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Research Article

Summation Characterization of the Recessive Solution for Half-Linear Difference Equations

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We show that the recessive solution of the second-order half-linear difference equation $\Delta(r_k\Phi(\Delta x_k)) + c_k\Phi(x_{k+1}) = 0$, $\Phi(x) := |x|^{p-2}x, p > 1$, where r, c are real-valued sequences, is closely related to the divergence of the infinite series $\sum^{\infty} (r_k x_k x_{k+1}|\Delta x_k|^{p-2})^{-1}$.

1. Introduction

We consider the second-order half-linear difference equation

$$\Delta(r_k\Phi(\Delta x_k)) + c_k\Phi(x_{k+1}) = 0, \quad \Phi(x) := |x|^{p-2}x, \quad p > 1, \tag{1.1}$$

where r, c are real-valued sequences and $r_k > 0$, and we investigate properties of its *recessive solution*.

Qualitative theory of (1.1) was established in the series of the papers of Řehák [1–5] and it is summarized in [6, Chapter 3]. It was shown there that the oscillation theory of (1.1) is very similar to that of the linear equation

$$\Delta(r_k\Delta x_k) + c_k x_{k+1} = 0, \tag{1.2}$$

which is the special case $p = 2$ in (1.1). We will recall basic facts of the oscillation theory of (1.1) in the following section.

The concept of the recessive solution of (1.1) has been introduced in [7]. There are several attempts in literature to find a summation characterization of this solution, see [8] and also related references [9, 10], which are based on the asymptotic analysis of solutions of (1.1). However, this approach requires the sign restriction of the sequence c_k and additional assumptions on the convergence (divergence) of certain infinite series involving sequences r and c, see Proposition 2.1 in the following section. Here we use a different approach which is based on estimates for a certain nonlinear function which appears in the Picone-type identity for (1.1).

The recessive solution of (1.1) is a discrete counterpart of the concept of the principal solution of the half-linear differential equation

$$\left(r(t)\Phi\left(x'\right)\right)' + c(t)\Phi(x) = 0, \tag{1.3}$$

which attracted considerable attention in recent years, we refer to the work in [11–15] and the references given therein.

Let us recall the main result of [11] whose discrete version we are going to prove in this paper.

Proposition 1.1. *Let $\tilde{x}$ be a solution of* (1.3) *such that $\tilde{x}'(t) \neq 0$ for large t.*

(i) *Let $p \in (1, 2]$. If*

$$I(\tilde{x}) := \int^{\infty} \frac{dt}{r(t)\tilde{x}^2(t)|\tilde{x}'(t)|^{p-2}} = \infty, \tag{1.4}$$

then $\tilde{x}$ is the principal solution of (1.3).

(ii) *If $p \geq 2$ and $I(\tilde{x}) < \infty$, then $\tilde{x}$ is not the principal solution of* (1.3).

The paper is organized as follows. In Section 2 we recall elements of the oscillation theory of (1.1). Section 3 is devoted to technical statements which we use in the proofs of our main results which are presented in Section 4. Section 5 contains formulation of open problems in our research.

2. Preliminaries

Oscillatory properties of (1.1) are defined using the concept of the generalized zero which is defined in the same way as for (1.2), see, for example, [6, Chapter 3],or [16, Chapter 7]. A solution x of (1.1) has a *generalized zero* in an interval $(m, m+1]$ if $x_m \neq 0$ and $x_m x_{m+1} r_m \leq 0$. Since we suppose that $r_k > 0$ (oscillation theory of (1.1) generally requires only $r_k \neq 0$), a generalized zero of x in $(m, m+1]$ is either a "real" zero at $k = m+1$ or the sign change between m and $m+1$. However, (1.1) is said to be *disconjugate* in a discrete interval $[m, n]$ if the solution x of (1.1) given by the initial condition $x_m = 0$, $x_{m+1} \neq 0$ has no generalized zero in $(m, n+1]$. However, (1.1) is said to be *nonoscillatory* if there exists $m \in \mathbb{N}$ such that it is disconjugate on $[m, n]$ for every $n > m$ and is said to be *oscillatory* in the opposite case.

If x is a solution of (1.1) such that $x_k \neq 0$ in some discrete interval $[m, \infty)$, then $w_k = r_k \Phi(\Delta x_k / x_k)$ is a solution of the associated Riccati type equation

$$\Delta w_k + c_k + w_k \left(1 - \frac{r_k}{\Phi\left(\Phi^{-1}(r_k) + \Phi^{-1}(w_k)\right)} \right) = 0, \tag{2.1}$$

where $\Phi^{-1}(x) = |x|^{q-2}x$ is the inverse function of Φ and $q = p/(p-1)$ is the conjugate number to p. Moreover, if x has no generalized zero in $[m, \infty)$, then $\Phi^{-1}(r_k) + \Phi^{-1}(w_k) > 0$, $k \in [m, \infty)$. If we suppose that (1.1) is nonoscillatory, among all solutions of (2.1) there exists the so-called *distinguished* solution $\tilde{w}$ which has the property that there exists an interval $[m, \infty)$ such that any other solution w of (2.1) for which $\Phi^{-1}(r_k) + \Phi^{-1}(w_k) > 0$, $k \in [m, \infty)$, satisfies $w_k > \tilde{w}_k$, $k \in [m, \infty)$. Therefore, the distinguished solution of (2.1) is, in a certain sense, minimal solution of this equation near ∞, and sometimes it is called the *minimal* solution of (2.1). If $\tilde{w}$ is the distinguished solution of (2.1), then the associated solution of (1.1) given by the formula

$$\tilde{x}_k = \prod_{j=m}^{k-1} \left[1 + \Phi^{-1}\left(\frac{\tilde{w}_j}{r_j} \right) \right] \tag{2.2}$$

is said to be the *recessive solution* of (1.1), see [7]. Note that in the linear case $p = 2$ a solution $\tilde{x}$ of (1.2) is recessive if and only if

$$\sum^{\infty} \frac{1}{r_k \tilde{x}_k \tilde{x}_{k+1}} = \infty. \tag{2.3}$$

At the end of this section, for the sake of comparison, we recall the main results of [8, 17], where summation characterizations of recessive solutions of (1.1) are investigated using the asymptotic analysis of the solution space of (1.1).

Proposition 2.1. *Let x be a solution of* (1.1).

(i) *Suppose that $c_k < 0$, then x is the recessive solution of* (1.1) *if and only if*

$$\sum^{\infty} \frac{1}{r_k^{q-1} x_k x_{k+1}} = \infty. \tag{2.4}$$

(ii) *Suppose that $c_k > 0$, $\sum^{\infty} r_k^{1-q} < \infty$, and*

$$\sum^{\infty} c_k \Phi\left(\sum_{j=k+1}^{\infty} r_j^{1-q} \right) < \infty. \tag{2.5}$$

If x is the recessive solution of (1.1), *then*

$$\sum^{\infty} \frac{1}{r_k x_k x_{k+1} |\Delta x_k|^{p-2}} = \infty. \tag{2.6}$$

(iii) *Suppose that $c_k > 0$, $\sum^{\infty} c_k < \infty$, and $\sum^{\infty} r_k^{1-q} < \infty$. Then x is the recessive solution if and only if* (2.4) *holds.*

In cases (i) and (iii), the previous proposition gives *necessary and sufficient condition* for a solution x to be recessive. The reason why under assumptions in (i) or (iii) it is possible to formulate such a condition is that there is a substantial difference in asymptotic behavior of recessive and dominant solutions (i.e., solutions which are linearly independent of the recessive solution). This difference enables to "separate" the recessive solution from dominant ones and to formulate for it a necessary and sufficient condition (2.4). We refer to [8, 17] and also to [9, 10] for more details.

3. Technical Results

Throughout the rest of the paper we suppose that (1.1) is nonoscillatory and h is its solution. Denote

$$\begin{gathered} v_k^* := r_k h_k (\Phi(h_k) + \Phi(\Delta h_k)), \quad R_k := \frac{2}{q} r_k h_k h_{k+1} |\Delta h_k|^{p-2}, \\ G_k := r_k h_k \Phi(\Delta h_k), \end{gathered} \tag{3.1}$$

and define the function

$$H(k,v) := v + r_k h_{k+1} \Phi(\Delta h_k) - \frac{r_k (v + G_k) |h_{k+1}|^p}{\Phi\big(|h_k|^q \Phi^{-1}(r_k) + \Phi^{-1}(v + G_k)\big)}. \tag{3.2}$$

Lemma 3.1. *Put*

$$v_k := |h_k|^p (w_k - \tilde{w}_k), \tag{3.3}$$

where $\tilde{w}_k = r_k \Phi(\Delta h_k / h_k)$ is a solution of (2.1) *and w_k is any sequence satisfying $r_k + w_k \neq 0$. Then the following statements hold:*

(i) *w_k is a solution of* (2.1) *if and only if v_k is a solution of*

$$\Delta v_k + H(k, v_k) = 0; \tag{3.4}$$

(ii) *$H(k,v) \geq 0$ for $v > -v_k^*$ with the equality if and only if $v = 0$;*

(iii) *$r_k + w_k > 0$ if and only if $v_k + v_k^* > 0$;*

(iv) *let v be a solution of* (3.4) *and suppose that $v_m < 0$ for some $m \in \mathbb{N}$, that is, $w_m < \tilde{w}_m$, then $v_{m+1} > 0$ if and only if $v_m + v_m^* < 0$.*

Proof. The statements (i), (ii) are consequences of [18, Lemma 2.5].

(iii) We have

$$\begin{aligned} r_k + w_k &= r_k + |h_k|^{-p} v_k + \tilde{w}_k \\ &= r_k + |h_k|^{-p} v_k + r_k \Phi\left(\frac{\Delta h_k}{h_k}\right) \\ &= |h_k|^{-p}\left[v_k + r_k h_k (\Phi(h_k) + \Phi(\Delta h_k))\right] \\ &= |h_k|^{-p}\left(v_k + v_k^*\right). \end{aligned} \tag{3.5}$$

(iv) We have

$$\begin{aligned} v_{m+1} &= v_m - H(m, v_m) \\ &= r_m h_{m+1}\left[\frac{\Phi(h_{m+1})(v_m + G_m)}{\Phi\left(|h_m|^q \Phi^{-1}(r_m) + \Phi^{-1}(v_m + G_m)\right)} - \Phi(\Delta h_m)\right] \\ &= r_m h_{m+1}\left[\frac{\Phi(h_{m+1}) w_m}{\Phi\left(\Phi^{-1}(r_m) + \Phi^{-1}(w_m)\right)} - \Phi(\Delta h_m)\right] \\ &= r_m h_{m+1} \Phi(h_m)\left[\Phi\left(\frac{h_{m+1}}{h_m}\right)\frac{w_m}{\Phi\left(\Phi^{-1}(r_m) + \Phi^{-1}(w_m)\right)} - \Phi\left(\frac{\Delta h_m}{h_m}\right)\right] \\ &= \frac{r_m h_{m+1} \Phi(h_m)}{\Phi\left(\Phi^{-1}(r_m) + \Phi^{-1}(w_m)\right)} \\ &\quad \times \left[\Phi\left(\frac{h_{m+1}\Phi^{-1}(w_m)}{h_m}\right) - \Phi\left(\frac{\Delta h_m}{h_m}\right)\Phi\left(\Phi^{-1}(r_m) + \Phi^{-1}(w_m)\right)\right]. \end{aligned} \tag{3.6}$$

Denote by A the expression in brackets, then

$$\begin{aligned} \operatorname{sgn} A &= \operatorname{sgn}\left[\frac{h_{m+1}\Phi^{-1}(w_m)}{h_m} - \left(\frac{h_{m+1}}{h_m} - 1\right)\left(\Phi^{-1}(r_m) + \Phi^{-1}(w_m)\right)\right] \\ &= \operatorname{sgn}\left[\Phi^{-1}(r_m) + \Phi^{-1}(w_m) - \frac{(h_m + \Delta h_m)\Phi^{-1}(r_m)}{h_m}\right] \\ &= \operatorname{sgn}\left[\Phi^{-1}(w_m) - \Phi^{-1}(\tilde{w}_m)\right] = \operatorname{sgn} v_m = -1. \end{aligned} \tag{3.7}$$

Consequently,

$$v_{m+1} > 0 \Longleftrightarrow \Phi^{-1}(r_m) + \Phi^{-1}(w_m) < 0, \tag{3.8}$$

that is, the statement holds according to the statement (iii) of this lemma. □

Lemma 3.2. *Let v^*, R, G, H be defined by* (3.1), (3.2) *and suppose that $h_k \Delta h_k < 0$ for large k. Then one has the following inequalities for large k.*

If $p \in (1,2]$, then $v_k^* \le R_k$ and

$$v - H(k,v) \le \frac{R_k v}{R_k + v} \quad \text{for } v \in \left(-v_k^*, 0\right]. \tag{3.9}$$

If $p \ge 2$, then $v_k^* \ge R_k$ and

$$v - H(k,v) \ge \frac{R_k v}{R_k + v} \quad \text{for } v \in (-R_k, 0]. \tag{3.10}$$

Proof. We have (with using the Lagrange mean value theorem)

$$\begin{aligned} v_k^* &= r_k h_k (\Phi(h_k) + \Phi(\Delta h_k)) \\ &= r_k h_k \Phi(h_{k+1}) \left[\Phi\left(\frac{h_k}{h_{k+1}}\right) - \Phi\left(-\frac{\Delta h_k}{h_{k+1}}\right)\right] \\ &= r_k h_k \Phi(h_{k+1}) \Phi'(\xi), \end{aligned} \tag{3.11}$$

where $-\Delta h_k / h_{k+1} \le \xi \le h_k / h_{k+1}$ and hence $\xi \ge |\Delta h_k / h_{k+1}|$.

Thus, if $p \in (1,2]$,

$$\begin{aligned} v_k^* &= (p-1) r_k h_k \Phi(h_{k+1}) |\xi|^{p-2} \le (p-1) r_k h_k \Phi(h_{k+1}) \left|\frac{\Delta h_k}{h_{k+1}}\right|^{p-2} \\ &= \frac{1}{q-1} r_k h_k h_{k+1} |\Delta h_k|^{p-2} \le R_k, \end{aligned} \tag{3.12}$$

and in the case $p \ge 2$, we obtain

$$v_k^* \ge R_k. \tag{3.13}$$

Next we proceed similarly as in [18, Lemma 2.6]. Inequalities (3.9), (3.10) can be written in the equivalent forms:

$$(R_k + v) H(k,v) \ge v^2, \quad v \in \left(-v_k^*, 0\right] \text{ for } p \in (1,2], \tag{3.14}$$

$$(R_k + v) H(k,v) \le v^2, \quad v \in (-R_k, 0] \text{ for } p \ge 2. \tag{3.15}$$

Denote $F(k,v) := (R_k+v)H(k,v)-v^2$ and let $v > -v_k^*$. Then

$$H_v(k,v) = 1 - \frac{r_k^q |h_k|^q |h_{k+1}|^p}{\left(|h_k|^q \Phi^{-1}(r_k) + \Phi^{-1}(v+G_k)\right)^p},$$
$$H_{vv}(k,v) = \frac{q r_k^q |h_k|^q |h_{k+1}|^p |v+G_k|^{q-2}}{\left(|h_k|^q \Phi^{-1}(r_k) + \Phi^{-1}(v+G_k)\right)^{p+1}}, \tag{3.16}$$
$$H_{vvv}(k,0) = \frac{q}{r_k^2 h_k^2 h_{k+1}^2 (\Delta h_k)^{2p-3}} \left[(q-2)h_{k+1} - (2q-1)\Delta h_k\right].$$

Consequently, $F(k,0) = F_v(k,0) = F_{vv}(k,0) = 0$ and

$$\begin{aligned} F_{vvv}(k,0) &= R_k H_{vvv}(k,0) + 3H_{vv}(k,0) \\ &= \frac{2}{r_k h_k h_{k+1} \Phi(\Delta h_k)}\left[(q-2)h_{k+1} - (2q-1)\Delta h_k\right] + \frac{3q}{r_k h_k h_{k+1} |\Delta h_k|^{p-2}} \\ &= \frac{1}{r_k h_k h_{k+1} \Phi(\Delta h_k)}\left[2(q-2)(h_k + \Delta h_k) + (2-q)\Delta h_k\right] \\ &= \frac{q-2}{r_k h_k h_{k+1} \Phi(\Delta h_k)}\left[h_k + h_k + \Delta h_k\right] \\ &= \frac{q-2}{r_k h_k h_{k+1} \Phi(\Delta h_k)}\left[h_k + h_{k+1}\right]. \end{aligned} \tag{3.17}$$

Hence, in view of the assumption $h_k \Delta h_k < 0$, $\operatorname{sgn} F_{vvv}(k,0) = -\operatorname{sgn}(q-2)$. It follows that

$$\operatorname{sgn} F(k,v) = \operatorname{sgn} F_{vv}(k,v) = \operatorname{sgn}(q-2) \tag{3.18}$$

in some left neighborhood of $v = 0$, and the function F is positive, decreasing, and convex for $p \in (1,2]$, and is negative, increasing, and concave for $p > 2$ (with respect to v). Hence, both the inequalities (3.14) and (3.15) are satisfied in some left neighborhood of $v = 0$. The proof will be completed by showing that $F_{vv}(k,v)$ has constant sign on the given intervals. By a direct computation,

$$\begin{aligned} F_{vv}(k,v) &= 2H_v(k,v) + (R_k+v)H_{vv}(k,v) - 2 \\ &= -\frac{2 r_k^q |h_k|^q |h_{k+1}|^p}{\left(|h_k|^q \Phi^{-1}(r_k) + \Phi^{-1}(v+G_k)\right)^p} + \frac{q r_k^q |h_k|^q |h_{k+1}|^p |v+G_k|^{q-2}(R_k+v)}{\left(|h_k|^q \Phi^{-1}(r_k) + \Phi^{-1}(v+G_k)\right)^{p+1}} \\ &= \frac{r_k^q |h_k|^q |h_{k+1}|^p}{\left(|h_k|^q \Phi^{-1}(r_k) + \Phi^{-1}(v+G_k)\right)^{p+1}} A(k,v), \end{aligned} \tag{3.19}$$

where

$$A(k,v) := -2|h_k|^q\Phi^{-1}(r_k) - 2\Phi^{-1}(v+G_k) + q|v+G_k|^{q-2}(R_k+v)$$
$$= (q-2)\Phi^{-1}(v+G_k) + q(R_k-G_k)|v+G_k|^{q-2} - 2|h_k|^q\Phi^{-1}(r_k). \tag{3.20}$$

Hence

$$\operatorname{sgn} A(k,v) = \operatorname{sgn} F_{vv}(k,v) \quad \text{for } v > -v_k^*, \tag{3.21}$$

and from (3.18)

$$\operatorname{sgn} A(k,v) = \operatorname{sgn}(q-2) \tag{3.22}$$

in some left neighborhood of $v = 0$.

Moreover, for $v < 0$

$$A_v(k,v) = (q-2)\operatorname{sgn}(v+G_k)|v+G_k|^{q-3}\left[(q-1)(v+G_k) + q(R_k-G_k)\right]$$
$$= -(q-2)|v+G_k|^{q-3}\left[(q-1)v - G_k + qR_k\right], \tag{3.23}$$

and $A_v(k,v) = 0$ (for $v < 0$) if and only if

$$v = \tilde{v}_k := \frac{1}{q-1}(G_k - qR_k) = -\frac{1}{q-1}r_k h_k|\Delta h_k|^{p-2}(h_k + h_{k+1}). \tag{3.24}$$

Next we distinguish between the cases $p \in (1,2]$ and $p \geq 2$.

If $p \in (1,2]$, then using (3.12),

$$\tilde{v}_k \leq -\frac{1}{q-1}r_k h_k h_{k+1}|\Delta h_k|^{p-2} \leq -v_k^*, \tag{3.25}$$

hence $A(k,v)$ is decreasing on $(-v_k^*, 0)$ and in view of (3.22) it means that $A(k,v)$ and consequently from (3.21) also $F_{vv}(k,v)$ is positive for $v \in (-v_k^*, 0)$. Hence, (3.14) holds.

Similarly, if $p \geq 2$, then

$$\tilde{v}_k \leq -\frac{1}{q-1}r_k h_k h_{k+1}|\Delta h_k|^{p-2} \leq -R_k, \tag{3.26}$$

hence $A(k,v)$ is increasing for $v \in (-R_k, 0)$ and from (3.22) we have that $A(k,v)$ and hence also $F_{vv}(k,v)$ is negative for $v \in (-R_k, 0)$. This means that (3.15) is satisfied. □

4. Main Results

Theorem 4.1. *Suppose $p \in (1,2]$ and let h be a solution of* (1.1) *such that $h_k \Delta h_k < 0$ for large k. If*

$$\sum^{\infty} \frac{1}{r_k h_k h_{k+1} |\Delta h_k|^{p-2}} = \infty, \tag{4.1}$$

then h is the recessive solution.

Proof. Denote by $\tilde{w}_k = r_k \Phi(\Delta h_k / h_k)$ the associated solution of (2.1) and let w_k be a solution of (2.1) generated by another solution (linearly independent of h) of (1.1). Then, it follows from Lemma 3.1 that $v_k = |h_k|^p (w_k - \tilde{w}_k)$ is a solution of (3.4), that is,

$$v_{k+1} = v_k - H(k, v_k), \tag{4.2}$$

and suppose that this solution satisfies the condition $v_N < 0$. This means that $w_N < \tilde{w}_N$ and to prove that h is the recessive solution of (1.1), we need to show that there exists $m \geq N$ such that $r_m + w_m \leq 0$, that is, according to Lemma 3.1, $v_m + v_m^* \leq 0$. Suppose by contradiction that $v_k + v_k^* > 0$ for $k \geq N$. According to Lemma 3.1 (iv), it means that $v_k < 0$ for $k \geq N$, that is, $v_k \in (-v_k^*, 0)$. Then we have from Lemma 3.2 that $v_k + R_k > 0$ and

$$v_{k+1} \leq \frac{R_k v_k}{R_k + v_k} \quad \text{for } k \geq N. \tag{4.3}$$

Next, consider the equation

$$u_{k+1} = \frac{R_k u_k}{R_k + u_k}, \tag{4.4}$$

and let u_k be its solution satisfying $u_N = v_N$. However, (4.4) is equivalent to

$$-\Delta u_k = \frac{u_k^2}{R_k + u_k}, \tag{4.5}$$

that is,

$$-\frac{\Delta u_k}{u_k u_{k+1}} = \frac{u_k}{u_{k+1}(R_k + u_k)} = \frac{1}{R_k}, \tag{4.6}$$

where we have substituted for u_{k+1} from (4.4) in the denominator. Hence

$$\frac{1}{u_{k+1}} = \frac{1}{u_k} + \frac{1}{R_k}, \tag{4.7}$$

and we obtain

$$u_k = \frac{1}{1/u_N + \sum_{j=N}^{k-1}(1/R_j)}. \tag{4.8}$$

Condition (4.1) implies that there exists $m \geq N$ such that $u_m < 0$ and either $u_{m+1} > 0$ or u_{m+1} is not defined. This means that $R_m + u_m \leq 0$ (from (4.4)). On the other hand, (4.3) together with (4.4) and the fact that $R_k x/(R_k + x)$ is increasing with respect to x on $(-v_k^*, 0)$ imply that $v_k \leq u_k$ for $k \geq N$. Since $v_k + R_k > 0$ for $k \geq N$, we have $u_k + R_k > 0$ for $k \geq N$, a contradiction. □

Theorem 4.2. *Suppose $p \geq 2$ and let h be a solution of* (1.1) *such that $h_k \Delta h_k < 0$ for large k. If*

$$\sum^{\infty} \frac{1}{r_k h_k h_{k+1} |\Delta h_k|^{p-2}} < \infty, \tag{4.9}$$

then h is not the recessive solution.

Proof. Similarly, as in the proof of Theorem 4.1, denote $\tilde{w}_k = r_k \Phi(\Delta h_k / h_k)$ and let w_k be a solution of (2.1) generated by another solution (linearly independent of h) of (1.1). Then $v_k = |h_k|^p (w_k - \tilde{w}_k)$ is a solution of (3.4), that is,

$$v_{k+1} = v_k - H(k, v_k), \tag{4.10}$$

and suppose that this solution satisfies the condition $v_N < 0$, $|v_N|$ being sufficiently small (will be specified later). Hence $w_N < \tilde{w}_N$ and we have to show that $r_k + w_k > 0$ for $k \geq N$, that is, $v_k + v_k^* > 0$ for $k \geq N$.

Let u_k be a solution of (4.4) and suppose that $u_N = v_N$. Hence, similarly as in the proof of Theorem 4.1, we obtain

$$u_k = \frac{1}{1/u_N + \sum_{j=N}^{k-1}(1/R_j)}. \tag{4.11}$$

If $|u_N|$ is sufficiently small, then condition (4.9) implies that $u_k < 0$ for $k \geq N$ and from (4.4), we have $R_k + u_k > 0$ for $k \geq N$. Consequently, from Lemma 3.2 we obtain that $v_k^* \geq R_k$ and

$$u_k - H(k, u_k) \geq \frac{R_k u_k}{R_k + u_k} = u_{k+1} \quad \text{for } k \geq N. \tag{4.12}$$

Moreover, since $x - H(k, x)$ is increasing with respect to x on $(-R_k, 0)$, we obtain from (4.12) that $v_k \geq u_k$ for $k \geq N$. Hence $R_k + v_k > 0$ for $k \geq N$ and hence also $v_k^* + v_k > 0$ for $k \geq N$. □

5. Applications and Open Problems

(i)Theorems 4.1 and 4.2, as formulated in the previous section, apply only to positive decreasing (or negative increasing) solutions of (1.1). The reason is that we have been able to prove inequalities (3.9), (3.10) only when $G = rh\Phi(\Delta h) < 0$. We conjecture that Theorems 4.1 and 4.2 remain to hold for *every* solution of (1.1) for which $\Delta h_k \neq 0$ for large k. To justify this conjecture, consider the function

$$\mathcal{F}_k(v) = H(k,v)/v(v - H(k,v)). \tag{5.1}$$

By an easy computation one can find that inequalities (3.9), (3.10) are equivalent to the inequalities

$$\mathcal{F}_k(v) \geq \frac{1}{R_k}, \quad p \in (1,2], \quad \mathcal{F}_k(v) \leq \frac{1}{R_k}, \quad p \in [2,\infty). \tag{5.2}$$

However, if $G_k > 0$, that is, $-G_k < 0$, we have

$$\mathcal{F}_k(-G_k) = \frac{1}{r_k h_k h_{k+1} |\Delta h_k|^{p-2}} = \frac{2}{qR_k}, \tag{5.3}$$

so inequalities (3.9), (3.10) are no longer valid in this case. Numerical computations together with a closer examination of the graph of the function $\mathcal{F}$ lead to the following conjecture.

Conjecture 5.1. *Let $h_k, h_{k+1} > 0$, $\Delta h_k \neq 0$, and $R_k^* := (q-1) r_k h_k h_{k+1} |\Delta h_k|^{p-2}$. Then for $v \in (-v_k^*, \infty)$ one has*

$$\mathcal{F}_k(v) \geq \frac{1}{R_k^*} \quad \text{for } p \in (1,2], \qquad \mathcal{F}_k(v) \leq \frac{1}{R_k^*} \quad \text{for } p \in [2,\infty). \tag{5.4}$$

To explain this conjecture in more details, consider the case $p \in (1,2]$, the case $p \geq 2$ can be treated analogically. We have (we skip the index k, only indices different from k are written explicitly)

$$\begin{aligned} \mathcal{F}(\infty) := \lim_{v\to\infty} \mathcal{F}(v) &= \frac{1}{rh_{k+1}[\Phi(h_{k+1}) - \Phi(\Delta h)]} \\ &= \frac{1}{r\Phi(h)h_{k+1}[\Phi(h_{k+1}/h) - \Phi(\Delta h/h)]} \\ &= \frac{(q-1)|\xi|^{2-p}}{r\Phi(h)h_{k+1}}, \end{aligned} \tag{5.5}$$

where $\Delta h/h \leq \xi \leq h_{k+1}/h$. If $\Delta h > 0$, the direct substitution yields

$$\mathcal{F}(\infty) \geq \frac{(q-1)}{rhh_{k+1}|\Delta h|^{p-2}} \geq \frac{1}{(q-1)rhh_{k+1}|\Delta h|^{p-2}} = \frac{1}{R^*}. \tag{5.6}$$

If $\Delta h < 0$, then $|\Delta h| < h$ and we proceed as follows. For $p \in (1, 2]$, the function Φ is concave for nonnegative arguments, so for $x, y \geq 0$, we have the inequality

$$\Phi\left(\frac{x+y}{2}\right) \geq \frac{1}{2}\left[\Phi(x) + \Phi(y)\right]. \tag{5.7}$$

We substitute $x = h_{k+1}/h$, $y = -\Delta h/h$, then $x + y = 1$, that is, $2^{2-p} \geq \Phi(x) + \Phi(y)$. Hence we have

$$\begin{aligned} \mathcal{F}(\infty) &= \frac{1}{r h_{k+1} \Phi(h)\left[\Phi(h_{k+1}/h) - \Phi(\Delta h/h)\right]} \\ &\geq \frac{1}{2^{2-p} r h_{k+1} \Phi(h)} = \frac{|h|^{2-p}}{2^{2-p} r h_{k+1} h}. \end{aligned} \tag{5.8}$$

Hence

$$\mathcal{F}(\infty) \geq \frac{|h|^{2-p}}{2^{2-p} r h_{k+1} h} \geq \frac{|\Delta h|^{2-p}}{2^{2-p} r h_{k+1} h}. \tag{5.9}$$

Next we prove that $(q-1) \geq 2^{2-p}$ for $p \in (1, 2]$. Denote $t = q - 1 = 1/(p-1)$, then we need to prove the inequality $g(t) := t - 2 \cdot 2^{-1/t} \geq 0$ for $t \in [1, \infty)$. A standard investigation of the graph of the function $t \to 2 \cdot 2^{-1/t}$ shows that the required inequality really holds, so we have

$$\mathcal{F}(\infty) \geq \frac{1}{(q-1) r h_{k+1} h |\Delta h|^{p-2}} = \frac{1}{R^*}. \tag{5.10}$$

By a similar computation we find that

$$\begin{gathered} \mathcal{F}(0) = \lim_{v \to 0} \mathcal{F}(v) = \frac{1}{R} \leq \frac{1}{R^*}, \quad v^* \leq R^*, \\ \mathcal{F}(-v^*+) = \lim_{v \to -v^*+} \mathcal{F}(v) = \frac{1}{v^*} \geq \frac{1}{R_k^*}, \quad \mathcal{F}'(-v^*+) < 0, \\ \mathcal{F}'(-G) < 0 \quad \text{if } G < 0, \quad \mathcal{F}'(-G) > 0 \quad \text{if } G > 0, \\ \mathcal{F}'(0) < 0 \quad \text{if } G < 0, \quad \mathcal{F}'(0) > 0 \quad \text{if } G > 0. \end{gathered} \tag{5.11}$$

These computations lead to the conjecture that $\mathcal{F}$ attains its global minimum at a point in $(-v^*, -G)$ if $G > 0$ and at a point in $(-G, \infty)$ if $G < 0$. Numerical computations suggest that this minimum is $1/(c r h h_{k+1} |\Delta h|^{p-2})$, where $1 \leq c \leq q - 1$.

Having proved inequalities (5.4), Theorems 4.1 and 4.2 could be proved for any positive h with $\Delta h \neq 0$ in the same way as in the previous section, it is only sufficient to replace $R = (2/q) r h h_{k+1} |\Delta h|^{p-2}$ by $R^* = (q-1) r h h_{k+1} |\Delta h|^{p-2}$.

(ii) A typical example of (1.1) to which Theorems 4.1 and 4.2 apply is (1.1) with

$$\sum^{\infty} r_k^{1-q} < \infty, \quad c_k > 0, \quad \sum^{\infty} c_k = \infty, \tag{5.12}$$

since under these assumption all positive solutions of (1.1) are decreasing, see [19]. However, one can apply *indirectly* Theorems 4.1 and 4.2 also to (1.1) with

$$\sum^{\infty} r_k^{1-q} = \infty, \qquad c_k > 0 \tag{5.13}$$

(and $\sum^{\infty} c_k < \infty$, otherwise (1.1) would be oscillatory, see [16, Theorem 8.2.14]), even if all positive solutions of (1.1) are *increasing* in this case. The method which enables to overcome this difficulty is the so-called *reciprocity principle*, which can be explained as follows.

Suppose that $c_k \neq 0$ in (1.1) and let $u_k := r_k \Phi(\Delta x_k)$. Then by a direct computation one can verify that u solves the so-called *reciprocal equation:*

$$\Delta\left(\frac{1}{\Phi^{-1}(c_k)}\Phi^{-1}(\Delta u_k)\right) + r_{k+1}^{1-q}\Phi^{-1}(u_{k+1}) = 0. \tag{5.14}$$

Moreover, if c_k does not change its sign for large k, (1.1) is nonoscillatory if and only if (5.14) is nonoscillatory, see [9]. The following statement relates recessive solutions of (1.1) and (5.14). A similar statement can be found in [9], but our proof differs from that given in [9].

Theorem 5.2. *Suppose that* (1.1) *is nonoscillatory and* (5.12) *or* (5.13) *holds. If a solution h of* (1.1) *is recessive, then $\tilde{u} := r\Phi(\Delta h)$ is the recessive solution of* (5.14).

Proof. First suppose that (5.13) holds and let $\tilde{w} = r\Phi(\Delta h/h)$ be the distinguished solution of (2.1). Assumption (5.13) implies that $\tilde{w}_k > 0$ for large k, see [7]. The solution v of the Riccati equation

$$v_{k+1} + r_{k+1}^{1-q} - \frac{c_k^{1-q} v_k}{\Phi^{-1}\left(c_k^{-1} + \Phi(v_k)\right)} = 0 \tag{5.15}$$

associated with (5.14) is given by $v = (c^{1-q}\Phi^{-1}(\Delta u))/\Phi^{-1}(u)$ and we have the following relationship between solutions of (5.15) and (2.1) (no index means again the index k):

$$\begin{aligned} v &= \frac{c^{1-q}\Phi^{-1}(\Delta u)}{\Phi^{-1}(u)} = \frac{c^{1-q}\Phi^{-1}(-c\Phi(x_{k+1}))}{\Phi^{-1}(r\Phi(\Delta x))} = -\frac{x_{k+1}}{\Phi^{-1}(r)\Delta x} = -\frac{x + \Delta x}{\Phi^{-1}(r)\Delta x} \\ &= -\frac{1 + \Delta x/x}{\Phi^{-1}(r)(\Delta x/x)} = -\frac{1 + \Phi^{-1}(w)/\Phi^{-1}(r)}{\Phi^{-1}(w)} = -\frac{\Phi^{-1}(r) + \Phi^{-1}(w)}{\Phi^{-1}(r)\Phi^{-1}(w)}. \end{aligned} \tag{5.16}$$

Since the function

$$x \longrightarrow -\frac{\Phi^{-1}(r) + \Phi^{-1}(x)}{\Phi^{-1}(r)\Phi^{-1}(x)} \tag{5.17}$$

is increasing for $x \in \mathbb{R} \setminus \{0\}$, the inequality $0 < \widetilde{w}_k < w_k$ for large k and for any solution $w \neq \widetilde{w}$ of (2.1) implies the inequality $0 > v_k > \widetilde{v}_k$, where

$$\widetilde{v} = \frac{c^{1-q}\Phi^{-1}(\Delta \widetilde{u}_k)}{\Phi^{-1}(\widetilde{u}_k)} = -\frac{\Phi^{-1}(r) + \Phi^{-1}(\widetilde{w})}{\Phi^{-1}(r)\Phi^{-1}(\widetilde{w})}, \tag{5.18}$$

and v is any other solution of (5.15). Consequently, $\widetilde{v}$ is the distinguished solution of (5.15) and hence $\widetilde{u}$ is the recessive solution of (5.14).

Now suppose that (5.12) holds. Then all solutions w of (2.1) satisfying $r_k + w_k > 0$ for large k are negative (see [19]), that is, $0 > w_k > \widetilde{w}_k$. Then using the same argument as in the first part of the proof we have $0 < \widetilde{v}_k < v_k$ for large k for any solution v of (5.15), that is, $\widetilde{u}$ is the recessive solution of (5.14). □

(iii) In [18], we posed the question whether the sequence $h_k := k^{(p-1)/p}$ is the recessive solution of the difference equation

$$\Delta(\Phi(\Delta x_k)) + c_k\Phi(x_{k+1}) = 0, \quad c_k := -\frac{\Delta(\Phi(\Delta h_k))}{\Phi(h_{k+1})}. \tag{5.19}$$

Now we can give the affirmative answer to this question for $p \geq 2$. It is shown in [18] that

$$\begin{aligned} u_k := \Phi(\Delta h_k) &= \left(\frac{p-1}{p}\right)^{p-1} k^{-(p-1)/p}\left[1 + \frac{p-1}{2pk} + o\left(k^{-1}\right)\right], \\ c_k &= \frac{\gamma_p}{(k+1)^p}\left[1 + O\left(k^{-1}\right)\right], \quad \gamma_p := \left(\frac{p-1}{p}\right)^p, \end{aligned} \tag{5.20}$$

both as $k \to \infty$. The sequence u is a solution of the equation

$$\Delta\left(c_k^{1-q}\Phi^{-1}(\Delta u_k)\right) + \Phi^{-1}(u_{k+1}) = 0, \tag{5.21}$$

which is reciprocal to (5.19) and $y_k = h_{k+1} = (k+1)^{(p-1)/p}$ is a solution of the equation

$$\Delta(\Phi(\Delta y_k)) + c_{k+1}\Phi(y_{k+1}) = 0, \tag{5.22}$$

which is reciprocal to (5.21) and differs from (5.19) only by the shift $k \to k+1$ in the sequence c. Since

$$\sum^{\infty}\left(c_k^{1-q}\right)^{1-p} = \sum^{\infty} c_k < \infty, \tag{5.23}$$

assumption (5.12) is satisfied (with q, c^{1-q}, and 1 instead of p, r, and c, resp.), hence positive solutions of (5.21) are decreasing, that is, Theorems 4.1 and 4.2 apply to this case. By a direct computation, we have

$$c_k^{1-q} u_k u_{k+1} |\Delta u_k|^{q-2} \sim k^{-p(1-q)} k^{-2(p-1)/p} k^{(-2p+1)(q-2)/p} = k. \tag{5.24}$$

This means, by Theorem 4.1, that if $q \in (1,2]$, then u is the recessive solution of (5.21) and hence $y_k = h_{k+1}$ is the recessive solution of (5.22). Consequently, $h_k = k^{(p-1)/p}$ is the recessive solution of (5.19) if $p \geq 2$.

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References

[1] P. Řehák, "Hartman-Wintner type lemma, oscillation, and conjugacy criteria for half-linear difference equations," *Journal of Mathematical Analysis and Applications*, vol. 252, no. 2, pp. 813–827, 2000.
[2] P. Řehák, "Oscillatory properties of second order half-linear difference equations," *Czechoslovak Mathematical Journal*, vol. 51, no. 2, pp. 303–321, 2001.
[3] P. Řehák, "Generalized discrete Riccati equation and oscillation of half-linear difference equations," *Mathematical and Computer Modelling*, vol. 34, no. 3-4, pp. 257–269, 2001.
[4] P. Řehák, "Oscillation criteria for second order half-linear difference equations," *Journal of Difference Equations and Applications*, vol. 7, no. 4, pp. 483–505, 2001.
[5] P. Řehák, "Oscillation and nonoscillation criteria for second order linear difference equations," *Fasciculi Mathematici*, no. 31, pp. 71–89, 2001.
[6] R. P. Agarwal, M. Bohner, S. R. Grace, and D. O'Regan, *Discrete Oscillation Theory*, Hindawi, New York, NY, USA, 2005.
[7] O. Došlý and P. Řehák, "Recessive solution of half-linear second order difference equations," *Journal of Difference Equations and Applications*, vol. 9, no. 1, pp. 49–61, 2003.
[8] M. Cecchi, Z. Došlá, and M. Marini, "Nonoscillatory half-linear difference equations and recessive solutions," *Advances in Difference Equations*, no. 2, pp. 193–204, 2005.
[9] M. Cecchi, Z. Došlá, and M. Marini, "Reciprocity principle and asymptotic properties of solutions of half-linear difference equations," submitted.
[10] M. Cecchi, Z. Došlá, M. Marini, and I. Vrkoč, "Asymptotic properties for half-linear difference equations," *Mathematica Bohemica*, vol. 131, no. 4, pp. 347–363, 2006.
[11] O. Došlý and Á. Elbert, "Integral characterization of the principal solution of half-linear second order differential equations," *Studia Scientiarum Mathematicarum Hungarica*, vol. 36, no. 3-4, pp. 455–469, 2000.
[12] M. Cecchi, Z. Došlá, and M. Marini, "Half-linear equations and characteristic properties of the principal solution," *Journal of Differential Equations*, vol. 208, no. 2, pp. 494–507, 2005.
[13] M. Cecchi, Z. Došlá, and M. Marini, "Corrigendum to: "Half-linear equations and characteristic properties of the principal solution"," *Journal of Differential Equations*, vol. 221, no. 1, pp. 272–274, 2006.
[14] M. Cecchi, Z. Došlá, and M. Marini, "Limit and integral properties of principal solutions for half-linear differential equations," *Archivum Mathematicum*, vol. 43, no. 1, pp. 75–86, 2007.
[15] Á. Elbert and T. Kusano, "Principal solutions of non-oscillatory half-linear differential equations," *Advances in Mathematical Sciences and Applications*, vol. 8, no. 2, pp. 745–759, 1998.
[16] O. Došlý and P. Řehák, *Half-linear Differential Equations*, vol. 202 of *North-Holland Mathematics Studies*, Elsevier, Amsterdam, The Netherlands, 2005.
[17] M. Cecchi, Z. Došlá, and M. Marini, "On recessive and dominant solutions for half-linear difference equations," *Journal of Difference Equations and Applications*, vol. 10, no. 9, pp. 797–808, 2004.

[18] O. Došlý and S. Fišnarová, “Linearized Riccati technique and (non-)oscillation criteria for half-linear difference equations,” *Advances in Difference Equations*, vol. 2008, Article ID 438130, 18 pages, 2008.

[19] W.-T. Li, “Classification schemes for nonoscillatory solutions of two-dimensional nonlinear difference systems,” *Computers & Mathematics with Applications*, vol. 42, no. 3–5, pp. 341–355, 2001.