$$

where $x = \{x_k\}_{k=1-n}^{N+n} \in H$ and $(\cdot, \cdot)$ is the inner product in H. In fact, we have

$$\begin{aligned} \phi(y + x) - \phi(y) &= \sum_{k=1-n}^{N} \frac{1}{2} p_k \left[(\Delta^n y_k + \Delta^n x_k)^2 - (\Delta^n y_k)^2 \right] - [F(k, y_k + x_k) - F(k, y_k)] \\ &= \sum_{k=1-n}^{N} \left[p_k(\Delta^n y_k)(\Delta^n x_k) + \frac{1}{2} p_k (\Delta^n x_k)^2 - f(k, y_k + \theta x_k) x_k \right], \quad \theta \in (0, 1). \end{aligned} \tag{2.16}$$

The continuity of f and the right-hand side of the inequality in Lemma 2.4 lead to (2.15).

Furthermore, for any $x \in H$, we have $\Delta^{n-j} x_{N+1} = \Delta^{n-j} x_{1-n} = 0$, $j = 1, 2, \ldots, n$. By using the following formula (e.g., see [14, page 28]):

$$\sum_{k=n_1}^{n_2} [g_k \Delta f_k + f_{k+1} \Delta g_k] = (f_k g_k)\big|_{n_1}^{n_2+1}, \tag{2.17}$$

we have

$$\begin{aligned}\sum_{1-n}^{N} p_k(\Delta^n y_k)(\Delta^n x_k) &= p_{k-1}(\Delta^n y_{k-1})\left(\Delta^{n-1}x_k\right)\Big|_{1-n}^{N+1} - \sum_{1-n}^{N}\Delta(p_{k-1}\Delta^n y_{k-1})\Delta^{n-1}x_k \\ &= -\sum_{1-n}^{N}\Delta(p_{k-1}\Delta^n y_{k-1})\Delta^{n-1}x_k \\ &= -\Delta(p_{k-2}\Delta^n y_{k-2})\,\Delta^{n-2}x_k\Big|_{1-n}^{N+1} + \sum_{1-n}^{N}\Delta^2(p_{k-2}\Delta^n y_{k-2})\Delta^{n-2}x_k \\ &= \sum_{1-n}^{N}\Delta^2(p_{k-2}\Delta^n y_{k-2})\Delta^{n-2}x_k.\end{aligned} \tag{2.18}$$

Repeating the above process, we obtain

$$\sum_{1-n}^{N} p_k(\Delta^n y_k)(\Delta^n x_k) = (-1)^n\sum_{1-n}^{N}\Delta^n(p_{k-n}\Delta^n y_{k-n})x_k. \tag{2.19}$$

Let $(\phi'(y), x) = 0$, that is,

$$\begin{aligned}&\sum_{1-n}^{N}\left[(-1)^n\Delta^n(p_{k-n}\Delta^n y_{k-n}) - f(k, y_k)\right]x_k \\ &\quad= (-1)^n\sum_{1-n}^{N}\left[\Delta^n(p_{k-n}\Delta^n y_{k-n}) + (-1)^{n+1}f(k, y_k)\right]x_k \\ &\quad= 0.\end{aligned} \tag{2.20}$$

Since $x \in H$ is arbitrary, we know that the solution of BVP (1.1)-(1.2) corresponds to the critical point of ϕ.

3. Main Results

Now we present our main results of this paper.

Theorem 3.1. *If there exist $M_1 > 0$, $a_1 > 0$, $a_2 \in \mathbb{R}$, and $\sigma > 2$ s.t.*

$$F(k, u) \geqslant a_1|u|^\sigma + a_2, \quad \forall\, |u| > M_1, \tag{3.1}$$

then BVP (1.1)-(1.2) *has at least one solution.*

Proof. In fact, we can choose a suitable $a_2 < 0$ such that

$$F(k, u) \geqslant a_1|u|^\sigma + a_2, \quad \forall u \in \mathbb{R}. \tag{3.2}$$

Since there exists $\sigma_1 > 0$ with $\sigma_1\|y\| \leqslant \|y\|_\sigma$, we have

$$\sum_{1-n}^{N} F(k, y_k) \geqslant a_1 \sum_{1}^{N} |y_k|^\sigma + a_2(N+n) \geqslant a_1\sigma_1^\sigma \|y\|^\sigma + a_2(N+n). \tag{3.3}$$

Then by Lemma 2.4, we obtain

$$\phi(y) \leqslant \frac{\overline{p}}{2} 4^n \|y\|^2 - a_1\sigma_1^\sigma \|y\|^\sigma - a_2(N+n). \tag{3.4}$$

Noticing that $\sigma > 2$, we have $\lim_{\|y\|\to+\infty}\phi(y) = -\infty$. From Lemma 2.1, the conclusion of this lemma follows. □

Corollary 3.2. *If there exists $M_2 > 0$ s.t. $uf(k,u) > 0$ for all $|u| > M_2$, and*

$$\inf_{k\in[1-n,N]} \lim_{u\to\infty} \frac{|f(k,u)|}{|u|^r} \geqslant r_1, \tag{3.5}$$

where r, r_1 satisfy either $r = 1$, $r_1 > 4^n\overline{p}$ or $r > 1, r_1 > 0$, then BVP (1.1)-(1.2) *has at least one solution.*

Proof. Assume that $r = 1$, $r_1 > 4^n\overline{p}$. Then for $\epsilon_1 = (r_1 - 4^n\overline{p})/2 > 0$, there exists $M_3 > M_2$, such that $|f(k,y)| \geqslant (r_1 - \epsilon_1)|y|$ as $|y| > M_3$. We have from the continuity of $f(k,u)$ that there is a $K > 0$ such that $-K \leq f(k,u) \leq K$ for all $k \in [1,N]$, $|u| \leq M_3$. When $y > 0$, one has $f(k,y) \geqslant (r_1 - \epsilon_1)y > 0$ for $y \in (M_3, +\infty)$, then

$$\int_0^y f(k,s)ds = \int_0^{M_3} f(k,s)ds + \int_{M_3}^{y} f(k,s)ds \geqslant -KM_3 + \frac{r_1-\epsilon_1}{2}y^2 - \frac{r_1-\epsilon_1}{2}M_3^2; \tag{3.6}$$

when $y < 0$, one has $f(k,y) \leqslant (r_1 - \epsilon_1)y < 0$ for $y \in (-\infty, -M_3)$, then

$$\int_0^y f(k,s)ds = \int_0^{-M_3} f(k,s)ds + \int_{-M_3}^{y} f(k,s)ds \geqslant -KM_3 + \frac{r_1-\epsilon_1}{2}y^2 - \frac{r_1-\epsilon_1}{2}M_3^2. \tag{3.7}$$

Let $c := -KM_3 - ((r_1-\epsilon_1)/2)M_3^2$, then we have $\int_0^y f(k,s)ds \geqslant ((r_1-\epsilon_1)/2)y^2 + c$ for $y \in \mathbb{R}$. Therefore, we have

$$\phi(y) \leqslant \frac{\overline{p}}{2}4^n\|y\|^2 - \frac{r_1-\epsilon_1}{2}\sum_{1}^{N}|y_k|^2 - c = \frac{4^n\overline{p} - r_1 + \epsilon_1}{2}\|y\|^2 - c = -\frac{\epsilon_1}{2}\|y\|^2 - c, \tag{3.8}$$

which implies $\lim_{\|y\|\to+\infty}\phi(y) = -\infty$, and by Lemma 2.1, the conclusion of this lemma follows.

Assume that $r > 1$, $r_1 > 0$. Then for $\epsilon_2 = r_1/2 > 0$, there exists $M_4 > M_2$, such that $|f(k,y)| \geqslant (r_1/2)|y|$ as $|y| > M_4$. We have from the continuity of $f(k,u)$ that there is a $\overline{K} > 0$ such that $-\overline{K} \leq f(k,u) \leq \overline{K}$ for all $k \in [1,N]$, $|u| \leq M_4$. When $y > 0$, one has $f(k,y) \geqslant (r_1/2)y^r > 0$, $y \in (M_4, +\infty)$, then we have

$$\begin{aligned}\int_0^y f(k,s)ds &= \int_0^{M_4} f(k,s)ds + \int_{M_4}^y f(k,s)ds \\ &\geqslant -\overline{K}M_4 + \frac{r_1}{2(r+1)}y^{r+1} - \frac{r_1}{2(r+1)}M_4^{r+1},\end{aligned} \tag{3.9}$$

when $y < 0$, one has $f(k,y) \leqslant -(r_1/2)|y|^r = -(r_1/2)(-y)^r < 0$, $y \in (-\infty, -M_4)$, then we have

$$\begin{aligned}\int_0^y f(k,s)ds &= \int_0^{-M_4} f(k,s)ds + \int_{-M_4}^y f(k,s)ds \\ &\geqslant -\overline{K}M_4 + \frac{r_1}{2}\int_{-M_4}^y (-s)^r d(-s) \\ &= -\overline{K}M_4 + \frac{r_1}{2(r+1)}|y|^{r+1} - \frac{r_1}{2(r+1)}(M_4)^{r+1}.\end{aligned} \tag{3.10}$$

Let $d := -\overline{K}M_4 - (r_1/2(r+1))M_4^{r+1}$, then we have $\int_0^y f(k,s)ds \geqslant (r_1/2(r+1))|y|^{r+1} + d$ for $|y| > M_4$. Therefore, by Theorem 3.1, the conclusion of this lemma follows. □

Theorem 3.3. *Assume that $p_k > 0$, $k = 1-n, \ldots, N$, and*

(i) $\sup_{k\in[1-n,N]}\lim_{u\to 0}(f(k,u)/u) \leqslant r_2 < \underline{p}\lambda$, $\lambda > 0$ *is defined in Lemma 2.4;*

(ii) F *satisfies* (3.1) *in Theorem 3.1 or* f *satisfies the assumptions in Corollary 3.2.*

Then BVP (1.1)-(1.2) *has at least two solutions.*

Proof. We first show that ϕ satisfies the P-S condition. Let $\{y^{(m)}\}_{m=1}^{\infty} \subset H$ satisfy that $\{\phi(y^{(m)})\}$ is bounded and $\lim_{m\to\infty}\phi'(y^{(m)}) = 0$. If $\{y^{(m)}\}$ is unbounded, it possesses a divergent subseries, say $y^{(m_k)} \to +\infty$ as $k \to \infty$. However from (ii), we get (3.4) or (3.8), hence $\phi(y^{(m_k)}) \to -\infty$ as $k \to \infty$, which is contradictory to the the fact that $\{\phi(y^{(m)})\}$ is bounded.

Next we use the mountain-pass lemma to finish the proof. By (i), for $\epsilon_3 = (\underline{p}\lambda - r_2)/2 > 0$, there exists $R_1 > 0$ such that $f(k,y)/y \leqslant r_2 + \epsilon_3$ for $|y| \leqslant R_1$. Then $\int_0^y f(k,s)ds \leqslant ((r_2 + \epsilon_3)/2)y^2$ for $|y| \leqslant R_1$. Now together with Lemma 2.3, we have

$$\phi(y) \geqslant \frac{\underline{p}}{2}\lambda\|y\|^2 - \frac{r_2+\epsilon_3}{2}\sum_{1-n}^{N}|y_k|^2 = \|y\|^2\left(\frac{\underline{p}\lambda - r_2 - \epsilon_3}{2}\right) = \frac{\epsilon_3}{2}\|y\|^2 > 0 \quad \text{for } |y| \leqslant R_1, \tag{3.11}$$

which implies that

$$\phi(y) \geqslant \frac{\epsilon_3}{2}R_1^2 > 0 = \phi(\theta), \quad y \in \partial\Omega, \tag{3.12}$$

where θ is the zero element in H, and $\Omega = \{y \in H \mid \|y\| < R_1\}$. Since we have from (3.4) or (3.8) that $\lim_{\|y\|\to+\infty}\phi(y) = -\infty$, there exists $y_1 \in H$ with $\|y_1\| > R_1$, that is, $y_1 \notin \overline{\Omega}$, but

$\phi(y_1) < \phi(\theta) = 0$. Using Lemma 2.2, we have shown that $\xi = \inf_{h\in\Gamma}\max_{t\in[0,1]}\phi(h(t))$ is the critical value of ϕ, with Γ defined as

$$\Gamma = \{h \mid h : [0,1] \longrightarrow H, h \text{ is continuous}, h(0) = \theta, h(1) = y_1\}. \tag{3.13}$$

We denote $\overline{y}$ as its corresponding critical point.

On the other hand, by Theorem 3.1 or Corollary 3.2, we know that there exists $y^* \in H$, s.t. $\phi(y^*) = \sup_{y\in H}\phi(y)$. If $y^* \neq \overline{y}$, the theorem is proved. If on the contrary, $y^* = \overline{y}$, that is, $\sup_{y\in H}\phi(y) = \inf_{h\in\Gamma}\max_{t\in[0,1]}\phi(h(t))$, that implies for any $h \in \Gamma$, $\max_{t\in[0,1]}\phi(h(t)) = \sup_{y\in H}\phi(y)$. Taking $h_1 \neq h_2$ in Γ with $\max_{t\in[0,1]}\phi(h_1(t)) = \max_{t\in[0,1]}\phi(h_2(t)) = \sup_{y\in H}\phi(y)$, by the continuity of $\phi(h(t))$, there exist $t_1, t_2 \in (0,1)$ s.t. $\phi(h_1(t_1)) = \max_{t\in[0,1]}\phi(h_1(t))$, $\phi(h_2(t_2)) = \max_{t\in[0,1]}\phi(h_2(t))$. Hence $h_1(t_1)$, $h_2(t_2)$ are two different critical points of ϕ, that is, BVP (1.1)-(1.2) has at least two different solutions. □

4. An Example

Consider the 6th-order boundary value problem for difference equation

$$\begin{gathered}\Delta^6 y_{k-3} + y_k^3 e^{y_k^2-9} = 0, \quad k \in [1,300], \\ y_{-2} = y_{-1} = y_0 = 0, \qquad y_{301} = y_{302} = y_{303} = 0.\end{gathered} \tag{4.1}$$

Let $f(k,u) = u^3 e^{u^2-9}$, we have $\lim_{u\to 0}(f(k,u)/u) = 0$, $\lim_{u\to\infty}(f(k,u)/u) = +\infty$. Hence $f(k,u)$ satisfies the conditions in Theorem 3.3, the boundary value problem (4.1) has at least two solutions.

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