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Research Article

On Approximate Cubic Homomorphisms

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We investigate the generalized Hyers-Ulam-Rassias stability of the system of functional equations: $f(xy) = f(x)f(y)$, $f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x)$, on Banach algebras. Indeed we establish the superstability of this system by suitable control functions.

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1. Introduction

A definition of stability in the case of homomorphisms between metric groups was suggested by a problem by Ulam [2] in 1940. Let $(G_1, \cdot)$ be a group and let $(G_2, *)$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x \cdot y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? In this case, the equation of homomorphism $h(x \cdot y) = h(x) * h(y)$ is called stable. On the other hand, we are looking for situations when the homomorphisms are stable, that is, if a mapping is an approximate homomorphism, then there exists an exact homomorphism near it. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, Hyers [3] gave a positive answer to the question of Ulam for Banach spaces. Let $f : E_1 \to E_2$ be a mapping between Banach spaces such that

$$\left\| f(x+y) - f(x) - f(y) \right\| \leq \delta \tag{1.1}$$

for all $x, y \in E_1$ and for some $\delta \geq 0$. Then there exists a unique additive mapping $T : E_1 \to E_2$ satisfying

$$\left\| f(x) - T(x) \right\| \leq \delta \tag{1.2}$$

for all $x \in E_1$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in E_1$, then the mapping T is linear. Rassias [4] succeeded in extending the result of Hyers' theorem by weakening the condition for the Cauchy difference controlled by $(\|x\|^p + \|y\|^p),\ p \in [0,1)$ to be unbounded. This condition has been assumed further till now, through the complete Hyers direct method, in order to prove linearity for generalized Hyers-Ulam stability problem forms. A number of mathematicians were attracted to the pertinent stability results of Rassias [4], and stimulated to investigate the stability problems of functional equations. The stability phenomenon that was introduced and proved by Rassias is called Hyers-Ulam-Rassias stability. Then the stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem, see [5–13].

Bourgin [14] is the first mathematician dealing with stability of (ring) homomorphism $f(xy) = f(x)f(y)$. The topic of approximate homomorphisms was studied by a number of mathematicians, see [15–22] and references therein.

Jun and Kim [1] introduced the following functional equation:

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x), \tag{1.3}$$

and they established the general solution and generalized Hyers-Ulam-Rassias stability problem for this functional equation. It is easy to see that the function $f(x) = cx^3$ is a solution of the functional equation (1.3). Thus, it is natural that (1.3) is called a cubic functional equation and every solution of the cubic functional equation is said to be a cubic function.

Let R be a ring. Then a mapping $f : R \to R$ is called a cubic homomorphism if f is a cubic function satisfying

$$f(ab) = f(a)f(b), \tag{1.4}$$

for all $a, b \in R$. For instance, let R be commutative, then the mapping $f : R \to R$, defined by $f(a) = a^3 (a \in R)$, is a cubic homomorphism. It is easy to see that a cubic homomorphism is a ring homomorphism if and only if it is zero function. In this paper, we study the stability of cubic homomorphisms on Banach algebras. Indeed, we investigate the generalized Hyers-Ulam-Rassias stability of the system of functional equations:

$$\begin{gathered} f(xy) = f(x)f(y), \\ f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x), \end{gathered} \tag{1.5}$$

on Banach algebras. To this end, we need two control functions for our stability. One control function for (1.3) and an other control function for (1.4). So this is the main difference between our hypothesis (where two-degree freedom appears in the election for two control functions ϕ_1 and ϕ_2 in Theorem 2.1 in what follows), and the conditions (with one control function) that appear, for example, in [1, Theorem 3.1].

2. Main Results

In the following we suppose that A is a normed algebra, B is a Banach algebra, and f is a mapping from A into B, and $\varphi,\ \varphi_1,\ \varphi_2$ are maps from $A \times A$ into $\mathbb{R}^+$. Also, we put $0^p = 0$ for $p \le 0$.

Theorem 2.1. *Let*

$$\|f(xy) - f(x)f(y)\| \le \varphi_1(x, y), \tag{2.1}$$

$$\|f(2x + y) + f(2x - y) - 2f(x + y) - 2f(x - y) - 12f(x)\| \le \varphi_2(x, y) \tag{2.2}$$

for all $x, y \in A$. Assume that the series

$$\Psi(x, y) = \sum_{i=0}^{\infty} \frac{\varphi_2(2^i x, 2^i y)}{2^{3i}} \tag{2.3}$$

converges, and that

$$\lim_{n\to\infty} \frac{\varphi_1(2^n x, 2^n y)}{2^{6n}} = 0 \tag{2.4}$$

for all $x, y \in A$. Then there exists a unique cubic homomorphism $T : A \to A$ such that

$$\|T(x) - f(x)\| \le \frac{1}{16}\Psi(x, 0) \tag{2.5}$$

for all $x \in A$.

Proof. Setting $y = 0$ in (2.2) yields

$$\left\|2f(2x) - 2^4 f(x)\right\| \le \varphi_2(x, 0), \tag{2.6}$$

and then dividing by 2^4 in (2.6), we obtain

$$\left\|\frac{f(2x)}{2^3} - f(x)\right\| \le \frac{\varphi_2(x, 0)}{2 \cdot 2^3} \tag{2.7}$$

for all $x \in A$. Now by induction we have

$$\left\|\frac{f(2^n x)}{2^{3n}} - f(x)\right\| \le \frac{1}{2 \cdot 2^3} \sum_{i=0}^{n-1} \frac{\varphi_2(2^i x, 0)}{2^{3i}}. \tag{2.8}$$

In order to show that the functions $T_n(x) = f(2^n x)/2^{3n}$are a convergent sequence, we use the Cauchy convergence criterion. Indeed, replace x by $2^m x$ and divide by 2^{3m} in (2.8), where m is an arbitrary positive integer. We find that

$$\left\| \frac{f(2^{n+m}x)}{2^{3(n+m)}} - \frac{f(2^m x)}{2^{3m}} \right\| \leq \frac{1}{2\cdot 2^3}\sum_{i=0}^{n-1}\frac{\varphi_2(2^{i+m}x,0)}{2^{3(i+m)}} = \frac{1}{2\cdot 2^3}\sum_{i=m}^{n+m-1}\frac{\varphi_2(2^i x,0)}{2^{3i}} \tag{2.9}$$

for all positive integers m, n. Hence by the Cauchy criterion, the limit $T(x) = \lim_{n\to\infty} T_n(x)$ exists for each $x \in A$. By taking the limit as $n \to \infty$ in (2.8), we see that $\|T(x) - f(x)\| \leq (1/2\cdot 2^3)\sum_{i=0}^{\infty}(\varphi_2(2^i x,0)/2^{3i}) = 1/16\Psi(x,0)$ and (2.5) holds for all $x \in A$. If we replace x by $2^n x$ and y by $2^n y$, respectively, in (2.2) and divide by 2^{3n}, we see that

$$\begin{aligned} &\left\| \frac{f(2\cdot(2^n x) + 2^n y)}{2^{3n}} + \frac{f(2\cdot(2^n x) - 2^n y)}{2^{3n}} - 2\frac{f(2^n x + 2^n y)}{2^{3n}} - 2\frac{f(2^n x - 2^n y)}{2^{3n}} - 12\frac{f(2^n x)}{2^{3n}} \right\| \\ &\quad \leq \frac{\varphi_2(2^n x, 2^n y)}{2^{3n}}. \end{aligned} \tag{2.10}$$

Taking the limit as $n \to \infty$, we find that T satisfies (1.3) [1, Theorem 3.1]. On the other hand we have

$$\begin{aligned} \|T(xy) - T(x)\cdot T(y)\| &= \left\| \lim_{n\to\infty}\frac{f(2^n xy)}{2^{3n}} - \lim_{n\to\infty}\frac{f(2^n x)}{2^{3n}} \cdot \lim_{n\to\infty}\frac{f(2^n y)}{2^{3n}} \right\| \\ &= \lim_{n\to\infty}\left\| \frac{f(2^n x 2^n y)}{2^{6n}} - \frac{f(2^n y)f(2^n y)}{2^{6n}} \right\| \\ &\leq \lim_{n\to\infty}\frac{\varphi_1(2^n x, 2^n y)}{2^{6n}} = 0 \end{aligned} \tag{2.11}$$

for all $x, y \in A$. We find that T satisfies (1.4). To prove the uniqueness property of T, let $T' : A \to A$ be a function satisfing $T'(2x+y) + T'(2x-y) = 2T'(x+y) + 2T'(x-y) + 12T'(x)$ and $\|T'(x) - f(x)\| \leq (1/16)\Psi(x,0)$. Since T, T' are cubic, then we have

$$T(2^n x) = 2^{3n}T(x), \qquad T'(2^n x) = 2^{3n}T'(x) \tag{2.12}$$

for all $x \in A$, hence,

$$\begin{aligned}
\|T(x) - T'(x)\| &= \frac{1}{2^{3n}}\|T(2^n x) - T'(2^n x)\| \\
&\le \frac{1}{2^{3n}}\left(\|T(2^n x) - f(2^n x)\| + \|T'(2^n x) - f(2^n x)\|\right) \\
&\le \frac{1}{2^{3n}}\left(\frac{1}{2\cdot 2^3}\Psi(2^n x, 0) + \frac{1}{2\cdot 2^3}\Psi(2^n x, 0)\right) \\
&= \frac{1}{2^{3(n+1)}}\Psi(2^n x, 0) = \frac{1}{2^{3(n+1)}}\sum_{i=0}^{\infty}\frac{1}{2^{3i}}\varphi_2\left(2^{i+n}x, 0\right) \\
&= \frac{1}{2^3}\sum_{i=0}^{\infty}\frac{1}{2^{3(i+n)}}\varphi_2\left(2^{i+n}x, 0\right) = \frac{1}{2^3}\sum_{i=n}^{\infty}\frac{1}{2^{3i}}\varphi_2\left(2^i x, 0\right).
\end{aligned} \tag{2.13}$$

By taking $n \to \infty$ we get $T(x) = T'(x)$. □

Corollary 2.2. *Let θ_1 and θ_2 be nonnegative real numbers, and let $p \in (-\infty, 3)$. Suppose that*

$$\begin{gathered}
\|f(xy) - f(x)f(y)\| \le \theta_1, \\
\|f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 12f(x)\| \le \theta_2\left(\|x\|^p + \|y\|^p\right)
\end{gathered} \tag{2.14}$$

for all $x, y \in A$. Then there exists a unique cubic homomorphism $T : A \to A$ such that

$$\|T(x) - f(x)\| \le \frac{1}{16}\frac{\theta_2\|x\|^p}{1 - 2^{p-3}} \tag{2.15}$$

for all $x, y \in A$.

Proof. In Theorem 2.1, let $\varphi_1(x, y) = \theta_1$ and $\varphi_2(x, y) = \theta_2(\|x\|^p + \|y\|^p)$ for all $x, y \in A$. □

Corollary 2.3. *Let θ_1 and θ_2 be nonnegative real numbers. Suppose that*

$$\begin{gathered}
\|f(xy) - f(x)f(y)\| \le \theta_1, \\
\|f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 12f(x)\| \le \theta_2
\end{gathered} \tag{2.16}$$

for all $x, y \in A$. Then there exists a unique cubic homomorphism $T : A \to A$ such that

$$\|T(x) - f(x)\| \le \frac{\theta_2}{14} \tag{2.17}$$

for all $x \in A$.

Proof. The proof follows from Corollary 2.2. □

Corollary 2.4. *Let $p \in (-\infty, 3)$ and let θ be a positive real number. Suppose that*

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{2^{6n}} = 0, \tag{2.18}$$

for all $x, y \in A$. Moreover, suppose that

$$\| f(xy) - f(x)f(y) \| \le \varphi(x, y), \tag{2.19}$$

and that

$$\| f(2x + y) + f(2x - y) - 2f(x + y) - 2f(x - y) - 12f(x) \| \le \theta \|y\|^p, \tag{2.20}$$

for all $x, y \in A$. Then f is a cubic homomorphism.

Proof. Letting $x = y = 0$ in (2.20), we get that $f(0) = 0$. So by $y = 0$, in (2.20), we get $f(2x) = 2^3 f(x)$ for all $x \in A$. By using induction we have

$$f(2^n x) = 2^{3n} f(x) \tag{2.21}$$

for all $x \in A$ and $n \in \mathbb{N}$. On the other hand, by Theorem 2.1, the mapping $T : A \to A$, defined by

$$T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^{3n}}, \tag{2.22}$$

is a cubic homomorphism. Therefore it follows from (2.21) that $f = T$. Hence it is a cubic homomorphism. □

Corollary 2.5. *Let $p,\ q,\ \theta \ge 0$, and $p + q < 3$. Let*

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{2^{6n}} = 0 \tag{2.23}$$

for all $x, y \in A$. Moreover, suppose that

$$\| f(xy) - f(x)f(y) \| \le \varphi(x, y), \tag{2.24}$$

and that

$$\| f(2x + y) + f(2x - y) - 2f(x + y) - 2f(x - y) - 12f(x) \| \le \theta \|x\|^q \|y\|^p \tag{2.25}$$

for all $x, y \in A$. Then f is a cubic homomorphism.

Proof. If $q = 0$, then by Corollary 2.4 we get the result. If $q \neq 0$, the following results from Theorem 2.1, by putting $\varphi_1(x, y) = \varphi(x, y)$ and $\varphi_2(x, y) = \theta(\|x\|^p\|y\|^p)$ for all $x, y \in A$. □

Corollary 2.6. *Let $p \in (-\infty, 3)$ and θ be a positive real number. Let*

$$\begin{gathered}\|f(xy) - f(x)f(y)\| \le \theta\|y\|^p,\\ \|f(2x + y) + f(2x - y) - 2f(x + y) - 2f(x - y) - 12f(x)\| \le \theta\|y\|^p\end{gathered} \tag{2.26}$$

for all $x, y \in A$. Then f is a cubic homomorphism.

Proof. Let $\varphi(x, y) = \theta\|y\|^p$. Then by Corollary 2.4, we get the result. □

Theorem 2.7. *Let*

$$\|f(xy) - f(x)f(y)\| \le \varphi_1(x, y), \tag{2.27}$$

$$\|f(2x + y) + f(2x - y) - 2f(x + y) - 2f(x - y) - 12f(x)\| \le \varphi_2(x, y) \tag{2.28}$$

for all $x, y \in A$. Assume that the series

$$\Psi(x, y) = \sum_{i=1}^{\infty} 2^{3i}\varphi_2\left(\frac{x}{2^i}, \frac{y}{2^i}\right) \tag{2.29}$$

converges and that

$$\lim_{n\to\infty} 2^{6n}\varphi_1\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0 \tag{2.30}$$

for all $x, y \in A$. Then there exists a unique cubic homomorphism $T : A \to A$ such that

$$\|T(x) - f(x)\| \le \frac{1}{16}\Psi(x, 0) \tag{2.31}$$

for all $x \in A$.

Proof. Setting $y = 0$ in (2.28) yields

$$\left\|2f(2x) - 2\cdot 2^3 f(x)\right\| \le \varphi_2(x, 0). \tag{2.32}$$

Replacing x by $x/2$ in (2.32), we get

$$\left\|f(x) - 2^3 f\left(\frac{x}{2}\right)\right\| \le \frac{1}{2}\varphi_2\left(\frac{x}{2}, 0\right) \tag{2.33}$$

for all $x \in A$. By (2.33) we use iterative methods and induction on n to prove the following relation

$$\left\| f(x) - 2^{3n} f\left(\frac{x}{2^n}\right) \right\| \leq \frac{1}{2 \cdot 2^3} \sum_{i=1}^{n} 2^{3i} \varphi_2\left(\frac{x}{2^i}, 0\right). \tag{2.34}$$

In order to show that the functions $T_n(x) = 2^{3n} f(x/2^n)$ are a convergent sequence, replace x by $x/2^m$ in (2.34), and then multiply by 2^{3m}, where m is an arbitrary positive integer. We find that

$$\begin{aligned} \left\| 2^{3m} f\left(\frac{x}{2^m}\right) - 2^{3(n+m)} f\left(\frac{x}{2^{n+m}}\right) \right\| &\leq \frac{1}{2 \cdot 2^3} \sum_{i=1}^{n} 2^{3(i+m)} \varphi_2\left(\frac{x}{2^{i+m}}, 0\right) \\ &= \frac{1}{2 \cdot 2^3} \sum_{i=1+m}^{n+m} 2^{3i} \varphi_2\left(\frac{x}{2^i}, 0\right) \end{aligned} \tag{2.35}$$

for all positive integers. Hence by the Cauchy criterion the limit $T(x) = \lim_{n \to \infty} T_n(x)$ exists for each $x \in A$. By taking the limit as $n \to \infty$ in (2.34), we see that $\|T(x) - f(x)\| \leq 1/2 \cdot 2^3 \sum_{i=1}^{\infty} 2^{3i} \varphi_2(x/2^i, 0) = (1/16)\Psi(x, 0)$, and (2.31) holds for all $x \in A$. The rest of proof is similar to the proof of Theorem 2.1. □

Corollary 2.8. *Let $p > 3$ and θ be a positive real number. Let*

$$\lim_{n \to \infty} 2^{6n} \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0, \tag{2.36}$$

for all $x, y \in A$. Moreover, suppose that

$$\| f(xy) - f(x)f(y) \| \leq \varphi(x, y), \tag{2.37}$$

$$\| f(2x + y) + f(2x - y) - 2f(x + y) - 2f(x - y) - 12f(x) \| \leq \theta \|y\|^p, \tag{2.38}$$

for all $x, y \in A$. Then f is a cubic homomorphism.

Proof. Letting $x = y = 0$ in (2.38), we get that $f(0) = 0$. So by $y = 0$, in (2.38), we get $f(2x) = 2^3 f(x)$ for all $x \in A$. By using induction, we have

$$f(x) = 2^{3n} f\left(\frac{x}{2^n}\right) \tag{2.39}$$

for all $x \in A$ and $n \in \mathbb{N}$. On the other hand, by Theorem 2.8, the mapping $T : A \to A$, defined by

$$T(x) = \lim_{n \to \infty} 2^{3n} f\left(\frac{x}{2^n}\right), \tag{2.40}$$

is a cubic homomorphism. Therefore, it follows from (2.39) that $f = T$. Hence f is a cubic homomorphism. □

Example 2.9. Let

$$\mathcal{A} := \begin{bmatrix} 0 & \mathbb{R} & \mathbb{R} & \mathbb{R} \\ 0 & 0 & \mathbb{R} & \mathbb{R} \\ 0 & 0 & 0 & \mathbb{R} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2.41}$$

then $\mathcal{A}$ is a Banach algebra equipped with the usual matrix-like operations and the following norm:

$$\left\| \begin{bmatrix} 0 & a_1 & a_2 & a_3 \\ 0 & 0 & a_4 & a_5 \\ 0 & 0 & 0 & a_6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right\| = \sum_{i=1}^{6} |a_i| \quad (a_i \in \mathbb{R}). \tag{2.42}$$

Let

$$a := \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2.43}$$

and we define $f : \mathcal{A} \to \mathcal{A}$ by $f(x) = x^3 + a$, and

$$\begin{gathered} \varphi_1(x, y) := \|f(xy) - f(x)f(y)\| = \|a\| = 4, \\ \varphi_2(x, y) := \|f(2x + y) + f(2x - y) - 2f(x + y) - 2f(x - y) - 12f(x)\| = 14\|a\| = 56 \end{gathered} \tag{2.44}$$

for all $x, y \in \mathcal{A}$. Then we have

$$\begin{gathered} \sum_{k=0}^{\infty} \frac{\varphi_2(2^k x, 2^k y)}{2^{3k}} = \sum_{k=0}^{\infty} \frac{56}{2^{3k}} = 64, \\ \lim_{n \to \infty} \frac{\varphi_1(2^n x, 2^n y)}{2^{6n}} = 0. \end{gathered} \tag{2.45}$$

Thus the limit $T(x) = \lim_{n \to \infty}(f(2^n x)/2^{3n}) = x^3$ exists. Also,

$$T(xy) = (xy)^3 = x^3 y^3 = T(x)T(y). \tag{2.46}$$

Furthermore,

$$\begin{aligned} T(2x+y)+T(2x-y) &= (2x+y)^3+(2x-y)^3 = 16x^3+12xy^2 \\ &= 2T(x+y)+2T(x-y)+12T(x). \end{aligned} \tag{2.47}$$

Hence T is cubic homomorphism.

Also from this example, it is clear that the superstability of the system of functional equations

$$\begin{gathered} f(xy) = f(x)f(y), \\ f(2x+y)+f(2x-y) = 2f(x+y)+2f(x-y)+12f(x), \end{gathered} \tag{2.48}$$

with the control functions in Corollaries 2.4, 2.5 and 2.6 does not hold.

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