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Research Article

On the Superstability Related with the Trigonometric Functional Equation

Gwang Hui Kim

Department of Mathematics, Kangnam University, Youngin, Gyeonggi 446-702, South Korea

Correspondence should be addressed to Gwang Hui Kim, ghkim@kangnam.ac.kr

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We will investigate the superstability of the (hyperbolic) trigonometric functional equation from the following functional equations: $f(x+y)\pm g(x-y)=\lambda f(x)g(y)$, $f(x+y)\pm g(x-y)=\lambda g(x)f(y)$, $f(x+y)\pm g(x-y)=\lambda f(x)f(y)$, $f(x+y)\pm g(x-y)=\lambda g(x)g(y)$, which can be considered the mixed functional equations of the sine function and cosine function, of the hyperbolic sine function and hyperbolic cosine function, and of the exponential functions, respectively.

1. Introduction

Baker et al. in [1] introduced the following: if f satisfies the inequality $|E_1(f)-E_2(f)|\le\varepsilon$, then either f is bounded or $E_1(f)=E_2(f)$. This is frequently referred to as superstability.

The superstability of the cosine functional equation (also called the d'Alembert equation):

$$f(x+y)+f(x-y)=2f(x)f(y) \tag{C}$$

and the sine functional equation

$$f(x)f(y)=f\left(\frac{x+y}{2}\right)^2-f\left(\frac{x-y}{2}\right)^2 \tag{S}$$

were investigated by Baker [2] and Cholewa [3], respectively. Their results were improved by Badora [4], Badora and Ger [5], Forti [6], and Găvruta [7], as well as by Kim [8, 9] and Kim and Dragomir [10]. The superstability of the Wilson equation

$$f(x+y)+f(x-y)=2f(x)g(y), \tag{C_{fg}}$$

was investigated by Kannappan and Kim [11].

The superstability of the trigonometric functional equation with the sine and the cosine equation

$$f(x+y)-f(x-y)=2f(x)f(y), \tag{T}$$

$$f(x+y)-f(x-y)=2f(x)g(y) \tag{T_{fg}}$$

was investigated by Kim [12].

The hyperbolic cosine function, hyperbolic sine function, hyperbolic trigonometric function, and some exponential functions satisfy the aforementioned equations; thus they can be called by the *hyperbolic* cosine (sine, trigonometric, exponential) functional equation, respectively.

The aim of this paper is to investigate the superstability of the (hyperbolic) sine functional equation (S) from the following functional equations:

$$f(x+y)+g(x-y)=\lambda f(x)g(y), \tag{C_{fgfg}}$$

$$f(x+y)+g(x-y)=\lambda g(x)f(y), \tag{C_{fggf}}$$

$$f(x+y)-g(x-y)=\lambda f(x)g(y), \tag{T_{fgfg}}$$

$$f(x+y)-g(x-y)=\lambda g(x)f(y), \tag{T_{fggf}}$$

on the abelian group. Consequently, we obtain the superstability of (S) from the following functional equations:

$$f(x+y)+g(x-y)=\lambda f(x)f(y), \tag{C_{fgff}}$$

$$f(x+y)+g(x-y)=\lambda g(x)g(y), \tag{C_{fggg}}$$

$$f(x+y)-g(x-y)=\lambda f(x)f(y), \tag{T_{fgff}}$$

$$f(x+y)-g(x-y)=\lambda g(x)g(y). \tag{T_{fggg}}$$

Furthermore, the obtained results of which can be extended to the Banach space.

In this paper, let $(G,+)$ be a uniquely 2-divisible Abelian group, $\mathbb{C}$ the field of complex numbers, and $\mathbb{R}$ the field of real numbers. Whenever we deal with (C), we do not need to assume that 2-divisibility of $(G,+)$ but the Abelian condition is enough.

We may assume that f and g are nonzero functions, and ε is a nonnegative real constant, $\varphi : G \to \mathbb{R}$. For the notation of the equation,

$$f(x+y)+f(x-y)=\lambda f(x)f(y), \tag{C^λ}$$

$$f(x+y)+f(x-y)=\lambda f(x)g(y), \tag{C^λ_{fg}}$$

$$g(x+y)+g(x-y)=\lambda g(x)g(y), \tag{C^λ_{g}}$$

$$g(x+y)+g(x-y)=\lambda g(x)f(y). \tag{C^λ_{gf}}$$

2. Superstability of the Functional Equations

In this section, we will investigate the superstability of the hyperbolic sine functional equation (S) from the functional equations (C_{fgfg}), (C_{fggf}), (C_{fgff}), (C_{fggg}), (T_{fgfg}), (T_{fggf}), (T_{fgff}), and (T_{fggg}).

Theorem 2.1. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$\left|f(x+y)+g(x-y)-\lambda f(x)g(y)\right| \le \varepsilon \quad \forall x, y \in G. \tag{2.1}$$

If g (or f) fails to be bounded, then

(i) *f with $f(0)=0$ satisfies* (S),

(ii) *g with $g(0)=0$ satisfies* (S),

(iii) *particularly, if g satisfies (C^λ), then f and g are solutions of the Wilson-type equation (C^λ_{fg}); if f satisfies (C^λ), then f and g are solutions of (C^λ_{gf}).*

Proof. Taking $y=0$ in the (2.1), then it implies that

$$\begin{gathered}\left|g(x)\right| \le \left|f(x)-\lambda f(x)g(0)\right|+\varepsilon,\\ \left|f(x)\right| \le \frac{\left|g(x)\right|+\varepsilon}{\left|1-\lambda g(0)\right|}.\end{gathered} \tag{2.2}$$

From (2.2), we can know that f is bounded if and only if g is bounded.

Let g be the unbounded solution of (2.1). Then, there exists a sequence $\{y_n\}$ in G such that $0 \ne |g(y_n)| \to \infty$ as $n \to \infty$.

(i) Taking $y=y_n$ in (2.1), dividing both sides by $|\lambda g(y_n)|$, and passing to the limit as $n \to \infty$, we obtain the following:

$$f(x)=\lim_{n\to\infty}\frac{f(x+y_n)+g(x-y_n)}{\lambda g(y_n)}, \quad x \in G. \tag{2.3}$$

Using (2.1), we have

$$\begin{aligned} &\big| f(x+(y+y_n)) + g(x-(y+y_n)) - \lambda f(x)g(y+y_n) \\ &\quad f(x+(-y+y_n)) + g(x-(-y+y_n)) - \lambda f(x)g(-y+y_n)\big| \le 2\varepsilon, \end{aligned} \tag{2.4}$$

so that

$$\begin{aligned} &\bigg| \frac{f((x+y)+y_n)+g((x+y)-y_n)}{\lambda g(y_n)} \\ &\quad + \frac{f((x-y)+y_n)+g((x-y)-y_n)}{\lambda g(y_n)} - \lambda f(x)\cdot \frac{g(y+y_n)+g(-y+y_n)}{\lambda g(y_n)} \bigg| \\ &\le \frac{2\varepsilon}{|\lambda||g(y_n)|} \quad \forall x,y \in G. \end{aligned} \tag{2.5}$$

We conclude that, for every $y \in G$, there exists a limit function

$$k_1(y) := \lim_{n\to\infty} \frac{g(y+y_n)+g(-y+y_n)}{\lambda g(y_n)}, \tag{2.6}$$

where the function $k_1 : G \to \mathbb{C}$ satisfies

$$f(x+y)+f(x-y) = \lambda f(x)k_1(y) \quad \forall x,y \in G. \tag{2.7}$$

Applying the case $f(0) = 0$ in (2.7), it implies that f is odd. Keeping this in mind, by means of (2.7), we infer the equality

$$\begin{aligned} f(x+y)^2 - f(x-y)^2 &= \lambda f(x)k_1(y)[f(x+y)-f(x-y)] \\ &= f(x)[f(x+2y)-f(x-2y)] \\ &= f(x)[f(2y+x)+f(2y-x)] \\ &= \lambda f(x)f(2y)k_1(x). \end{aligned} \tag{2.8}$$

Putting $y = x$ in (2.7), we obtain the equation

$$f(2x) = \lambda f(x)k_1(x), \quad x \in G. \tag{2.9}$$

This, in return, leads to the equation

$$f(x+y)^2 - f(x-y)^2 = f(2x)f(2y) \tag{2.10}$$

valid for all $x, y \in G$, which, in the light of the unique 2-divisibility of G, states nothing else but (S).

Due to the necessary and sufficient conditions for the boundedness of f and g, the unboundedness of f is assumed. For the unbounded f of (2.1), we can choose a sequence $\{x_n\}$ in G such that $0 \neq |f(x_n)| \to \infty$ as $n \to \infty$.

(ii) Taking $x = x_n$ in (2.1), dividing both sides by $|\lambda f(x_n)|$, and passing to the limit as $n \to \infty$, we obtain

$$g(y) = \lim_{n\to\infty} \frac{f(x_n + y) + g(x_n - y)}{\lambda f(x_n)}, \quad x \in G. \tag{2.11}$$

Replacing x by $x_n + x$ and $x_n - x$ in (2.1), dividing by $|\lambda f(x_n)|$, it then gives us the existence of a limit function

$$k_2(x) := \lim_{n\to\infty} \frac{f(x_n + x) + f(x_n - x)}{\lambda f(x_n)}, \tag{2.12}$$

where the function $k_2 : G \to \mathbb{C}$ satisfies

$$g(y + x) + g(y - x) = \lambda k_2(x) g(y) \quad \forall x, y \in G. \tag{2.13}$$

Applying the case $g(0) = 0$ in (2.13), it implies that g is odd.

A similar procedure to that applied in (i) in (2.13) allows us to show that g satisfies (S).

(iii) In the case g satisfies (C^λ), the limit k_1 states nothing else but g; thus, (2.7) validates the required equation (C^λ_{fg}). Also in the case f satisfies (C^λ), since the limit k_2 states nothing else but f, the functions g and f are solutions of (C^λ_{gf}) from (2.13). □

Corollary 2.2. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$\left| f(x + y) + g(x - y) - \lambda f(x) f(y) \right| \le \varepsilon \quad \forall x, y \in G. \tag{2.14}$$

Then, either f with $f(0) = 0$ is bounded or f satisfies (S).

Proof. Substituting $f(y)$ for $g(y)$ in the stability inequality (2.1) of Theorem 2.1, the process of the proof is the same as (i) of Theorem 2.1.

Namely, for f be unbounded, there exists a sequence $\{y_n\}$ in G such that $0 \neq |f(y_n)| \to \infty$ as $n \to \infty$. Taking $y = y_n$ in (2.1), dividing both sides by $|\lambda f(y_n)|$, and passing to the limit as $n \to \infty$, we obtain

$$f(x) = \lim_{n\to\infty} \frac{f(x + y_n) + g(x - y_n)}{\lambda f(y_n)}, \quad x \in G. \tag{2.15}$$

An obvious slight change in the proof steps applied after formula (2.3) allows one to the required result via (2.7). □

Theorem 2.3. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$\left|f(x+y) + g(x-y) - \lambda g(x)f(y)\right| \leq \varepsilon \quad \forall x, y \in G. \tag{2.16}$$

If f (or g) fails to be bounded, then

(i) *g with $g(0) = 0$ satisfies* (S),

(ii) *f with $f(0) = 0$ satisfies* (S),

(iii) *particularly, if g satisfies (C^{λ}), then f and g are solutions of the Wilson equation (C^{λ}_{fg}), and also if f satisfies (C^{λ}), then g and f are solutions of (C^{λ}_{gf}).*

Proof. The process of the proof is similar as Theorem 2.1. Therefore, we will only write an brief proof for the case (i). Indeed, the necessary and sufficient conditions for the boundedness of f and g are same.

(i) For the unbounded f, we can choose a sequence $\{y_n\}$ in G such that $0 \neq |f(y_n)| \to \infty$ as $n \to \infty$.

A similar reasoning as the proof applied in Theorem 2.1 for (2.16) with $y = y_n$ gives us

$$g(x) = \lim_{n\to\infty} \frac{f(x+y_n) + g(x-y_n)}{\lambda f(y_n)}, \quad x \in G. \tag{2.17}$$

Substituting $y + y_n$ and $-y + y_n$ for y in (2.16), and dividing by $|\lambda f(y_n)|$, it then gives us the existence of a limit function

$$k_3(y) := \lim_{n\to\infty} \frac{f(y+y_n) + f(-y+y_n)}{\lambda f(y_n)}, \tag{2.18}$$

where the function $k_3 : G \to \mathbb{C}$ satisfies the equation

$$g(x+y) + g(x-y) = \lambda g(x)k_3(y) \quad \forall x, y \in G. \tag{2.19}$$

Applying the case $g(0) = 0$ in (2.19), it implies that g is odd.

A similar procedure to that applied in (i) of Theorem 2.1 in (2.19) allows us to show that g satisfies (S).

The proofs for (ii) and (iii) also run along those of Theorem 2.1. □

Corollary 2.4. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$\left|f(x+y) + g(x-y) - \lambda g(x)g(y)\right| \leq \varepsilon \quad \forall x, y \in G. \tag{2.20}$$

Then, either g with $g(0) = 0$ is bounded or g satisfies (S).

Proof. Substituting $g(x)$ for $f(x)$ in (2.16) of Theorem 2.3, the next of the proof runs along that of the Theorem 2.3. □

Since the proofs of the functional equations (T_{fgfg}), (T_{fggf}), (T_{fgff}), and (T_{fggg}) are very similar to above mentioned proofs, we will give a brief proof for Theorem 2.5.

Theorem 2.5. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$\left|f(x+y) - g(x-y) - \lambda f(x)g(y)\right| \leq \varepsilon \quad \forall x, y \in G. \tag{2.21}$$

If g (or f) fails to be bounded, then

(i) *f with $f(0) = 0$ satisfies* (S),

(ii) *g with $g(0) = 0$ satisfies* (S),

(iii) *particularly, if g satisfies (C^{λ}), then f and g are solutions of the Wilson equation (C^{λ}_{fg}), and also if f satisfies (C^{λ}), then f and g are solutions of (C^{λ}_{gf}).*

Proof. Using the same method as the proof of Theorem 2.1, we can know that f is bounded if and only if g is bounded.

(i) For the unbounded g, we can choose a sequence $\{y_n\}$ in G such that $0 \neq |g(y_n)| \to \infty$ as $n \to \infty$.

A similar reasoning as the proof applied in Theorem 2.1 for (2.21) with $y = y_n$ gives us

$$f(x) = \lim_{n \to \infty} \frac{f(x+y_n) - g(x-y_n)}{\lambda g(y_n)}, \quad x \in G. \tag{2.22}$$

Substituting $y + y_n$ and $-y + y_n$ for y in (2.21), and dividing by $|\lambda f(y_n)|$, it then gives us the existence of a limit function

$$k_4(y) := \lim_{n \to \infty} \frac{\lambda g(y+y_n) + g(-y+y_n)}{\lambda g(y_n)}, \tag{2.23}$$

where the function $k_4 : G \to \mathbb{C}$ satisfies the equation

$$f(x+y) + f(x-y) = \lambda f(x) k_4(y) \quad \forall x, y \in G. \tag{2.24}$$

The next of the proof runs along the same procedure as before.

(ii) For unbounded f, let $x = x_n$ in (2.21), dividing both sides by $|\lambda f(x_n)|$, and passing to the limit as $n \to \infty$, we obtain

$$g(y) = \lim_{n \to \infty} \frac{f(x_n+y) - g(x_n-y)}{\lambda f(x_n)}, \quad x \in G. \tag{2.25}$$

Replacing x by $x + x_n$ and $-x + x_n$ in (2.21) and dividing it by $|\lambda f(y_n)|$, which gives us the existence of a limit function

$$k_5(x) := \lim_{n \to \infty} \frac{f(x+x_n) + f(-x+x_n)}{\lambda f(x_n)}, \tag{2.26}$$

where the function $k_5 : G \to \mathbb{C}$, satisfy

$$g(y+x) + g(y-x) = \lambda k_5(x) g(y) \quad \forall x, y \in G. \tag{2.27}$$

The next of the proof and (iii) also run along the same procedure as before. □

Corollary 2.6. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$\left| f(x+y) - g(x-y) - \lambda f(x) f(y) \right| \le \varepsilon \quad \forall x, y \in G. \tag{2.28}$$

Then, either f with $f(0) = 0$ is bounded or f satisfies (S).

Theorem 2.7. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$\left| f(x+y) - g(x-y) - \lambda g(x) f(y) \right| \le \varepsilon \quad \forall x, y \in G. \tag{2.29}$$

If g (or f) fails to be bounded, then

(i) *f with $f(0) = 0$ satisfies* (S),

(ii) *g with $g(0) = 0$ satisfies* (S),

(iii) *particularly, if g satisfies (C^{λ}), then f and g are solutions of the Wilson equation (C^{λ}_{fg}), and also if f satisfies (C^{λ}), then f and g are solutions of (C^{λ}_{gf}).*

Proof. As in Theorem 2.5, the proof steps in Theorem 2.1 should be followed. □

Corollary 2.8. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$\left| f(x+y) - g(x-y) - \lambda g(x) g(y) \right| \le \varepsilon \quad \forall x, y \in G. \tag{2.30}$$

Then, either g with $g(0) = 0$ is bounded or g satisfies (S).

Remark 2.9. Let us consider the case $\lambda = 2$.

(i) Substituting f for g of the second term of the stability inequalities in the aforementioned results, which imply the (hyperbolic) cosine type functional equations (C, C_{fg}), and the (hyperbolic) trigonometric-type functional equation (T, T_{fg}). Their stability was founded in papers [8, 10, 12, 13].

(ii) Substituting f for g in the aforementioned results, Theorems 2.1 and 2.3 and Corollaries 2.2 and 2.4 imply the (hyperbolic) cosine functional equation (C), the stability of which is established in the work in [4–7]. Furthermore, Theorems 2.5 and 2.7 and Corollaries 2.6 and 2.8 imply the (hyperbolic) trigonometric functional equation (T), the stability of which is established in [14].

3. Extension to the Banach Space

In all the results presented in Section 2, the range of functions on the abelian group can be extended to the Banach space. For simplicity, we will only prove case (i) of Theorem 3.1.

Theorem 3.1. *Let* $(E, \|\cdot\|)$ *be a semisimple commutative Banach space. Assume that* $f, g : G \to E$ *satisfy one of each inequalities*

$$\|f(x+y) \pm g(x-y) - \lambda f(x)g(y)\| \leq \varepsilon, \tag{3.1}$$

$$\|f(x+y) \pm g(x-y) - \lambda g(x)f(y)\| \leq \varepsilon, \quad \forall x, y \in G. \tag{3.2}$$

For an arbitrary linear multiplicative functional $x^* \in E^*$,
if $x^* \circ g$ *(or* $x^* \circ f$*) fails to be bounded, then*

(i) f *with* $f(0) = 0$ *satisfies* (S),

(ii) g *with* $g(0) = 0$ *satisfies* (S),

(iii) *particularly, if* g *satisfies* (C^λ)*, then* f *and* g *are solutions of the Wilson equation* (C^λ_{fg})*, and also if* f *satisfies* (C^λ)*, then* f *and* g *are solutions of* (C^λ_{gf}).

Proof. As + and − have the same procedure, we will show only case + in (3.1).

(i) Assume that (3.1) holds and arbitrarily fixes a linear multiplicative functional $x^* \in E^*$. As is well known, we have $\|x^*\| = 1$, hence, for every $x, y \in G$, we have

$$\begin{aligned} \varepsilon &\geq \|f(x+y) + g(x-y) - \lambda f(x)g(y)\| \\ &= \sup_{\|y^*\|=1} \left| y^*(f(x+y) + g(x-y) - \lambda f(x)g(y)) \right| \\ &\geq \left| x^*(f(x+y)) + x^*(g(x-y)) - \lambda x^*(f(x))x^*(g(y)) \right|, \end{aligned} \tag{3.3}$$

which states that the superpositions $x^* \circ f$ and $x^* \circ g$ yield a solution of inequality (2.1). Since, by assumption, the superposition $x^* \circ g$ is unbounded, an appeal to Theorem 2.1 shows that three results hold. Namely, (i) the function $x^* \circ f$ with $f(0) = 0$ solves (S), (ii) the function $x^* \circ g$ with $g(0) = 0$ solves (S), and (iii), in particular, if $x^* \circ g$ satisfies (C^λ), then $x^* \circ f$ and $x^* \circ g$ are solutions of the Wilson equation (C^λ_{fg}), and also if $x^* \circ f$ satisfies (C^λ), then $x^* \circ f$ and $x^* \circ g$ are solutions of (C^λ_{gf}).

To put case (i) another way, bearing the linear multiplicativity of x^* in mind, for all $x, y \in G$, the difference $\mathfrak{D} : G \times G \to \mathbb{C}$, defined by

$$\mathfrak{D}S(x,y) := f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 - f(x)f(y), \tag{DS}$$

falls into the kernel of x^*. Therefore, in view of the unrestricted choice of x^*, we infer that

$$\mathfrak{D}S(x,y) \in \bigcap \{\ker x^* : x^* \text{ is a multiplicative member of } E^*\} \quad \forall x, y \in G. \tag{3.4}$$

Since the algebra E has been assumed to be semisimple, the last term of the above formula coincides with the singleton $\{0\}$, that is,

$$\mathfrak{D}S(x,y)=0 \quad \forall x,y\in G, \tag{3.5}$$

as claimed. The other cases also are the same. □

Theorem 3.2. *Let $(E,\|\cdot\|)$ be a semisimple commutative Banach space. Assume that $f,g:G\to E$ satisfy one of each inequalities*

$$\|f(x+y)\pm g(x-y)-\lambda f(x)f(y)\|\le\varepsilon, \tag{3.6}$$

$$\|f(x+y)\pm g(x-y)-\lambda g(x)g(y)\|\le\varepsilon,\quad \forall x,y\in G. \tag{3.7}$$

For an arbitrary linear multiplicative functional $x^*\in E^*$,

(i) *in case* (3.6)*, either $x^*\circ f$ is bounded or f satisfies* (S)*,*

(ii) *in case* (3.7)*, either $x^*\circ g$ is bounded or g satisfies* (S)*.*

Remark 3.3. By applying the same procedure as in Remark 2.9, we obtain the superstability for aforemensioned theorems on the Banach space, which are also in [4, 5, 7–10, 12, 14].

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