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Research Article

Multiple Positive Solutions of m-Point BVPs for Third-Order p-Laplacian Dynamic Equations on Time Scales

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This paper is concerned with the existence of multiple positive solutions for the third-order p-Laplacian dynamic equation $(\phi_p(u^{\Delta\nabla}(t)))^{\nabla} + a(t)f(t, u(t), u^{\Delta}(t)) = 0, t \in [0, T]_{\mathbb{T}}$ with the multipoint boundary conditions $u^{\Delta}(0) = u^{\Delta\nabla}(0) = 0, u(T) + B_0(\sum_{i=1}^{m-2} b_i u^{\Delta}(\xi_i)) = 0$, where $\phi_p(u) = |u|^{p-2}u$ with $p > 1$. Using the fixed point theorem due to Avery and Peterson, we establish the existence criteria of at least three positive solutions to the problem. As an application, an example is given to illustrate the result. The interesting points are that not only do we consider third-order p-Laplacian dynamic equation but also the nonlinear term f is involved with the first-order delta derivative of the unknown function.

1. Introduction

The theory of dynamic equations on time scales was introduced by Stefan Hilger in 1988 [1]. This theory has attracted many researchers' attention and interest since it cannot only unify differential and difference equations but also provides accurate information of phenomena that manifest themselves partly in continuous time and partly in discrete time. In addition, time-scale calculus would allow exploration of a variety of situations in economic, biological, heat transfer, stock market, and epidemic models [2, 3], and so forth.

Recently, there has been much attention paid to the existence of positive solutions for second-order nonlinear boundary value problems on time scales; see [4–10] and the references therein. On the one hand, higher-order nonlinear boundary value problems have been studied extensively; see [11–14] and the references therein. On the other hand, the boundary value problems with p-Laplacian operator have also been discussed extensively in literature; for example, see [15–17]. However, very little work has been done to the third-order p-Laplacian dynamic equations on time scales [18, 19].

For convenience, throughout this paper, we denote $\phi_p(s)$ as p-Laplacian operator, that is, $\phi_p(s) = |s|^{p-2}s$ for $p > 1$ with $\phi_p^{-1} = \phi_q$ and $1/p + 1/q = 1$. We also assume that $\mathbb{T}$ is a closed subset of $\mathbb{R}$ with $0, T \in \mathbb{T}$; an interval $(0,T)_{\mathbb{T}}$ always means $(0,T) \cap \mathbb{T}$. Other types of intervals are defined similarly.

For example, Sun and Li [16] studied the two-point boundary value problem:

$$\begin{gathered} \left(\phi_p\left(u^{\Delta}(t)\right)\right)^{\Delta} + h(t)f(u^{\sigma}(t)) = 0, \quad t \in [a,b]_{\mathbb{T}}, \\ u(a) - B_0\left(u^{\Delta}(a)\right) = 0, \quad u^{\Delta}(\sigma(b)) = 0. \end{gathered} \tag{1.1}$$

They established the existence theory for positive solutions by using various fixed point theorems [20, 21].

In [15], Su et al. investigated the existence of positive solutions for the following singular p-Laplacian m-point boundary value problem on time scales:

$$\begin{gathered} \left(\phi_p\left(u^{\Delta}(t)\right)\right)^{\nabla} + a_1(t)f(u(t)) = 0, \quad t \in (0,T)_{\mathbb{T}}, \\ u(0) = 0, \qquad u(T) - \sum_{i=1}^{m-2}\psi_i(u(\xi_i)) = 0. \end{gathered} \tag{1.2}$$

The main techniques are Schauder fixed point theorem and upper and lower solutions method.

In [19], Han and Kang considered the following third-order p-Laplacian dynamic equation on time scales:

$$\begin{gathered} \left(\phi_p\left(u^{\Delta\Delta}(t)\right)\right)^{\nabla} + f(t,u(t)) = 0, \quad t \in [a,b], \\ \alpha u(\rho(a)) - \beta u^{\Delta}(\rho(a)) = 0, \qquad \gamma u(b) + \delta u^{\Delta}(b) = 0, \qquad u^{\Delta\Delta}(\rho(a)) = 0. \end{gathered} \tag{1.3}$$

By using fixed point theorems in cones, the existence criteria of multiple positive solutions are established.

In [10], Zhao and Sun studied the following second-order nonlinear three-point boundary value problem on time scales:

$$\begin{gathered} u^{\Delta\nabla}(t) + q(t)f\left(t,u(t),u^{\Delta}(t)\right) = 0, \quad t \in (0,T)_{\mathbb{T}}, \\ \beta u(0) - \gamma u^{\Delta}(0) = 0, \qquad \alpha u(\eta) = u(T). \end{gathered} \tag{1.4}$$

They gave sufficient condition for the existence of three positive solutions by using a fixed point theorem due to Avery and Peterson [22].

Motivated by [10, 15, 16, 19], in this paper we consider the following third-order p-Laplacian dynamic equation on time scales:

$$\left(\phi_p\left(u^{\Delta\nabla}(t)\right)\right)^{\nabla} + a(t)f\left(t, u(t), u^{\Delta}(t)\right) = 0, \quad t \in [0,T]_{\mathbb{T}} \tag{1.5}$$

subject to the boundary condition

$$u^{\Delta}(0) = u^{\Delta\nabla}(0) = 0, \qquad u(T) + B_0\left(\sum_{i=1}^{m-2} b_i u^{\Delta}(\xi_i)\right) = 0, \tag{1.6}$$

where $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < \rho(T)$, $b_i \in [0,\infty)$ for $i = 1, 2, \ldots, m-2$. By using fixed point theorem due to Avery and Peterson [22], we prove that the boundary value problems (1.5) and (1.6) have at least three positive solutions under suitable assumptions. The interesting points are that not only do we consider third-order p-Laplacian dynamic equation on time scales but also the nonlinear term f is involved with the first-order delta derivative of the unknown function.

Throughout this paper, it is assumed that

(H1) $a \in C_{ld}([0,T]_{\mathbb{T}}, \mathbb{R}^+)$ and $f \in C([0,T]_{\mathbb{T}} \times \mathbb{R}^+ \times \mathbb{R}, \mathbb{R}^+)$, both a and f do not vanish identically on any closed subinterval of $[0,T]_{\mathbb{T}}$, and there exists $l \in (0, \xi_1]_{\mathbb{T}}$ such that $\int_0^l a(\tau)\nabla\tau > 0$ hold;

(H2) there exist nonnegative constants B_1 and B_2 satisfying $B_1 x \le B_0(x) \le B_2 x$ for $x \in \mathbb{R}$.

2. Preliminary

To prove the main results in this paper, we will employ several lemmas. And the following lemma is based on the linear BVP:

$$\left(\phi_p\left(u^{\Delta\nabla}(t)\right)\right)^{\nabla} + h(t) = 0, \quad t \in [0,T]_{\mathbb{T}}, \tag{2.1}$$

$$u^{\Delta}(0) = u^{\Delta\nabla}(0) = 0, \qquad u(T) + B_0\left(\sum_{i=1}^{m-2} b_i u^{\Delta}(\xi_i)\right) = 0. \tag{2.2}$$

Lemma 2.1. *If $h \in C_{ld}([0,T]_{\mathbb{T}}, \mathbb{R}^+)$, then the problems* (2.1) *and* (2.2) *have the unique nonnegative solution:*

$$\begin{aligned} u(t) = {} & \int_0^t (t-s)\phi_q\left(-\int_0^s h(\tau)\nabla\tau\right)\nabla s + \int_0^T (T-s)\phi_q\left(\int_0^s h(\tau)\nabla\tau\right)\nabla s \\ & - B_0\left(\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \phi_q\left(\int_0^s - h(\tau)\nabla\tau\right)\nabla s\right). \end{aligned} \tag{2.3}$$

Proof. For any $h \in C_{ld}([0,T]_{\mathbb{T}}, \mathbb{R}^+)$, suppose that u is a solution of the BVPs (2.1) and (2.2). By integrating (2.1) from 0 to t, and combining the boundary condition, it follows that

$$u^{\Delta\nabla}(t) = \phi_q\left(-\int_0^t h(s)\nabla s\right), \qquad u^{\Delta}(t) = \int_0^t \phi_q\left(-\int_0^s h(\tau)\nabla\tau\right)\nabla s,$$
$$u(t) - u(0) = \int_0^t \left(\int_0^s \phi_q\left(-\int_0^\tau h(r)\nabla r\right)\nabla\tau\right)\Delta s = \int_0^t (t-s)\phi_q\left(-\int_0^s h(\tau)\nabla\tau\right)\nabla s. \tag{2.4}$$

Using (2.2), we can easily obtain

$$u(0) = \int_0^T (T-s)\phi_q\left(\int_0^s h(\tau)\nabla\tau\right)\nabla s - B_0\left(\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \phi_q\left(\int_0^s - h(\tau)\nabla\tau\right)\nabla s\right). \tag{2.5}$$

So

$$u(t) = \int_0^t (t-s)\phi_q\left(-\int_0^s h(\tau)\nabla\tau\right)\nabla s + \int_0^T (T-s)\phi_q\left(\int_0^s h(\tau)\nabla\tau\right)\nabla s$$
$$- B_0\left(\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \phi_q\left(\int_0^s - h(\tau)\nabla\tau\right)\nabla s\right). \tag{2.6}$$

Then it is easy to see that

$$\int_0^t (t-s)\phi_q\left(-\int_0^s h(\tau)\nabla\tau\right)\nabla s + \int_0^T (T-s)\phi_q\left(\int_0^s h(\tau)\nabla\tau\right)\nabla s \geq 0,$$
$$-B_0\left(\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \phi_q\left(\int_0^s - h(\tau)\nabla\tau\right)\Delta s\right) \geq B_2 \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \phi_q\left(\int_0^s h(\tau)\nabla\tau\right)\Delta s \geq 0. \tag{2.7}$$

So $u(t) \geq 0$. On the other hand, it is easy to verify that if u is as in (2.3), then u is a solution of (2.1) and (2.2). Thus u in (2.3) is the unique solution of (2.1) and (2.2). □

Let $X = C_{ld}^{\Delta}[0,T]_{\mathbb{T}}$ be endowed with the norm

$$\|u\|_1 = \max\left\{\sup_{t\in[0,T]_{\mathbb{T}}} |u(t)|, \sup_{t\in[0,T]_{\mathbb{T}^\kappa}} \left|u^{\Delta}(t)\right|\right\}, \quad u \in X. \tag{2.8}$$

It follows that $(X, \|\cdot\|_1)$ is a Banach space. Define the cone $P \subset X$ by

$$P = \left\{u \in X \mid u(t) \geq 0, u^{\Delta}(0) \leq 0, u(T) + B_0\left(\sum_{i=1}^{m-2} b_i u^{\Delta}(\xi_i)\right) \leq 0, u^{\Delta\nabla}(t) \leq 0, t \in (0,T)_{\mathbb{T}}\right\}. \tag{2.9}$$

Lemma 2.2. *If $u \in P$, then there exists a constant K such that*

$$\sup_{t\in[0,T]_{\mathbb{T}}} |u(t)| \le K \sup_{t\in[0,T]_{\mathbb{T}^k}} \left|u^{\Delta}(t)\right|. \tag{2.10}$$

Proof. For $u \in P$, $u^{\Delta\nabla}(t) \le 0$ implies that

$$u(t) - u(T) \le u^{\Delta}(T)(t-T) \le T \sup_{t\in[0,T]_{\mathbb{T}}} \left|u^{\Delta}(t)\right|. \tag{2.11}$$

In addition, since

$$u(T) \le -B_0\left(\sum_{i=1}^{m-2} b_i u^{\Delta}(\xi_i)\right) \le -B_1\sum_{i=1}^{m-2} b_i u^{\Delta}(\xi_i) \le B_1\sum_{i=1}^{m-2} b_i \sup_{t\in[0,T]_{\mathbb{T}}} \left|u^{\Delta}(t)\right|, \tag{2.12}$$

then we have

$$\sup_{t\in[0,T]_{\mathbb{T}}} |u(t)| \le \left(T + B_1\sum_{i=1}^{m-2} b_i\right) \sup_{t\in[0,T]_{\mathbb{T}}} \left|u^{\Delta}(t)\right|. \tag{2.13}$$

Therefore, We can choose $K = T + B_1\sum_{i=1}^{m-2} b_i$ and the proof is complete. □

Lemma 2.3. *If $u \in P$, then $u(t) \ge ((T-t)/T)\sup_{t\in[0,T]_{\mathbb{T}}}|u(t)|$ for $t \in [0,T]_{\mathbb{T}}$.*

Proof. If $u \in P$, then $u^{\Delta}(t)$ is decreasing and $u^{\Delta}(0) \le 0$, and thus $u^{\Delta}(t) \le 0$ and $u(t)$ are decreasing. So we have

$$\sup_{t\in[0,T]_{\mathbb{T}}} |u(t)| = u(0). \tag{2.14}$$

By the concavity of $u(t)$, for $t \in (0,T)_{\mathbb{T}}$, there is

$$\begin{gathered} \frac{u(T)-u(0)}{T-0} \ge \frac{u(T)-u(t)}{T-t}, \\ u(0)(T-t) \le u(t)T - tu(T) \le u(t)T. \end{gathered} \tag{2.15}$$

Then we have

$$u(t) \ge \frac{T-t}{T} \sup_{t\in[0,T]_{\mathbb{T}}} |u(t)|. \tag{2.16}$$

The proof is complete. □

Let γ and θ be nonnegative continuous convex functionals on P, let α be a nonnegative continuous concave functional on P, and let ψ be a nonnegative continuous functional on P. Then for positive real numbers a, b, c, and d, we define the following convex sets:

$$\begin{aligned} P(\gamma, d) &= \{x \in P \mid \gamma(x) < d\}, \\ \widetilde{P}(\gamma, \alpha, b, d) &= \{x \in P \mid b \leq \alpha(x), \gamma(x) \leq d\}, \\ \overline{P}(\gamma, \theta, \alpha, b, c, d) &= \{x \in P \mid b \leq \alpha(x), \theta(x) \leq c, \gamma(x) \leq d\}, \end{aligned} \tag{2.17}$$

and a closed set

$$R\ (\gamma, \psi, a, d) = \{x \in P \mid a \leq \psi(x), \gamma(x) \leq d\}. \tag{2.18}$$

The following fixed point theorem due to Avery and Peterson is fundamental in the proof of our main results.

Lemma 2.4 (see [22]). *Let P be a cone in a real Banach space E. Let γ and θ be nonnegative continuous convex functionals on P, let α be a nonnegative continuous concave functional on P, and let ψ be a nonnegative continuous functional on P satisfying $\psi(\lambda x) \leq \lambda \psi(x)$ for $0 \leq \lambda \leq 1$, such that for some positive numbers $\overline{M}$ and d,*

$$\alpha(x) \leq \psi(x), \qquad \|x\| \leq \overline{M}\gamma(x) \tag{2.19}$$

for all $x \in \overline{P(\gamma, d)}$. Suppose that $A : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$ is completely continuous and there exist positive numbers a, b, and c with $a < b$ such that

(S_1) *$\{x \in \overline{P}(\gamma, \theta, \alpha, b, c, d) \mid \alpha(x) > b\} \neq \emptyset$ and $\alpha(Ax) > b$ for $x \in \overline{P}\ (\gamma, \theta, \alpha, b, c, d)$;*

(S_2) *$\alpha(Ax) > b$ for $x \in \widetilde{P}(\gamma, \alpha, b, d)$ with $\theta(Ax) > c$;*

(S_3) *$0 \notin R(\gamma, \psi, a, d)$ and $\psi(Ax) < a$ for $x \in R(\gamma, \psi, a, d)$ with $\psi(x) = a$.*

Then A has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, d)}$, such that

$$\begin{aligned} &\gamma(x_i) \leq d \quad \text{for } i = 1, 2, 3, \qquad b < \alpha(x_1), \\ &a < \psi(x_2) \quad \text{with } \alpha(x_2) < b, \qquad \psi(x_3) < a. \end{aligned} \tag{2.20}$$

3. Existence Results

In this section, by using the Avery-Peterson fixed point theorem, we shall give the sufficient conditions for the existence of at least three positive solutions to the BVPs (1.5) and (1.6).

Firstly, we define the nonnegative continuous concave functional α, the nonnegative continuous convex functionals θ, γ, and the nonnegative continuous functional ψ on P, respectively, by

$$\begin{aligned}
\gamma(u) &= \sup_{t\in[0,T]_{\mathbb{T}^k}} \left|u^{\Delta}(t)\right| = \max\left\{\left|u^{\Delta}(0)\right|, \left|u^{\Delta}(T)\right|\right\},\\
\psi(u) &= \theta(u) = \sup_{t\in[0,T]_{\mathbb{T}}} |u(t)| \le \|u\|_1,\\
\alpha(u) &= \inf_{t\in[0,l]_{\mathbb{T}}} |u(t)| = u(l) \quad \text{for } u \in P.
\end{aligned} \tag{3.1}$$

For notation convenience, we denote

$$\begin{aligned}
L &= T\phi_q\left(\int_0^T a(\tau)\nabla\tau\right), \qquad M = B_2\sum_{i=1}^{m-2} b_i \int_0^l \phi_q\left(\int_0^s a(\tau)\nabla\tau\right)\nabla s,\\
N &= \left(T^2 + TB_1\sum_{i=1}^{m-2} b_i\right)\phi_q\left(\int_0^T a(\tau)\nabla\tau\right), \qquad \rho = \frac{2T}{T-l}.
\end{aligned} \tag{3.2}$$

Now we state and prove our main result.

Theorem 3.1. *Let $0 < a < Nb/M \le \min\{Nd(T-l)/2M, dN/L\}$ and suppose that f satisfies the following conditions:*

(A_1) $f(t,u,v) \le \phi_p(d/L)$ *for* $(t,u,v) \in [0,T]_{\mathbb{T}} \times [0,Kd] \times [-d,d]$,

(A_2) $f(t,u,v) > \phi_p(b/M)$ *for* $(t,u,v) \in [0,l]_{\mathbb{T}} \times [b,\rho b] \times [-d,d]$,

(A_3) $f(t,u,v) < \phi_p(a/N)$ *for* $(t,u,v) \in [0,T]_{\mathbb{T}} \times [0,a] \times [-d,d]$.

Then problems (1.5) *and* (1.6) *have at least three positive solutions u_1, u_2, and u_3 such that*

$$\begin{aligned}
&\sup_{t\in[0,T]_{\mathbb{T}^k}} \left|u_i^{\Delta}(t)\right| \le d \quad \text{for } i = 1,2,3, \qquad b < \inf_{t\in[0,l]_{\mathbb{T}}} |u_1(t)|, \qquad \sup_{t\in[0,T]_{\mathbb{T}}} |u_1(t)| \le Kd,\\
&a < \sup_{t\in[0,T]_{\mathbb{T}}} |u_2(t)| \quad \text{with } \inf_{t\in[0,l]_{\mathbb{T}}} |u_2(t)| < b, \qquad \sup_{t\in[0,T]_{\mathbb{T}}} |u_3(t)| < a.
\end{aligned} \tag{3.3}$$

Proof. Define an integral operator $A : P \to X$ by

$$\begin{aligned}
(Au)(t) = &\int_0^t (t-s)\phi_q\left(\int_0^s -a(\tau)f\left(\tau, u(\tau), u^{\Delta}(\tau)\right)\nabla\tau\right)\nabla s\\
&+ \int_0^T (T-s)\phi_q\left(\int_0^s a(\tau)f\left(\tau, u(\tau), u^{\Delta}(\tau)\right)\nabla\tau\right)\nabla s\\
&- B_0\left(\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \phi_q\left(\int_0^s -a(\tau)f\left(\tau, u(\tau), u^{\Delta}(\tau)\right)\nabla\tau\right)\nabla s\right)
\end{aligned} \tag{3.4}$$

for $t \in [0,T]_{\mathbb{T}}$. It is easy to obtain that $A : P \to P$ is a completely continuous operator and every fixed point of A is a solution of (1.5) and (1.6).

Thus we set out to verify that the operator A satisfies Avery-Peterson fixed point theorem which will prove the existence of three fixed points of A. Now the proof is divided into some steps.

By virtue of $\alpha(u) = \inf_{t\in[0,l]_{\mathbb{T}}}|u(t)|$, $\gamma(u) = \sup_{t\in[0,T]_{\mathbb{T}^k}}|u^{\Delta}(t)|$, $\psi(u) = \sup_{t\in[0,T]_{\mathbb{T}}}|u(t)|$ and Lemma 2.2 we know that there exists a constant $\overline{M}$ such that

$$\alpha(u) \le \psi(u), \qquad \|u\|_1 \le \overline{M}\gamma(u) \quad \text{for } u \in \overline{P(\gamma,d)}. \tag{3.5}$$

We first show that (A_1) implies that

$$A : \overline{P(\gamma,d)} \longrightarrow \overline{P(\gamma,d)}. \tag{3.6}$$

In fact, for $u \in \overline{P(\gamma,d)}$, $\gamma(u) = \sup_{t\in[0,T]_{\mathbb{T}}}|u^{\Delta}(t)| \le d$, by Lemma 2.2, there is $\sup_{t\in[0,T]_{\mathbb{T}}}|u(t)| \le Kd$. It follows from (A_1) that

$$\begin{aligned}
\gamma(A(u)) &= \sup_{t\in[0,T]_{\mathbb{T}}}\left|(Au)^{\Delta}(t)\right| = \sup_{t\in[0,T]_{\mathbb{T}}}\left|\int_0^t \phi_q\left(\int_0^s a(\tau) f\left(\tau,u(\tau),u^{\Delta}(\tau)\right)\nabla\tau\right)\nabla s\right| \\
&\le \int_0^T \phi_q\left(\int_0^T a(\tau) f\left(\tau,u(\tau),u^{\Delta}(\tau)\right)\nabla\tau\right)\nabla s \\
&\le \frac{Td}{L}\phi_q\left(\int_0^T a(\tau)\nabla\tau\right) = d.
\end{aligned} \tag{3.7}$$

Thus (3.6) holds.

Next we show that condition (S_1) in Lemma 2.4 holds. Let $\overline{u}(t) = 2b(T-t)/(T-l)$. Then it is easy to see that $\overline{u}(t) \ge 0$, $\overline{u}(T) + B_0(\sum_{i=1}^{m-2} b_i \overline{u}^{\Delta}(\xi_i)) \le 0$, $\overline{u}^{\Delta}(0) \le 0$, and $\overline{u}^{\Delta\nabla}(t) \le 0$ for $t \in [0,T]_{\mathbb{T}}$, so $\overline{u} \in P$. Also, we have

$$\begin{aligned}
\gamma(\overline{u}) &= \sup_{t\in[0,T]_{\mathbb{T}^k}}\left|u^{\Delta}(t)\right| = \frac{2b}{T-l} \le d, \\
\theta(\overline{u}) &= \sup_{t\in[0,T]_{\mathbb{T}}}|u(t)| = \frac{2bT}{T-l} \le \rho b, \\
\alpha(\overline{u}) &= \inf_{t\in[0,l]_{\mathbb{T}}}|u(t)| = 2b > b.
\end{aligned} \tag{3.8}$$

So $\overline{u} \in \overline{P}(\gamma,\theta,\alpha,b,\rho b,d)$. Hence $\{u \in \overline{P}(\gamma,\theta,\alpha,b,\rho b,d) \mid \alpha(u) > b\} \neq \emptyset$.

If $u \in \overline{P}(\gamma,\theta,\alpha,b,\rho b,d)$, then $b \le u(t) \le \rho b,\ |u^{\Delta}(t)| \le d$ for $0 \le t \le l$. It follows from condition (A_2) that

$$\begin{aligned}
\alpha(Au) &= \inf_{t\in[0,l]_{\mathbb{T}}} |(Au)(t)| = (Au)(l) \\
&= \int_0^l (l-s)\phi_q\left(\int_0^s -a(\tau)f\left(\tau,u(\tau),u^{\Delta}(\tau)\right)\nabla\tau\right)\nabla s \\
&\quad + \int_0^T (T-s)\phi_q\left(\int_0^s a(\tau)f\left(\tau,u(\tau),u^{\Delta}(\tau)\right)\nabla\tau\right)\nabla s \\
&\quad - B_0\left(\sum_{i=1}^{m-2} b_i\int_0^{\xi_i}\phi_q\left(\int_0^s -a(\tau)f\left(\tau,u(\tau),u^{\Delta}(\tau)\right)\nabla\tau\right)\nabla s\right) \\
&\ge -B_0\left(\sum_{i=1}^{m-2} b_i\int_0^{\xi_i}\phi_q\left(\int_0^s -a(\tau)f\left(\tau,u(\tau),u^{\Delta}(\tau)\right)\nabla\tau\right)\nabla s\right) \\
&\ge B_2\sum_{i=1}^{m-2} b_i\int_0^l \phi_q\left(\int_0^s a(\tau)f\left(\tau,u(\tau),u^{\Delta}(\tau)\right)\nabla\tau\right)\nabla s \\
&> \frac{b}{M}B_2\sum_{i=1}^{m-2} b_i\int_0^l \phi_q\left(\int_0^s a(\tau)\nabla\tau\right)\nabla s = b.
\end{aligned} \tag{3.9}$$

Therefore we have

$$\alpha(Au) > b \quad \text{for } u \in \overline{P}(\gamma,\theta,\alpha,b,\rho b,d). \tag{3.10}$$

That is, condition (S_1) in Lemma 2.4 is satisfied.

We now prove that (S_2) in Lemma 2.4 holds. In fact, since $\alpha(Au) = Au(l),\ \theta(Au) = \sup_{t\in[0,T]_{\mathbb{T}}}|Au(t)|$, then with Lemma 2.3 it follows that

$$\alpha(Au) \ge \frac{T-l}{T}\theta(Au) > \frac{T-l}{T}\rho b \ge b \tag{3.11}$$

for $u \in \widetilde{P}(\gamma,\alpha,b,d)$ with $\theta(Au) > \rho b$. Hence condition (S_2) in Lemma 2.4 is satisfied.

Finally, we assert that (S_3) in Lemma 2.4 also holds.

Observe that $\psi(0) = 0 < a$, so $0 \notin R(\gamma, \psi, a, d)$. Suppose $u \in R(\gamma, \psi, a, d)$ with $\psi(u) = a$. Then, by hypothesis (A_3) we have

$$\begin{aligned}
\psi(A(u)) &= \sup_{t\in[0,T]_{\mathbb{T}}} |(Au)(t)| = (Au)(0) \\
&= \int_0^T (T-s)\phi_q\left(\int_0^s a(\tau) f\left(\tau, u(\tau), u^{\Delta}(\tau)\right)\nabla\tau\right)\nabla s \\
&\quad - B_0\left(\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \phi_q\left(\int_0^s - a(\tau) f\left(\tau, u(\tau), u^{\Delta}(\tau)\right)\nabla\tau\right)\nabla s\right) \\
&< \int_0^T T\phi_q\left(\int_0^s a(\tau) f\left(\tau, u(\tau), u^{\Delta}(\tau)\right)\nabla\tau\right)\nabla s \\
&\quad + B_1 \sum_{i=1}^{m-2} b_i \int_0^T \phi_q\left(\int_0^s a(\tau) f\left(\tau, u(\tau), u^{\Delta}(\tau)\right)\nabla\tau\right)\nabla s \\
&\le \frac{T^2 a}{N}\phi_q\left(\int_0^T a(\tau)\nabla\tau\right) + \frac{TB_1 a}{N}\sum_{i=1}^{m-2} b_i\phi_q\left(\int_0^T a(\tau)\nabla\tau\right) \\
&= \left(T^2 + TB_1\sum_{i=1}^{m-2} b_i\right)\frac{a}{N}\phi_q\left(\int_0^T a(\tau)\nabla\tau\right) = a.
\end{aligned} \tag{3.12}$$

Thus condition (S_3) in Lemma 2.4 holds.

Therefore an application of Lemma 2.4 implies that the BVPs (1.5) and (1.6) have at least three positive solutions u_1, u_2, and u_3 such that (3.3) holds. □

4. Example

In this section, we present an example to explain our result.

Let $\mathbb{T} = \{0\} \bigcup \{1/2^n, n \in \mathbb{N}_0\}$, $a(t) \equiv 1$, $p = 3$, and $m = 4$, $\xi_1 = 1/4$, $\xi_2 = 1/2$, $b_1 = b_2 = 1$, $B_0(x) = 2x$. We consider the following boundary value problem:

$$\begin{gathered}
\left(\left|u^{\Delta\nabla}(t)\right| u^{\Delta\nabla}(t)\right)^{\nabla} + f\left(t, u(t), u^{\Delta}(t)\right) = 0, \quad t \in [0,1]_{\mathbb{T},} \\
u^{\Delta}(0) = u^{\Delta\nabla}(0) = 0, \qquad u(1) + 2\left(u^{\Delta}\frac{1}{4} + u^{\Delta}\frac{1}{2}\right) = 0,
\end{gathered} \tag{4.1}$$

where

$$f(t,u,v)=\begin{cases}\dfrac{1}{10^3}t+\dfrac{1}{5}\times u^{12}+\dfrac{1}{7}\left(\dfrac{v}{2.4\times10^7}\right)^2, & u\le 2,\\[2ex] \dfrac{1}{10^3}t+\dfrac{1}{5}\times 2^{12}+\dfrac{1}{7}\left(\dfrac{v}{2.4\times10^7}\right)^2, & u>2.\end{cases}\tag{4.2}$$

Choosing $a=1/2$, $b=2$, $l=1/4$, $d=2.4\times10^6$, direct calculation shows that

$$\rho=\frac{8}{3},\qquad K=5,\qquad L=1,\qquad M=\frac{4+\sqrt{2}}{56},\qquad N=5.\tag{4.3}$$

Consequently, $f(t,u,v)$ satisfies

(i) $f(t,u,v)<822<\phi_3(d/L)=5.76\times10^{12}$ for $(t,u,v)\in[0,1]_{\mathbb{T}}\times[0,1.2\times10^7]\times[-2.4\times10^6,2.4\times10^6]$;

(ii) $f(t,u,v)=4096/5>\phi_3(b/M)=128(9-4\sqrt{2})$ for $(t,u,v)\in[0,1/4]_{\mathbb{T}}\times[2,16/3]\times[-2.4\times10^6,2.4\times10^6]$;

(iii) $f(t,u,v)\le 4/1000<\phi_3(a/N)=1/100$ for $(t,u,v)\in[0,1]_{\mathbb{T}}\times[0,1/2]\times[-2.4\times10^6,2.4\times10^6]$.

Then all conditions of Theorem 3.1 hold. Thus with Theorem 3.1, the BVP (4.1) has at least three positive solutions u_1, u_2, and u_3 such that

$$\begin{gathered}\sup_{t\in[0,1]_{\mathbb{T}^\kappa}}\left|u_i^{\Delta}(t)\right|\le 2.4\times10^6\quad\text{for } i=1,2,3,\qquad 2<\inf_{t\in[0,1/4]_{\mathbb{T}}}|u_1(t)|,\qquad \sup_{t\in[0,1]_{\mathbb{T}}}|u_1(t)|\le 1.2\times10^7,\\ \frac{1}{2}<\sup_{t\in[0,1]_{\mathbb{T}}}|u_2(t)|\quad\text{with}\quad\inf_{t\in[0,1/4]_{\mathbb{T}}}|u_2(t)|<2,\qquad \sup_{t\in[0,1]_{\mathbb{T}}}|u_3(t)|<\frac{1}{2}.\end{gathered}\tag{4.4}$$

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