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Research Article

Doubly Periodic Traveling Waves in a Cellular Neural Network with Linear Reaction

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Szekeley observed that the dynamic pattern of the locomotion of salamanders can be explained by periodic vector sequences generated by logical neural networks. Such sequences can mathematically be described by "doubly periodic traveling waves" and therefore it is of interest to propose dynamic models that may produce such waves. One such dynamic network model is built here based on reaction-diffusion principles and a complete discussion is given for the existence of doubly periodic waves as outputs. Since there are 2 parameters in our model and 4 a priori unknown parameters involved in our search of solutions, our results are nontrivial. The reaction term in our model is a linear function and hence our results can also be interpreted as existence criteria for solutions of a nontrivial linear problem depending on 6 parameters.

1. Introduction

Szekely in [1] studied the locomotion of salamanders and showed that a bipolar neural network may generate dynamic rhythms that mimic the "sequential" contraction and relaxation of four muscle pools that govern the movements of these animals. What is interesting is that we may explain the correct sequential rhythm by means of the transition of state values of four different (artificial) neurons and the sequential rhythm can be explained in terms of an 8-periodic vector sequence and subsequently in terms of a "doubly periodic traveling wave solution" of the dynamic bipolar cellular neural network.

Similar dynamic (locomotive) patterns can be observed in many animal behaviors and therefore we need not repeat the same description in [1]. Instead, we may use "simplified" snorkeling or walking patterns to motivate our study here. When snorkeling, we need to float on water with our faces downward, stretch out our arms forward, and expand our legs backward. Then our legs must move alternatively. More precisely, one leg kicks downward and another moves upward alternatively.

Let v_0 and v_1 be two neuron pools controlling our right and left legs, respectively, so that our leg moves upward if the state value of the corresponding neuron pool is 1, and

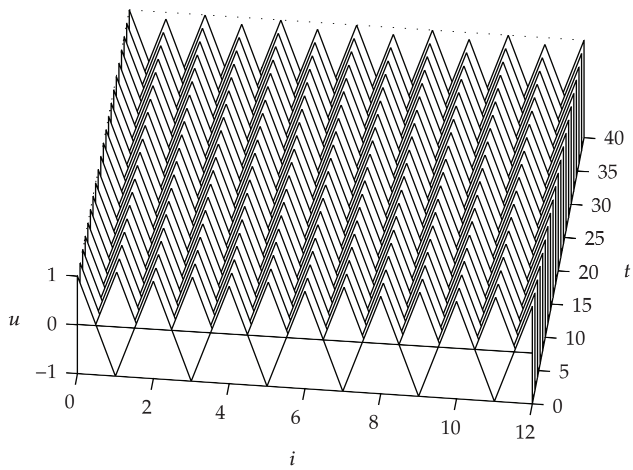

Figure 1: Doubly periodic traveling wave.

downward if the state value of the corresponding neuron pool is -1. Let $v_0^{(t)}$ and $v_1^{(t)}$ be the state values of v_0 and v_1 during the time stage t, where $t \in N = \{0,1,2,\ldots\}$. Then the movements of our legs in terms of $(v_0^{(t)}, v_1^{(t)}), t \in N$, will form a 2-periodic sequential pattern

$$(-1,1) \longrightarrow (1,-1) \longrightarrow (-1,1) \rightarrow (1,-1) \longrightarrow \cdots \tag{1.1}$$

or

$$(1,-1) \longrightarrow (-1,1) \longrightarrow (1,-1) \longrightarrow (1,1) \longrightarrow \cdots . \tag{1.2}$$

If we set $v_i^{(t)} = v_{i \bmod 2}^{(t)}$ for any $t \in N$ and $i \in \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$, then it is easy to check that

$$\begin{aligned} v_i^{(t+1)} &= v_{i+1}^{(t)} \quad \forall i \in \mathbb{Z},\ t \in N \ (\text{temporal-spatial transition condition}), \\ v_i^{(t+2)} &= v_i^{(t)} \quad \forall i \in \mathbb{Z},\ t \in N \ (\text{temporal periodicity condition}), \\ v_i^{(t)} &= v_{i+2}^{(t)} \quad \forall i \in \mathbb{Z},\ t \in N \ (\text{spatial periodicity condition}). \end{aligned} \tag{1.3}$$

Such a sequence $\{v_i^{(t)}\}$ may be called a "doubly periodic traveling wave" (see Figure 1). Now we need to face the following important issue (as in neuromorphic engineering). Can we build artificial neural networks which can support dynamic patterns similar to $\{(v_0^{(t)}, v_1^{(t)})\}_{t \in N}$? Besides this issue, there are other related questions. For example, can we build (nonlogical) networks that can support different types of *graded* dynamic patterns (remember an animal can walk, run, jump, and so forth, with *different strength*)?

To this end, in [2], we build a (nonlogical) neural network and showed the exact conditions such doubly periodic traveling wave solutions may or may not be generated by it. The network in [2] has a linear "diffusion part" and a nonlinear "reaction part." However, the reaction part consists of a quadratic polynomial so that the investigation is reduced to a linear and homogeneous problem. It is therefore of great interests to build networks with *general polynomials* as reaction terms. This job is carried out in two stages. The first stage results in the present paper and we consider linear functions as our reaction functions. In a subsequent paper, as a report of the second stage investigation, we consider polynomials with more general form (see the statement after (2.11)).

2. The Model

We briefly recall the diffusion-reaction network in [2]. In the following, we set $N = \{0, 1, 2, \ldots\}, \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ and $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$. For any $x \in \mathbb{R}$, we also use $[x]$ to denote the greatest integer part of x. Suppose that $v_0, \ldots, v_{\Upsilon-1}$ are neuron pools, where $\Upsilon \geq 1$, placed (in a counterclockwise manner) on the vertices of a regular polygon such that each neuron pool v_i has exactly two neighbors, v_{i-1} and v_{i+1}, where $i \in \{0, \ldots, \Upsilon - 1\}$. For the sake of convenience, we have set $v_0 = v_{-1}$ and $v_1 = v_\Upsilon$ to reflect the fact that these neuron pools are placed on the vertices of a regular polygon. For the same reason, we define $v_i = v_{i \bmod \Upsilon}$ for any $i \in \mathbb{Z}$ and let each $v_i^{(t)}$ be the state value of the ith unit v_i in the time period $t \in N$. During the time period t, if the value $v_i^{(t)}$ of the ith unit is higher than $v_{i-1}^{(t)}$, we assume that "information" will flow from the ith unit to its neighbor. The subsequent change of the state value of the ith unit is $v_i^{(t+1)} - v_i^{(t)}$, and it is reasonable to postulate that it is proportional to the difference $v_i^{(t)} - v_{i-1}^{(t)}$, say, $\alpha(v_i^{(t)} - v_{i-1}^{(t)})$, where α is a proportionality constant. Similarly, information is assumed to flow from the $(i+1)$-unit to the ith unit if $v_{i+1}^{(t)} > v_i^{(t)}$. Thus, it is reasonable that the total effect is

$$v_i^{(t+1)} - v_i^{(t)} = \alpha\left(v_{i-1}^{(t)} - v_i^{(t)}\right) + \alpha\left(v_{i+1}^{(t)} - v_i^{(t)}\right) = \alpha\left(v_{i+1}^{(t)} - 2v_i^{(t)} + v_{i-1}^{(t)}\right), \quad i \in \mathbb{Z},\ t \in N. \tag{2.1}$$

If we now assume further that a control or reaction mechanism is imposed, a slightly more complicated nonhomogeneous model such as the following

$$v_i^{(t+1)} - v_i^{(t)} = \alpha\left(v_{i+1}^{(t)} - v_i^{(t)}\right) + \alpha\left(v_{i-1}^{(t)} - v_i^{(t)}\right) + g\left(v_i^{(t)}\right) \quad \forall i \in \mathbb{Z},\ t \in N \tag{2.2}$$

may result. In the above model, we assume that g is a function and $\alpha \in \mathbb{R}$.

The existence and uniqueness of (real) solutions of (2.2) is easy to see. Indeed, if the (real) initial distribution $\{v_i^{(0)}\}_{i\in\mathbb{Z}}$ is known, then we may calculate successively the sequence

$$v_{-1}^{(1)}, v_0^{(1)}, v_1^{(1)}; v_{-2}^{(1)}, v_{-1}^{(2)}, v_0^{(2)}, v_1^{(2)}, v_2^{(1)}, \ldots \tag{2.3}$$

in a unique manner, which will give rise to a unique solution $\{v_i^{(t)}\}_{t\in N, i\in\mathbb{Z}}$ of (2.2). Motivated by our example above, we want to find solutions that satisfy

$$v_i^{(t+\tau)} = v_{i+\delta}^{(t)} \quad \forall i \in \mathbb{Z},\ t \in N, \tag{2.4}$$

$$v_i^{(t+\Delta)} = v_i^{(t)} \quad \forall i \in \mathbb{Z},\ t \in N, \tag{2.5}$$

$$v_i^{(t)} = v_{i+\Upsilon}^{(t)} \quad \forall i \in \mathbb{Z},\ t \in N, \tag{2.6}$$

where $\tau, \Delta, \Upsilon \in \mathbb{Z}^+$ and $\delta \in \mathbb{Z}$. It is clear that equations in (1.3) are special cases of (2.4), (2.5), and (2.6), respectively.

Suppose that $\mathbf{v} = \{v_i^{(t)}\}$ is a double sequence satisfying (2.4) for some $\tau \in \mathbb{Z}^+$ and $\delta \in \mathbb{Z}$. Then it is clear that

$$v_i^{(t+k\tau)} = v_{i+k\delta}^{(t)} \quad \text{for any } i \in \mathbb{Z},\ t \in N, \tag{2.7}$$

where $k \in \mathbb{Z}^+$. Hence when we want to find any solution $\{v_i^{(t)}\}$ of (2.2) satisfying (2.4), it is sufficient to find the solution of (2.2) satisfying

$$v_i^{(t+\tau/q)} = v_{i+\delta/q}^{(t)}, \tag{2.8}$$

where q is the greatest common divisor (τ, δ) of τ and δ. For this reason, we will pay attention to the condition that $(\tau, \delta) = 1$. Formally, given any $\tau \in \mathbb{Z}^+$ and $\delta \in \mathbb{Z}$ with $(\tau, \delta) = 1$, a real double sequence $\{v_i^{(t)}\}_{t\in N, i\in\mathbb{Z}}$ is called a traveling wave with velocity $-\delta/\tau$ if

$$v_i^{(t+\tau)} = v_{i+\delta}^{(t)}, \quad t \in N,\ i \in \mathbb{Z}. \tag{2.9}$$

In case $\delta = 0$ and $\tau = 1$, our traveling wave is also called a *standing wave*.

Next, recall that a positive integer ω is called a period of a sequence $\varphi = \{\varphi_m\}$ if $\varphi_{m+\omega} = \varphi_m$ for all $m \in \mathbb{Z}$. Furthermore, if $\omega \in \mathbb{Z}^+$ is the least among all periods of a sequence φ, then φ is said to be ω-periodic. It is clear that if a sequence φ is periodic, then the least number of all its (positive) periods exists. It is easy to see the following relation between the least period and a period of a periodic sequence.

Lemma 2.1. *If $\mathbf{y} = \{y_i\}$ is ω-periodic and ω_1 is a period of $\mathbf{y}$, then ω is a factor of ω_1, or $\omega \mod \omega_1 = 0$.*

We may extend the above concept of periodic sequences to double sequences. Suppose that $\mathbf{v} = \{v_i^{(t)}\}$ is a real double sequence. If $\xi \in \mathbb{Z}^+$ such that $v_{i+\xi}^{(t)} = v_i^{(t)}$ for all i and t, then ξ is called a spatial period of $\mathbf{v}$. Similarly, if $\eta \in \mathbb{Z}^+$ such that $v_i^{(t+\eta)} = v_i^{(t)}$ for all i and t, then η is called a temporal period of $\mathbf{v}$. Furthermore, if ξ is the least among all spatial periods of $\mathbf{v}$, then $\mathbf{v}$ is called spatial ξ-periodic, and if η is the least among all temporal periods of $\mathbf{v}$, then $\mathbf{v}$ is called temporal η-periodic.

In seeking solutions of (2.2) that satisfy (2.5) and (2.6), in view of Lemma 2.1, there is no loss of generality to assume that the numbers Δ and Υ are the least spatial and the least temporal periods of the sought solution. Therefore, from here onward, we will seek such doubly-periodic traveling wave solutions of (2.2). More precisely, given any function $g, \alpha \in \mathbb{R}, \delta \in \mathbb{Z}$ and $\Delta, \Upsilon, \tau \in \mathbb{Z}^+$ with $(\tau, \delta) = 1$, in this paper, we will mainly be concerned with the traveling wave solutions of (2.2) with velocity $-\delta/\tau$ which are also spatial Υ-periodic and temporal Δ-periodic. For convenience, we call such solutions (Δ, Υ)-periodic traveling wave solutions of (2.2) with velocity $-\delta/\tau$.

In general, the control function g in (2.2) can be selected in many different ways. But naturally, we should start with the trivial polynomial and general polynomials of the form

$$g(x) = \kappa f(x) := \kappa(x - r_1)(x - r_2) \cdots (x - r_n), \tag{2.10}$$

where $r_1, r_2, \ldots, r_n$ are real numbers, and κ is a real parameter. In [2], the trivial polynomial and the quadratic polynomial $f(x) = x^2$ are considered. In this paper, we will consider the linear case, namely,

$$f(x) = 1 \quad \text{for } x \in \mathbb{R} \quad \text{or} \quad f(x) = x - r \quad \text{for } x \in \mathbb{R}, \text{ where } r \in \mathbb{R}, \tag{2.11}$$

while the cases where $r_1, r_2, \ldots, r_n$ are mutually distinct and $n \geq 2$ will be considered in a subsequent paper (for the important reason that quite distinct techniques are needed).

Since the trivial polynomial is considered in [2], we may avoid the case where $\kappa = 0$. A further simplification of (2.11) is possible in view of the following translation invariance.

Lemma 2.2. *Let $\tau, \Delta, \Upsilon \in \mathbb{Z}^+, \delta \in \mathbb{Z}$ with $(\tau, \delta) = 1$ and $\alpha, \kappa, r \in \mathbb{R}$ with $\kappa \neq 0$. Then $\mathbf{v} = \{v_i^{(t)}\}$ is a (Δ, Υ)-periodic traveling wave solution with velocity $-\delta/\tau$ for the following equation:*

$$v_i^{(t+1)} - v_i^{(t)} = \alpha\left(v_{i+1}^{(t)} - 2v_i^{(t)} + v_{i-1}^{(t)}\right) + \kappa\left(v_i^{(t)} - r\right), \quad i \in \mathbb{Z}, \ t \in N, \tag{2.12}$$

if, and only if, $\mathbf{y} = \{y_i^{(t)}\} = \{v_i^{(t)} - r\}$ is a (Δ, Υ)-periodic traveling wave solution with velocity $-\delta/\tau$ for the following equation

$$y_i^{(t+1)} - y_i^{(t)} = \alpha\left(y_{i+1}^{(t)} - 2y_i^{(t)} + y_{i-1}^{(t)}\right) + \kappa y_i^{(t)}, \quad i \in \mathbb{Z}, \ t \in N. \tag{2.13}$$

Therefore, from now on, we assume in (2.2) that

$$\alpha \in \mathbb{R}, \qquad g = \kappa f, \tag{2.14}$$

where

$$\begin{gathered} \kappa \neq 0, \\ f(x) = 1 \quad \text{for } x \in \mathbb{R}, \quad \text{or,} \quad f(x) = x \quad \text{for } x \in \mathbb{R}. \end{gathered} \tag{2.15}$$

As for the traveling wave solutions, we also have the following reflection invariance result (a direct verification is easy and can be found in [2]).

Lemma 2.3 (cf. proof of [2, Theorem 3]). *Given any* $\delta \in \mathbb{Z} \setminus \{0\}$ *and* $\tau \in \mathbb{Z}^+$ *with* $(\tau, \delta) = 1$. *If* $\{v_i^{(t)}\}$ *is a traveling wave solution of* (2.2) *with velocity* $-\delta/\tau$, *then* $\{w_i^{(t)}\} = \{v_{-i}^{(t)}\}$ *is also a traveling wave solution of* (2.2) *with velocity* δ/τ.

Let $-\delta \in \mathbb{Z}^+$ and $\Delta, \Upsilon, \tau \in \mathbb{Z}^+$, where $(\tau, \delta) = 1$. Suppose that $\mathbf{v} = \{v_i^{(t)}\}$ is a (Δ, Υ)-periodic traveling wave solution of (2.2) with velocity $-\delta/\tau$. Then it is easy to check that $\mathbf{w} = \{w_i^{(t)}\} = \{v_{-i}^{(t)}\}$ is also temporal Δ-periodic and spatial Υ-periodic. From this fact and Lemma 2.3, when we want to consider the (Δ, Υ)-periodic traveling wave solutions of (2.2) with velocity $-\delta/\tau$, it is sufficient to consider the (Δ, Υ)-periodic traveling wave solutions of (2.2) with velocity δ/τ. In conclusion, from now on, we may restrict our attention to the case where

$$\tau \in \mathbb{Z}^+, \quad \delta \in N \quad \text{with } (\tau, \delta) = 1. \tag{2.16}$$

3. Basic Facts

Some additional basic facts are needed. Let us state these as follows. First, let A_ξ be a circulant matrix defined by

$$A_2 = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix},$$

$$A_\xi = \begin{bmatrix} 2 & -1 & & & 0 & -1 \\ -1 & 2 & -1 & & & 0 \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ 0 & & & -1 & 2 & -1 \\ -1 & 0 & & & -1 & 2 \end{bmatrix}_{\xi \times \xi}, \quad \xi \geq 3. \tag{3.1}$$

Second, we set

$$\lambda^{(i,\xi)} = 4\sin^2\left(\frac{i\pi}{\xi}\right), \quad i \in \mathbb{Z}, \ \xi \in \mathbb{Z}^+, \tag{3.2}$$

$$u_m^{(i,\xi)} = \frac{1}{\sqrt{\xi}(\cos(2mi\pi/\xi) + \sin(2mi\pi/\xi))}, \quad m, i \in \mathbb{Z}; \ \xi \in \mathbb{Z}^+. \tag{3.3}$$

It is known (see, e.g., [3]) that for any $\xi \geq 2$, the eigenvalues of A_ξ are $\lambda^{(1,\xi)}, \ldots, \lambda^{(\xi,\xi)}$ and the eigenvector corresponding to $\lambda^{(i,\xi)}$ is

$$u^{(i,\xi)} = \left(u_1^{(i,\xi)}, \ldots, u_\xi^{(i,\xi)}\right)^\dagger \quad \text{for } i \in \{1, \ldots, \xi\}, \tag{3.4}$$

and that $u^{(1,\xi)}, u^{(2,\xi)}, \ldots, u^{(\xi,\xi)}$ are orthonormal. It is also clear that $u^{(0,\xi)} = u^{(\xi,\xi)}, \lambda^{(0,\xi)} = \lambda^{(\xi,\xi)}, \lambda^{(i,\xi)} = \lambda^{(\xi-i,\xi)}$, and

$$u_m^{(\xi-i,\xi)} = \frac{1}{\sqrt{\xi}(\cos(2mi\pi/\xi) - \sin(2mi\pi/\xi))} \quad \forall m, i \in \mathbb{Z}. \tag{3.5}$$

Therefore, $\lambda^{(0,\xi)}, \ldots, \lambda^{([\xi/2],\xi)}$ are all distinct eigenvalues of A_ξ with corresponding eigenspaces $\operatorname{span}\{u^{(\xi)}\}, \operatorname{span}\{u^{(1)}, u^{(\xi-1)}\}, \ldots, \operatorname{span}\{u^{([\xi/2])}, u^{(\xi-[\xi/2])}\}$, respectively.

Given any finite sequence $v = \{v_1, v_2, \ldots, v_\xi\}$ (or vector $v = (v_1, v_2, \ldots, v_\xi)^\dagger$), where $\xi \geq 1$, its (periodic) *extension* is the sequence $\widehat{v} = \{\widehat{v}_i\}_{i\in\mathbb{Z}}$ defined by

$$\widehat{v}_i = v_{i \bmod \xi}, \quad i \in \mathbb{Z}. \tag{3.6}$$

Suppose that $\Upsilon, \Delta \in \mathbb{Z}^+$ and τ, δ satisfy (2.16). When we want to know whether a double sequence is a (Δ, Υ)-periodic traveling wave solution of (2.2) with velocity $-\delta/\tau$, the following two results will be useful.

Lemma 3.1. *Let $\xi, \eta \in \mathbb{Z}^+$ with $\xi \geq 2$ and let $u^{(i,\xi)}$ be defined by* (3.4).

(i) *Suppose $\xi \geq 4$. Let $j, k \in \{1, \ldots, [\xi/2]\}$ with $j \neq k$ and $a, b, c, d \in \mathbb{R}$ such that $au^{(j,\xi)} + bu^{(\xi-j,\xi)}$ and $cu^{(k,\xi)} + du^{(\xi-k,\xi)}$ are both nonzero vectors. Then η is a period of the extension of the vector $au^{(j,\xi)} + bu^{(\xi-j,\xi)} + cu^{(k,\xi)} + du^{(\xi-k,\xi)}$ if and only if $\eta j/\xi \in \mathbb{Z}^+$ and $\eta k/\xi \in \mathbb{Z}^+$.*

(ii) *Suppose $\xi \geq 3$. Let $j \in \{1, \ldots, [\xi/2]\}$ and $a, b, c \in \mathbb{R}$ such that $bu^{(\xi-j,\xi)} + cu^{(j,\xi)}$ is a nonzero vector. Then $au^{(\xi,\xi)} + bu^{(\xi-j,\xi)} + cu^{(j,\xi)}$ is ξ-periodic if and only if $(j, \xi) = 1$.*

(iii) *Suppose $\xi = 2$. Let $a, b \in \mathbb{R}$ such that $b \neq 0$. Then $au^{(2,2)} + bu^{(1,2)}$ is 2-periodic.*

Proof. To see (i), we need to consider five mutually exclusive and exhaustive cases: (a) $j, k \in \{1, \ldots, [\xi/2] - 1\}$; (b) ξ is odd, $j \in \{1, \ldots, [\xi/2] - 1\}$ and $k = (\xi - 1)/2$; (c) ξ is odd, $k \in \{1, \ldots, [\xi/2] - 1\}$ and $j = (\xi - 1)/2$; (d) ξ is even, $j \in \{1, \ldots, [\xi/2] - 1\}$ and $k = \xi/2$; (e) ξ is even, $k \in \{1, \ldots, [\xi/2] - 1\}$ and $j = \xi/2$.

Suppose that case (a) holds. Take

$$u = au^{(j,\xi)} + bu^{(\xi-j,\xi)} + cu^{(k,\xi)} + du^{(\xi-k,\xi)}, \tag{3.7}$$

where $a, b, c, d \in \mathbb{R}$ such that $au^{(j,\xi)} + bu^{(\xi-j,\xi)}$ and $cu^{(k,\xi)} + du^{(\xi-k,\xi)}$ are both nonzero vectors. Let $\widehat{u} = \{\widehat{u}_i\}_{i\in\mathbb{Z}}$ be the extension of u, so that $\widehat{u}_i = u_{i \bmod \xi}$ for $i \in \mathbb{Z}$. Then it is clear that for any $i \in \mathbb{Z}$,

$$\begin{aligned}\widehat{u}_i &= au_i^{(j,\xi)} + bu_i^{(\xi-j,\xi)} + cu_i^{(k,\xi)} + du_i^{(\xi-k,\xi)} \\ &= \frac{1}{\sqrt{\xi}}\left[(a+b)\cos\frac{2ij\pi}{\xi} + (a-b)\sin\frac{2ij\pi}{\xi} + (c+d)\cos\frac{2ik\pi}{\xi} + (c-d)\sin\frac{2ik\pi}{\xi}\right].\end{aligned} \tag{3.8}$$

By direct computation, we also have

$$\begin{aligned}\hat{u}_{i+\eta} = {} & \frac{1}{\sqrt{\xi}}\cos\frac{2\eta j\pi}{\xi}\left[(a+b)\cos\frac{2ij\pi}{\xi}+(a-b)\sin\frac{2ij\pi}{\xi}\right]\\ & +\frac{1}{\sqrt{\xi}}\sin\frac{2\eta j\pi}{\xi}\left[(a-b)\cos\frac{2ij\pi}{\xi}-(a+b)\sin\frac{2ij\pi}{\xi}\right]\\ & +\frac{1}{\sqrt{\xi}}\cos\frac{2\eta k\pi}{\xi}\left[(c+d)\cos\frac{2ik\pi}{\xi}+(c-d)\sin\frac{2ik\pi}{\xi}\right]\\ & +\frac{1}{\sqrt{\xi}}\sin\frac{2\eta k\pi}{\xi}\left[(c-d)\cos\frac{2ik\pi}{\xi}-(c+d)\sin\frac{2ik\pi}{\xi}\right].\end{aligned}\tag{3.9}$$

By (3.8) and (3.9), we see that η is a period of $\hat{u}$, that is, $\hat{u}_i-\hat{u}_{i+\eta}=0$ for all $i\in\mathbb{Z}$, if, and only if, given any $i\in\mathbb{Z}$,

$$\begin{aligned}0 = {} & \frac{1}{\sqrt{\xi}}\left(\cos\frac{2\eta j\pi}{\xi}-1\right)\left[(a+b)\cos\frac{2ij\pi}{\xi}+(a-b)\sin\frac{2ij\pi}{\xi}\right]\\ & +\frac{1}{\sqrt{\xi}}\left(\cos\frac{2\eta k\pi}{\xi}-1\right)\left[(c+d)\cos\frac{2ik\pi}{\xi}+(c-d)\sin\frac{2ik\pi}{\xi}\right]\\ & +\frac{1}{\sqrt{\xi}}\sin\frac{2\eta j\pi}{\xi}\left[(a-b)\cos\frac{2ij\pi}{\xi}-(a+b)\sin\frac{2ij\pi}{\xi}\right]\\ & +\frac{1}{\sqrt{\xi}}\sin\frac{2\eta k\pi}{\xi}\left[(c-d)\cos\frac{2ik\pi}{\xi}-(c+d)\sin\frac{2ik\pi}{\xi}\right].\end{aligned}\tag{3.10}$$

By (3.3) and (3.5), we may rewrite (3.10) as

$$\begin{aligned}0 = {} & \left(\cos\frac{2\eta j\pi}{\xi}-1\right)\left(au_i^{(j,\xi)}+bu_i^{(\xi-j,\xi)}\right)+\sin\frac{2\eta j\pi}{\xi}\left(-bu_i^{(j,\xi)}+au_i^{(\xi-j,\xi)}\right)\\ & +\left(\cos\frac{2\eta k\pi}{\xi}-1\right)\left(cu_i^{(k,\xi)}+du_i^{(\xi-k,\xi)}\right)+\sin\frac{2\eta k\pi}{\xi}\left(-cu_i^{(k,\xi)}+du_i^{(\xi-k,\xi)}\right).\end{aligned}\tag{3.11}$$

By (3.3) again, we have $u_{m+\xi}^{(i,\xi)}=u_m^{(i,\xi)}$ for each $i,m\in\mathbb{Z}$. Hence we see that η is a period of $\hat{u}$ if, and only if,

$$\begin{aligned}(0,\dots,0)^{\dagger} = {} & \left(\cos\frac{2\eta j\pi}{\xi}-1\right)\left(au^{(j,\xi)}+bu^{(\xi-j,\xi)}\right)+\sin\frac{2\eta j\pi}{\xi}\left(-bu^{(j,\xi)}+au^{(\xi-j,\xi)}\right)\\ & \times\left(\cos\frac{2\eta k\pi}{\xi}-1\right)\left(cu^{(k,\xi)}+du^{(\xi-k,\xi)}\right)+\sin\frac{2\eta k\pi}{\xi}\left(-cu^{(k,\xi)}+du^{(\xi-k,\xi)}\right).\end{aligned}\tag{3.12}$$

Note that $j \in \{1, \ldots, [\xi/2]-1\}$ implies that $u^{(j,\xi)}$ and $u^{(\xi-j,\xi)}$ are distinct and hence they are linearly independent. Thus, the fact that $au^{(j,\xi)}+bu^{(\xi-j,\xi)}$ is not a zero vector implies $|a|+|b| \neq 0$. Similarly, we also have $|c|+|d| \neq 0$. Then it is easy to check that $au^{(j,\xi)}+bu^{(\xi-j,\xi)}, -bu^{(j,\xi)}+au^{(\xi-j,\xi)}, cu^{(k,\xi)}+du^{(\xi-k,\xi)}$ and $-cu^{(k,\xi)}+du^{(\xi-k,\xi)}$ are linear independent. Hence we have that η is a period of $\widehat{u}$ if and only if

$$\cos\frac{2\eta j\pi}{\xi}-1=\cos\frac{2\eta k\pi}{\xi}-1=\sin\frac{2\eta j\pi}{\xi}=\sin\frac{2\eta k\pi}{\xi}=0. \tag{3.13}$$

In other words, η is a period of $\widehat{u}$ if, and only if, $\eta j/\xi \in \mathbb{Z}^+$ and $\eta k/\xi \in \mathbb{Z}^+$.

The other cases (b)–(e) can be proved in similar manners and hence their proofs are skipped.

To prove (ii), we first set $u = au^{(\xi,\xi)}+bu^{(\xi-j,\xi)}+cu^{(j,\xi)}$. As in (i), we also know that η is a period of $\widehat{u}=\{\widehat{u}_i\}_{i\in\mathbb{Z}}$, where $\widehat{u}_i = u_{i \bmod \xi}$, if and only if $\eta j/\xi \in \mathbb{Z}^+$. That is,

$$\left\{\eta \in \mathbb{Z}^+ \mid \frac{\eta j}{\xi} \in \mathbb{Z}^+\right\} = \left\{\eta \in \mathbb{Z}^+ \mid \eta \text{ is a period of } \widehat{u}\right\}. \tag{3.14}$$

Suppose $(\xi, j)=1$. If $\eta j/\xi \in \mathbb{Z}^+$ for some $\eta \in \mathbb{Z}^+$, then we have $\eta = 0 \bmod \xi$ because $(\xi, j)=1$. Hence we have

$$\min\left\{\eta \in \mathbb{Z}^+ \mid \eta \text{ is a period of } \widehat{u}\right\} = \min\left\{\eta \in \mathbb{Z}^+ \mid \frac{\eta j}{\xi} \in \mathbb{Z}^+\right\} = \xi. \tag{3.15}$$

In other words, if $(\xi, j)=1$, then $\widehat{u}$ is ξ-periodic. Next, suppose $(\xi, j)=\eta_1 \neq 1$; that is, there exists some $\xi_1, j_1 > 1$ such that $\xi = \eta_1\xi_1$ and $j=\eta_1 j_1$. Note that $j<\xi$ and hence we also have $\eta_1 < \xi_1$. Since $\xi=\eta_1\xi_1, \eta_1<\xi$ and $\eta_1>1$, we have $1<\xi_1<\xi$. Taking $\eta=\xi_1$, then we have $\eta j/\xi \in \mathbb{Z}^+$. Hence η is a period of $\widehat{u}$ and $\eta<\xi$. That is, $\widehat{u}$ is not ξ-periodic. In conclusion, if $\widehat{u}$ is ξ-periodic, then $(\xi, j)=1$.

The proof of (iii) is done by recalling that $u^{(2,2)} = (1/\sqrt{2})(1,1)^\dagger$ and $u^{(1,2)} = (1/\sqrt{2})(-1,1)^\dagger$ and checking that $au^{(2,2)}+bu^{(1,2)}$ is truly 2-periodic. The proof is complete. □

The above can be used, as we will see later, to determine the spatial periods of some special double sequences.

Lemma 3.2. *Let $u^{(i,\xi)}$ be defined by* (3.4). *Let $\xi \geq 3, j \in \{0,1,\ldots,[\xi/2]\}$ and $k \in \{1,\ldots,[\xi/2]\}$ with $j \neq k$. Let further*

$$\begin{aligned} u &= au^{(j,\xi)}+bu^{(\xi-j,\xi)}+cu^{(k,\xi)}+du^{(\xi-k,\xi)}, \\ u' &= -au^{(j,\xi)}-bu^{(\xi-j,\xi)}+cu^{(k,\xi)}+du^{(\xi-k,\xi)}, \end{aligned} \tag{3.16}$$

where $a,b,c,d \in \mathbb{R}$ such that $au^{(j,\xi)}+bu^{(\xi-j,\xi)}$ is a nonzero vector. Define $\mathbf{v}=\{v_i^{(t)}\}$ by

$$\left\{v_i^{(t)}\right\}_{i\in\mathbb{Z}} = \begin{cases} \widehat{u}, & \text{if } t \text{ is odd}, \\ \widehat{u'}, & \text{if } t \text{ is even}. \end{cases} \tag{3.17}$$

(i) *Suppose that* $j \neq 0$ *and* $cu^{(k,\xi)} + du^{(\xi-k,\xi)}$ *is a nonzero vector. Then* $\mathbf{v}$ *is spatial* ξ*-periodic if, and only if,* $\eta j/\xi \notin \mathbb{Z}^+$ *or* $\eta k/\xi \notin \mathbb{Z}^+$ *for any* $\eta \in \{1, \dots, \xi-1\}$ *with* $\eta \mid \xi$.

(ii) *Suppose that* $j = 0$ *and* $cu_i^{(k,\xi)} + du_i^{(\xi-k,\xi)}$ *is a nonzero vector. Then* $\mathbf{v}$ *is spatial* ξ*-periodic if, and only if,* $(k, \xi) = 1$.

(iii) *Suppose that* $cu^{(k,\xi)} + du^{(\xi-k,\xi)}$ *is a zero vector. Then* $\mathbf{v}$ *is spatial* ξ*-periodic if, and only if,* $(j, \xi) = 1$.

Proof. To see (i), suppose that $j \neq 0$ and $cu^{(k,\xi)} + du^{(\xi-k,\xi)}$ is a nonzero vector. Note that the fact that $j, k \in \{1, \dots, [\xi/2]\}$ with $j \neq k$ implies $\xi \geq 4$. By Lemma 3.1(i), η is a period of $\widehat{u}$ if, and only if, $\eta j/\xi \in \mathbb{Z}^+$ and $\eta k/\xi \in \mathbb{Z}$. By Lemma 3.1(i)again, $\eta j/\xi \in \mathbb{Z}^+$ and $\eta k/\xi \in \mathbb{Z}^+$ if, and only if, η is a period of $\widehat{u'}$. Hence the least period of $\widehat{u}$ is the same as $\widehat{u'}$ and $\mathbf{v}$ is spatial ξ-periodic if, and only if, $\widehat{u}$ is ξ-periodic. Note that ξ is a period of $\widehat{u}$. By Lemma 2.1 and Lemma 3.1(i), we have $\widehat{u}$ is ξ-periodic if and only if $\eta j/\xi \notin \mathbb{Z}^+$ or $\eta k/\xi \notin \mathbb{Z}^+$ for any $\eta \in \{1, \dots, \xi-1\}$ with $\eta \mid \xi$.

The assertions (ii) and (iii) can be proved in similar manners. The proof is complete. □

Lemma 3.3. *Let* ξ *be even with* $\xi \geq 3$ *and let* $u^{(i,\xi)}$ *be defined by* (3.4). *Let* $j, k \in \{1, \dots, [\xi/2]\}$ *and*

$$\begin{aligned} u &= au^{(j,\xi)} + bu^{(\xi-j,\xi)} + cu^{(k,\xi)} + du^{(\xi-k,\xi)}, \\ u' &= -au^{(j,\xi)} - bu^{(\xi-j,\xi)} + cu^{(k,\xi)} + du^{(\xi-k,\xi)}, \end{aligned} \tag{3.18}$$

where $a, b, c, d \in \mathbb{R}$ *such that* $au_i^{(j,\xi)} + bu_i^{(\xi-j,\xi)}$ *is a nonzero vector. Let* $\mathbf{v} = \{v_i^{(t)}\}$ *be defined by*

$$\left\{v_i^{(t)}\right\}_{i\in\mathbb{Z}} = \begin{cases} \widehat{u}, & \text{if } t \text{ is odd}, \\ \widehat{u'}, & \text{if } t \text{ is even}. \end{cases} \tag{3.19}$$

(i) *Suppose that* $cu^{(k,\xi)} + du^{(\xi-k,\xi)}$ *is a nonzero vector. Then* $v_i^{(t+1)} = v_{i+\xi/2}^{(t)}$ *for all* $i \in \mathbb{Z}$ *and* $t \in N$ *if and only if* j *is odd and* k *is even.*

(ii) *Suppose that* $cu^{(k,\xi)} + du^{(\xi-k,\xi)}$ *is a zero vector. Then* $v_i^{(t+1)} = v_{i+\xi/2}^{(t)}$ *for all* $i \in \mathbb{Z}$ *and* $t \in N$ *if and only if* j *is odd.*

Proof. To see (i), we first suppose that j is odd and k is even. Note that

$$v_i^{(0)} = \frac{1}{\sqrt{\xi}}\left[(a+b)\cos\frac{2ij\pi}{\xi} + (a-b)\sin\frac{2ij\pi}{\xi} + (c+d)\cos\frac{2ik\pi}{\xi} + (c-d)\sin\frac{2ik\pi}{\xi}\right], \tag{3.20}$$

$$v_i^{(1)} = \frac{1}{\sqrt{\xi}}\left[-(a+b)\cos\frac{2ij\pi}{\xi} - (a-b)\sin\frac{2ij\pi}{\xi} + (c+d)\cos\frac{2ik\pi}{\xi} + (c-d)\sin\frac{2ik\pi}{\xi}\right]. \tag{3.21}$$

For any $s, i \in \mathbb{Z}$, it is clear that

$$\cos\frac{2(i+\xi/2)s\pi}{\xi} = \begin{cases} \cos\dfrac{2is\pi}{\xi} & \text{if } s \text{ is even,} \\ -\cos\dfrac{2is\pi}{\xi} & \text{if } s \text{ is odd,} \end{cases}$$
$$\sin\frac{2(i+\xi/2)s\pi}{\xi} = \begin{cases} \sin\dfrac{2is\pi}{\xi} & \text{if } s \text{ is even,} \\ -\sin\dfrac{2is\pi}{\xi} & \text{if } s \text{ is odd.} \end{cases} \tag{3.22}$$

Since j is odd and k is even, by (3.22), it is easy to see that $\widehat{v^{(1)}}_i = \widehat{v^{(0)}}_{i+\xi/2}$ for all $i \in \mathbb{Z}$. By the definition of $\mathbf{v}$, we also have

$$v_i^{(t+2)} = v_i^{(t)}, \quad v_i^{(t)} = v_{i+\xi}^{(t)} \quad \forall i \in \mathbb{Z},\ t \in N. \tag{3.23}$$

In particular, we have $v_i^{(t+1)} = v_{i+\xi/2}^{(t)}$ for all $i \in \mathbb{Z}$ and $t \in N$.

For the converse, suppose that j is even or k is odd. We first focus on the case that j and k are both even. By (3.20) and (3.22), we have

$$v_{i+\xi/2}^{(0)} = \frac{1}{\sqrt{\xi}}\left[(a+b)\cos\frac{2ij\pi}{\xi} + (a-b)\sin\frac{2ij\pi}{\xi} + (c+d)\cos\frac{2ik\pi}{\xi} + (c-d)\sin\frac{2ik\pi}{\xi}\right]. \tag{3.24}$$

If $v_i^{(t+1)} = v_{i+\xi/2}^{(t)}$ for all $i \in \mathbb{Z}$ and $t \in N$, it is clear that $v_i^{(1)} = v_{i+\xi/2}^{(0)}$ for all $i \in \mathbb{Z}$. By (3.21) and (3.24), we have

$$2(a+b)\cos\frac{2ij\pi}{\xi} + 2(a-b)\sin\frac{2ij\pi}{\xi} = 0 \quad \forall i \in \mathbb{Z}. \tag{3.25}$$

That is, $2(au^{(j,\xi)} + bu^{(\xi-j,\xi)}) = 0$. This is contrary to our assumption. That is, if j, k are both even, then we have $v_i^{(t+1)} \neq v_{i+\xi/2}^{(t)}$ for some $t \in N$ and $i \in \mathbb{Z}$. By similar arguments, in case where j, k are both odd or where j is even and k is odd, we also have $v_i^{(t+1)} \neq v_{i+\xi/2}^{(t)}$ for some $t \in N$ and $i \in \mathbb{Z}$. In summary, if $v_i^{(t+1)} = v_{i+\xi/2}^{(t)}$ for all i, t, then j is odd and k is even.

The assertion (ii) is proved in a manner similar to that of (i). The proof is complete. □

4. Necessary Conditions

Let $\Upsilon, \Delta \in \mathbb{Z}^+$, in this section, we want to find the necessary and sufficient conditions for (Δ, Υ)-periodic traveling wave solutions of (2.2) with velocity $-\delta/\tau$, under the assumptions (2.14), (2.15), and (2.16).

We first consider the case where $f(x) = 1$ for all $x \in \mathbb{R}$. Then we may rewrite (2.2) as

$$v_i^{(t+1)} - v_i^{(t)} = \alpha\left(v_{i-1}^{(t)} - 2v_i^{(t)} + v_{i+1}^{(t)}\right) + \kappa, \quad i \in \mathbb{Z},\ t \in N,\ \kappa \neq 0. \tag{4.1}$$

Suppose that $\mathbf{v} = \{v_i^{(t)}\}$ is a (Δ, Υ)-periodic traveling wave solutions of (4.1) with velocity $-\delta/\tau$. For any $l, s \in \mathbb{Z}$, it is clear that

$$\sum_{t=1}^{\Delta}\sum_{i=1}^{\Upsilon} v_{i+l}^{(t+s)} = \sum_{t=1}^{\Delta}\sum_{i=1}^{\Upsilon} v_i^{(t)}. \tag{4.2}$$

Then we have

$$0 = \sum_{t=1}^{\Delta}\sum_{i=1}^{\Upsilon}\left(v_i^{(t+1)} - v_i^{(t)}\right) = \alpha\sum_{t=1}^{\Delta}\sum_{i=1}^{\Upsilon}\left(v_{i-1}^{(t)} - 2v_i^{(t)} + v_{i+1}^{(t)}\right) + \sum_{t=1}^{\Delta}\sum_{i=1}^{\Upsilon}\kappa = \Delta\Upsilon\kappa \neq 0. \tag{4.3}$$

This is a contradiction. In other words, (Δ, Υ)-periodic traveling wave solutions of (4.1) with velocity $-\delta/\tau$ do not exist.

Next, we consider the case $f(x) = x$ and focus on the equation

$$v_i^{(t+1)} - v_i^{(t)} = \alpha\left(v_{i-1}^{(t)} - 2v_i^{(t)} + v_{i+1}^{(t)}\right) + \kappa v_i^{(t)}, \quad i \in \mathbb{Z},\ t \in N,\ \kappa \neq 0. \tag{4.4}$$

Before dealing with this case, we give some necessary conditions for the existence of (Δ, Υ)-periodic traveling wave solutions of (4.4) with velocity $-\delta/\tau$.

Lemma 4.1. *Let $\alpha, \kappa \in \mathbb{R}$ with $\kappa \neq 0$ and τ, δ satisfy* (2.16)*. Suppose that $\mathbf{v} = \{v_i^{(t)}\}$ is a (Δ, Υ)-periodic traveling wave solution of* (4.4) *with velocity $-\delta/\tau$, where $\Delta = 1$ and $\Upsilon > 1$. Then the matrix $\kappa I_\Upsilon - \alpha A_\Upsilon$ is not invertible and $\left(v_1^{(0)}, \ldots, v_\Upsilon^{(0)}\right)^\dagger$ is a nonzero vector in* $\ker(\kappa I_\Upsilon - \alpha A_\Upsilon)$.

Proof. Let $\mathbf{v} = \{v_i^{(t)}\}$ be a $(1, \Upsilon)$-periodic traveling wave solution of (2.2) with velocity $-\delta/\tau$. It is clear that

$$v_i^{(t+1)} = v_i^{(t)}, \quad v_i^{(t)} = v_{i+\Upsilon}^{(t)} \quad \forall i, t. \tag{4.5}$$

Since $\mathbf{v}$ satisfies (4.4), by (4.5), we have

$$(\kappa I_\Upsilon - \alpha A_\Upsilon)\left(v_1^{(t)}, \ldots, v_\Upsilon^{(t)}\right)^\dagger = (0, \ldots, 0)^\dagger \quad \forall t \in N. \tag{4.6}$$

This fact implies that $\left(v_1^{(t)}, \ldots, v_\Upsilon^{(t)}\right)^\dagger$ is a vector in $\ker(\kappa I_\Upsilon - \alpha A_\Upsilon)$. If $\kappa I_\Upsilon - \alpha A_\Upsilon$ is invertible or $\left(v_1^{(t)}, \ldots, v_\Upsilon^{(t)}\right)^\dagger = 0$, by direct computation, we have

$$\left(v_1^{(t)}, \ldots, v_\Upsilon^{(t)}\right)^\dagger = (0, \ldots, 0)^\dagger \quad \forall t \in N, \tag{4.7}$$

and hence $v_i^{(t)} = 0$ for all $t \in N$ and $i \in \mathbb{Z}$. This is contrary to Δ being the least among all spatial periods and $\Delta > 1$. That is, $\kappa I_\Upsilon - \alpha A_\Upsilon$ is not invertible and $\left(v_1^{(0)}, \dots, v_\Upsilon^{(0)}\right)^\dagger$ is a nonzero vector in $\ker(\kappa I_\Upsilon - \alpha A_\Upsilon)$. The proof is complete. □

Lemma 4.2. *Let $\alpha, \kappa \in \mathbb{R}$ with $\kappa \neq 0$ and τ, δ satisfy* (2.16). *Suppose that $\mathbf{v} = \{v_i^{(t)}\}$ is a (Δ, Υ)-periodic traveling wave solution of* (4.4) *with velocity $-\delta/\tau$, where $\Delta > 1$ and $\Upsilon = 1$. Then $\Delta = 2, \kappa = -2$, each $v_i^{(0)} \neq 0$ and*

$$v_i^{(t)} = \begin{cases} v_i^{(0)}, & \text{if } t \text{ is even}, \ i \in \mathbb{Z}, \\ -v_i^{(0)}, & \text{if } t \text{ is odd}, \ i \in \mathbb{Z}. \end{cases} \tag{4.8}$$

Proof. From the assumption of $\mathbf{v}$, we have

$$v_i^{(t+\Delta)} = v_i^{(t)}, \quad v_i^{(t)} = v_{i+1}^{(t)} \quad \forall i, t. \tag{4.9}$$

Note that $\mathbf{v}$ also satisfies (4.4). Hence by (4.9) and computation, we have

$$v_i^{(t+1)} = (1+\kappa)^{t+1} v_i^{(0)}, \quad i \in \mathbb{Z}, \ t \in N. \tag{4.10}$$

If $v_j^{(0)} = 0$ for some j, by (4.9) and (4.10), we have $v_i^{(0)} = 0$ for all i and $v_i^{(t)} = 0$ for any i, t. This is contrary to Δ being the least among all temporal periods and $\Delta > 1$. Hence we have $v_i^{(0)} \neq 0$ for all i. Then it is clear that $v_i^{(t)}$ is divergent as $t \to \infty$ if $|1+\kappa| > 1$ and $v_i^{(t)} \to \lambda$ as $t \to \infty$ for all $i \in \mathbb{Z}$ if $|1+\kappa| < 1$. This is impossible because $\mathbf{v}$ is temporal Δ-periodic and $\Delta > 1$. Thus we know that $|1+\kappa| = 1$. Since $\kappa \neq 0$, we know that $\kappa = -2$. By (4.10), we have

$$v_i^{(t)} = \begin{cases} v_i^{(0)}, & \text{if } t \text{ is even}, \ i \in \mathbb{Z}, \\ -v_i^{(0)}, & \text{if } t \text{ is odd}, \ i \in \mathbb{Z}. \end{cases} \tag{4.11}$$

□

Lemma 4.3. *Let $\alpha, \kappa \in \mathbb{R}$ with $\kappa \neq 0, \tau, \delta$ satisfy* (2.16) *and $\lambda^{(i,\xi)}$ are defined by* (3.2). *Suppose that $\mathbf{v} = \{v_i^{(t)}\}$ is a (Δ, Υ)-periodic traveling wave solution of* (4.4) *with velocity $-\delta/\tau$, where $\Delta > 1$ and $\Upsilon > 1$. Then the following results are true.*

(i) *For any $t \in N$, one has*

$$\left(v_1^{(t+1)}, \dots, v_\Upsilon^{(t+1)}\right)^\dagger = [(1+\kappa)I_\Upsilon - \alpha A_\Upsilon]^{t+1} \left(v_1^{(0)}, \dots, v_\Upsilon^{(0)}\right)^\dagger. \tag{4.12}$$

(ii) *The vector $\left(v_1^{(0)}, \dots, v_\Upsilon^{(0)}\right)^\dagger$ is the sum of the vectors u and w, where u is an eigenvector of $(1+\kappa)I_\Upsilon - \alpha A_\Upsilon$ corresponding to the eigenvalue -1 and w is either the zero vector or an eigenvector of $(1+\kappa)I_\Upsilon - \alpha A_\Upsilon$ corresponding to the eigenvalue 1.*

(iii) *The matrix $(1+\kappa)I_\Upsilon - \alpha A_\Upsilon$ has an eigenvalue -1, that is, $(1+\kappa) - \alpha 4 \sin^2(j\pi/\Upsilon) = -1$ for some $j \in \{0, 1, \dots, [\Upsilon/2]\}$.*

(iv) $\Delta = 2$.

Proof. To see (i), note that the assumption on $\mathbf{v}$ implies

$$v_i^{(t+\Delta)} = v_i^{(t)}, \quad v_i^{(t)} = v_{i+\Upsilon}^{(t)} \quad \forall i, t. \tag{4.13}$$

Since $\mathbf{v}$ is a solution, by (4.13), we know that

$$\left(v_1^{(t+1)}, \ldots, v_\Upsilon^{(t+1)}\right)^\dagger = [(1+\kappa)I_\Upsilon - \alpha A_\Upsilon]\left(v_1^{(t)}, \ldots, v_\Upsilon^{(t)}\right)^\dagger, \quad t \in N. \tag{4.14}$$

By direct computation, we have

$$\left(v_1^{(t+1)}, \ldots, v_\Upsilon^{(t+1)}\right)^\dagger = [(1+\kappa)I_\Upsilon - \alpha A_\Upsilon]^{t+1}\left(v_1^{(0)}, \ldots, v_\Upsilon^{(0)}\right)^\dagger, \quad t \in N. \tag{4.15}$$

For (ii) and (iii), by taking $t+1 = \Delta$ in (4.12), it is clear from (4.13) that

$$\left(v_1^{(0)}, \ldots, v_\Upsilon^{(0)}\right)^\dagger = \left(v_1^{(\Delta)}, \ldots, v_\Upsilon^{(\Delta)}\right)^\dagger = [(1+\kappa)I_\Upsilon - \alpha A_\Upsilon]^{\Delta}\left(v_1^{(0)}, \ldots, v_\Upsilon^{(0)}\right)^\dagger. \tag{4.16}$$

Thus $\left(v_1^{(0)}, \ldots, v_\Upsilon^{(0)}\right)^\dagger$ is an eigenvector of $[(1+\kappa)I_\Upsilon - \alpha A_\Upsilon]^\Delta$ corresponding to the eigenvalue 1. This implies that the matrix $(1+\kappa)I_\Upsilon - \alpha A_\Upsilon$ must have eigenvalue 1 or -1, and

$$\left(v_1^{(0)}, \ldots, v_\Upsilon^{(0)}\right)^\dagger = u + w, \tag{4.17}$$

where u is either the zero vector or an eigenvector of $(1+\kappa)I_\Upsilon - \alpha A_\Upsilon$ corresponding to the eigenvalue -1 and w is either a zero vector or an eigenvector of $(1+\kappa)I_\Upsilon - \alpha A_\Upsilon$ corresponding to the eigenvalue 1. Suppose that u is the zero vector, or, -1 is not an eigenvalue of $(1+\kappa)I_\Upsilon - \alpha A_\Upsilon$. Then 1 must be an eigenvalue of $(1+\kappa)I_\Upsilon - \alpha A_\Upsilon$ and w must be an eigenvector corresponding to the eigenvalue 1; otherwise, $\left(v_1^{(0)}, \ldots, v_\Upsilon^{(0)}\right)^\dagger = 0$ and this is impossible. Thus, 1 is a temporal period of $\mathbf{v}$. This is contrary to Δ being the least among all periods and $\Delta > 1$. In conclusion, $(1+\kappa)I_\Upsilon - \alpha A_\Upsilon$ has eigenvalue -1 and $\left(v_1^{(0)}, \ldots, v_\Upsilon^{(0)}\right)^\dagger = u + w$, where u is an eigenvector of $(1+\kappa)I_\Upsilon - \alpha A_\Upsilon$ corresponding to the eigenvalue -1, and w is either a zero vector or an eigenvector of $(1+\kappa)I_\Upsilon - \alpha A_\Upsilon$ corresponding to the eigenvalue 1. Since $(1+\kappa) - \alpha\lambda^{(0,\Upsilon)}, \ldots, (1+\kappa) - \alpha\lambda^{([\Upsilon/2],\Upsilon)}$ are all distinct eigenvalues of $(1+\kappa)I_\Upsilon - \alpha A_\Upsilon$, there exists some $j \in \{0, 1, \ldots, [\Upsilon/2]\}$ such that $(1+\kappa) - \alpha\lambda^{(j,\Upsilon)} = -1$.

To see (iv), recall the result in (ii). We have

$$\left(v_1^{(0)}, \ldots, v_\Upsilon^{(0)}\right)^\dagger = [(1+\kappa)I_\Upsilon - \alpha A_\Upsilon]^2\left(v_1^{(0)}, \ldots, v_\Upsilon^{(0)}\right)^\dagger. \tag{4.18}$$

It is also clear that

$$\left(v_1^{(t+2)}, \ldots, v_\Upsilon^{(t+2)}\right)^\dagger = [(1+\kappa)I_\Upsilon - \alpha A_\Upsilon]^2\left(v_1^{(t)}, \ldots, v_\Upsilon^{(t)}\right)^\dagger, \quad t \in N. \tag{4.19}$$

That is, 2 is a temporal period of $\mathbf{v}$. By the definition of Δ and $\Delta > 1$, we have $\Delta = 2$. The proof is complete. □

Next, we consider one result about the relation between δ and Υ under the assumption that doubly-periodic traveling wave solutions of (4.4) exist.

Lemma 4.4. *Let $\alpha, \kappa \in \mathbb{R}$ with $\kappa \neq 0$ and τ, δ satisfy* (2.16)*. Suppose that $\mathbf{v} = \{v_i^{(t)}\}$ is a (Δ, Υ)-periodic traveling wave solution of* (4.4) *with velocity $-\delta/\tau$, where $\Delta = 2$ and $\Upsilon \geq 1$.*

(i) *If τ is even, then $\delta = T_1\Upsilon$ for some odd integer T_1 and Υ is odd.*

(ii) *If τ is odd, then Υ is even and $\delta = T_1\Upsilon/2$ for some odd integer T_1.*

Proof. By the assumption on $\mathbf{v}$, we have

$$v_i^{(t+2)} = v_i^{(t)}, \quad v_i^{(t)} = v_{i+\Upsilon}^{(t)} \quad \forall i, t. \tag{4.20}$$

Since $\mathbf{v}$ is a traveling wave, we also know that

$$v_i^{(t+\tau)} = v_{i+\delta}^{(t)} \quad \forall i, t. \tag{4.21}$$

To see (i), suppose that τ is even. Then from (4.20) and (4.21), we have

$$v_i^{(t)} = v_i^{(t+2)} = \cdots = v_i^{(t+\tau)} = v_{i+\delta}^{(t)} \quad \forall i, t. \tag{4.22}$$

That is, δ is also a spatial period of $\mathbf{v}$. By Lemma 2.1 and the definition of Υ, it is easy to see that $\delta = 0 \mod \Upsilon$. Since $\delta = 0 \mod \Upsilon, \tau$ is even and $(\tau, \delta) = 1$, we have $\delta = T_1\Upsilon$ for some odd integer T_1 and Υ is odd.

For (ii), suppose that τ is odd. Then from (4.20) and (4.21), we have

$$v_i^{(t+1)} = \cdots = v_i^{(t+\tau)} = v_{i+\delta}^{(t)} \quad \forall i, t. \tag{4.23}$$

By (4.20) and (4.23), we know that

$$v_i^{(t)} = v_i^{(t+2)} = v_{i+\delta}^{(t+1)} = v_{i+2\delta}^{(t)} \quad \forall i, t. \tag{4.24}$$

That is, 2δ is also a spatial period of $\mathbf{v}$. By Lemma 2.1 and the definition of Υ, it is easy to see that $2\delta = 0 \mod \Upsilon$. If $\delta = 0 \mod \Upsilon$. From (4.23), we have

$$v_i^{(t+1)} = v_{i+\delta}^{(t)} = \cdots = v_i^{(t)} \quad \forall i, t. \tag{4.25}$$

Then 1 is a temporal period of $\mathbf{v}$ and this is contrary to $\Delta = 2$. Thus $\delta \neq 0 \mod \Upsilon$. Since $2\delta = 0 \mod \Upsilon$, the fact that Υ is odd implies $\delta = 0 \mod \Upsilon$. This leads to a contradiction. So we must have that Υ is even and $\delta = 0 \mod (\Upsilon/2)$. Note that $\delta = 0 \mod (\Upsilon/2)$ and $\delta \neq 0 \mod \Upsilon$ implies $\delta = T_1\Upsilon/2$ for some odd integer T_1. The proof is complete. □

5. Existence Criteria

Now we turn to our main problem. First of all, let $\alpha, \kappa \in \mathbb{R}$ with $\kappa \neq 0$ and τ, δ satisfy (2.16). If $\Upsilon, \Delta \in \mathbb{Z}^+$ with $\Delta > 1$ and if (4.4) has a (Δ, Υ)-periodic traveling wave solution of (4.4) with velocity $-\delta/\tau$, by Lemmas 4.2 and 4.3, Δ must be 2. For this reason, we just need to consider five mutually exclusive and exhaustive cases: (i) $\Upsilon = \Delta = 1$; (ii) $\Delta = 1$ and $\Upsilon > 1$; (iii) $\Delta = 2$ and $\Upsilon = 1$; (iv) $\Delta = 2$ and $\Upsilon = 2$ and (v) $\Delta = 2$ and $\Upsilon \geq 3$.

The condition $\Upsilon = \Delta = 1$ is easy to handle.

Theorem 5.1. *Let $\alpha, \kappa \in \mathbb{R}$ with $\kappa \neq 0$ and τ, δ satisfy* (2.16). *Then the unique* $(1, 1)$*-periodic traveling wave solution of* (4.4) *is* $\{v_i^{(t)} = 0\}$.

Proof. If $\mathbf{v} = \{v_i^{(t)}\}$ is a $(1, 1)$-periodic traveling wave solution of (4.4), then $v_i^{(t)} = c$ for all $i \in \mathbb{Z}$ and $t \in N$, where $c \in \mathbb{R}$. Substituting $\{v_i^{(t)} = c\}$ into (4.4), we have $c = 0$. Conversely, it is clear that $\{v_i^{(t)} = 0\}$ is a $(1, 1)$-periodic traveling wave solution. □

Theorem 5.2. *Let $\alpha, \kappa \in \mathbb{R}$ with $\kappa \neq 0$ and τ, δ satisfy* (2.16). *Let $\lambda^{(i,\xi)}$ and $u^{(i,\xi)}$ be defined by* (3.2) *and* (3.4)*, respectively. Then the following results hold.*

(i) *For any $\Delta = 1$ and any $\Upsilon \geq 2$,* (4.4) *has a* $(1, \Upsilon)$*-periodic traveling wave solutions of* (2.2) *with velocity $-\delta/\tau$ if, and only if, $\delta = 0 \bmod \Upsilon$, and $\kappa - \alpha\lambda^{(j,\Upsilon)} = 0$ for some $j \in \{1, \ldots, [\Upsilon/2]\}$ with $(j, \Upsilon) = 1$.*

(ii) *Every* $(1, \Upsilon)$*-periodic traveling wave solution $\mathbf{v} = \{v_i^{(t)}\}$ is of the form*

$$\left\{v_i^{(t)}\right\} = \left\{v_i^{(0)}\right\}_{i\in\mathbb{Z}} = \widehat{u} \quad \forall t \in N, \tag{5.1}$$

where $u = au^{(j)} + bu^{(\Upsilon-j)}$ for some $a, b \in \mathbb{R}$ such that $au^{(j)} + bu^{(\Upsilon-j)}$ is a nonzero vector, and the converse is true.

Proof. For (i), let $\mathbf{v} = \{v_i^{(t)}\}$ be a $(1, \Upsilon)$-periodic traveling wave solution of (2.2) with velocity $-\delta/\tau$. From the assumption on $\mathbf{v}$, we have $\{v_i^{(t+1)}\}_{i\in\mathbb{Z}} = \{v_i^{(t)}\}_{i\in\mathbb{Z}}$ for all $t \in N$ and Υ is the least spatial period. Hence given any $t \in N$, it is easy to see that the extension $\{v_i^{(t)}\}_{i\in\mathbb{Z}}$ of $(v_1^{(t)}, \ldots, v_\Upsilon^{(t)})$ is Υ-periodic. Note that we also have

$$v_i^{(t+1)} = v_i^{(t)}, \quad v_i^{(t)} = v_{i+\Upsilon}^{(t)} \quad \forall i, t. \tag{5.2}$$

Since $\mathbf{v}$ is a traveling wave, from (5.2), we know that

$$v_{i+\delta}^{(t)} = v_i^{(t+\tau)} = v_i^{(t+\tau-1)} \cdots = v_i^{(t)} \quad \forall i, t. \tag{5.3}$$

Therefore, given any $t \in N, \delta$ is a period of $\{v_i^{(t)}\}_{i\in\mathbb{Z}}$. By Lemma 2.1, we have $\delta = 0 \bmod \Upsilon$.

By Lemma 4.1, we also know that $\kappa I_\Upsilon - \alpha A_\Upsilon$ is not invertible and $(v_1^{(0)}, \ldots, v_\Upsilon^{(0)})$ is a nonzero vector in $\ker(\kappa I_\Upsilon - \alpha A_\Upsilon)$. Note that $\kappa - \alpha\lambda^{(0,\Upsilon)}, \ldots, \kappa - \alpha\lambda^{([\Upsilon/2],\Upsilon)}$ are all distinct eigenvalues of $\kappa I_\Upsilon - \alpha A_\Upsilon$ with corresponding eigenspaces span$\{u^{(\Upsilon,\Upsilon)}\},\ldots,$ span$\{u^{([\Upsilon/2],\Upsilon)}, u^{(\Upsilon-[\Upsilon/2],\Upsilon)}\}$, respectively. Since $\kappa-\alpha\lambda^{(0,\Upsilon)} = \kappa \neq 0$ and $\kappa I_\Upsilon-\alpha A_\Upsilon$ is not invertible, we have $\kappa-\alpha\lambda^{(j,\Upsilon)} = 0$ for some $j \in \{1,\ldots,[\Upsilon/2]\}$. Hence $\ker(\kappa I_\Upsilon-\alpha A_\Upsilon) = \operatorname{span}\{u^{(j,\Upsilon)}, u^{(\Upsilon-j,\Upsilon)}\}$ and it is clear that

$$\left(v_1^{(0)},\ldots,v_\Upsilon^{(0)}\right)^\dagger = au^{(j,\Upsilon)} + bu^{(\Upsilon-j,\Upsilon)}, \tag{5.4}$$

where $a,b \in \mathbb{R}$ such that $au^{(j,\Upsilon)} + bu^{(\Upsilon-j,\Upsilon)}$ is a nonzero vector. If $\Upsilon = 2$, we see that j must be 1 since $j \in \{1,\ldots,[\Upsilon/2]\}$. It is clear that $(j,\Upsilon) = (1,2) = 1$. Suppose $\Upsilon \geq 3$ and recall that the extension $\{v_i^{(0)}\}_{i\in\mathbb{Z}}$ of $(v_1^{(0)},\ldots,v_\Upsilon^{(0)})$ is Υ-periodic. By Lemma 3.1(ii),the extension $\{v_i^{(0)}\}$ of $(v_1^{(0)},\ldots,v_\Upsilon^{(0)})$ is Υ-periodic if and only if $(j,\Upsilon) = 1$.

Conversely, suppose $\delta = 0 \bmod \Upsilon$; there exists some $j \in \{1,\ldots,[\Upsilon/2]\}$ such that $\kappa - \alpha\lambda^{(j,\Upsilon)} = 0$ and $(j,\Upsilon) = 1$ when $\Upsilon \geq 3$. Let $\mathbf{v} = \{v_i^{(t)}\}$ satisfy (5.1). By the definition of $\mathbf{v}$, it is clear that $\mathbf{v}$ is temporal 1-periodic and Υ is a spatial period of $\mathbf{v}$. Suppose $\Upsilon = 2$ and then we have that $u = (a+b)u^{(1)}$. The fact that u is not a zero vector implies $a+b \neq 0$. By Lemma 3.1(iii), we have that $\hat{u}$ is 2-periodic. By (5.1), it is clear that $\mathbf{v}$ is spatial 2-periodic. Suppose $\Upsilon \geq 3$. Since $(j,\Upsilon) = 1$, by Lemma 3.1(ii), we have $\hat{u}$ is Υ-periodic. By (5.1) again, it is also clear that $\mathbf{v}$ is spatial Υ-periodic. In conclusion, we have that $\mathbf{v}$ is spatial Υ-periodic, that is,

$$v_{i+\Upsilon}^{(t)} = v_i^{(t)} \quad \forall t \in N,\ i \in \mathbb{Z}. \tag{5.5}$$

Since $u \in \ker(\kappa I_\Upsilon - \alpha A_\Upsilon)$, from the definition of $\mathbf{v}$, it is easy to check that $\mathbf{v}$ is a solution of (4.4). Finally, since $\delta = 0 \mod \Upsilon$, by (5.1) and (5.5), we know that

$$v_i^{(t+\tau)} = \cdots = v_i^{(t+1)} = v_i^{(t)} = v_{i+\Upsilon}^{(t)} = \cdots = v_{i+\delta}^{(t)}; \tag{5.6}$$

that is, $\mathbf{v}$ is traveling wave with velocity $-\delta/\tau$.

To see (ii), note that from the second part of the proof in (i), it is easy to see that any $\mathbf{v} = \{v_i^{(t)}\}$ satisfying (5.1) is a $(1,\Upsilon)$-periodic traveling wave solution of (4.4) with velocity $-\delta/\tau$. Also, by the first part of the proof in (i), the converse is also true. The proof is complete. □

We remark that any $(1,\Upsilon)$-periodic traveling wave solution $\mathbf{v} = \{v_i^{(t)}\}$ of (4.4) is a standing wave since this $\mathbf{v}$ is also a traveling wave with velocity 0, that is, $v_i^{(t+1)} = v_i^{(t)}$ for all $i \in \mathbb{Z}$ and $t \in N$.

Theorem 5.3. *Let $\Upsilon = 1, \Delta = 2, \alpha, \kappa \in \mathbb{R}$ with $\kappa \neq 0$ and τ, δ satisfy* (2.16). *Then*

(i) (4.4) *has a (Δ,Υ)-periodic traveling wave solution with velocity $-\delta/\tau$ if, and only if, $\kappa = -2$ and τ is even;*

(ii) *furthermore, every such solution $\mathbf{v} = \{v_i^{(t)}\}$ is of the form*

$$v_i^{(t)} = \begin{cases} c, & \text{if } t \text{ is even},\ i \in \mathbb{Z}, \\ -c, & \text{if } t \text{ is odd},\ i \in \mathbb{Z}. \end{cases} \tag{5.7}$$

where $c \neq 0$, and the converse is true.

Proof. To see (i), let $\mathbf{v} = \{v_i^{(t)}\}$ be a (Δ, Υ)-periodic traveling wave solution of (4.4) with velocity $-\delta/\tau$. By Lemma 4.2, we have each $v_i^{(0)} \neq 0, \kappa = -2$, and

$$v_i^{(t)} = \begin{cases} v_i^{(0)}, & \text{if } t \text{ is even, } i \in \mathbb{Z}, \\ -v_i^{(0)}, & \text{if } t \text{ is odd, } i \in \mathbb{Z}. \end{cases} \tag{5.8}$$

We just need to show that τ is even. Suppose to the contrary that τ is odd. Since $\Delta = 2$ is a spatial period of $\mathbf{v}$ and $\mathbf{v}$ is a traveling wave, we have

$$v_i^{(t+1)} = \cdots = v_i^{(t+\tau)} = v_{i+\delta}^{(t)} = \cdots = v_i^{(t)} \quad \forall i \in \mathbb{Z}. \tag{5.9}$$

This is contrary to the fact that $\Delta = 2$ is least among all temporal periods. That is, τ is even. For the converse, suppose that $\kappa = -2$ and τ is even. Let $\mathbf{v} = \{v_i^{(t)}\}$ be defined by (5.7). Since $c \neq 0$, by the definition of $\mathbf{v}$, it is clear that $2(= \Delta)$ is the least temporal period and 1 is the least spatial period. That is,

$$v_i^{(t+2)} = v_i^{(t)}, \quad v_i^{(t)} = v_{i+1}^{(t)} \quad \forall i, t. \tag{5.10}$$

Since τ is even, by (5.10), it is clear that

$$v_i^{(t+\tau)} = \cdots = v_i^{(t)} = \cdots = v_{i+\delta}^{(t)} \quad \forall i, t. \tag{5.11}$$

For (ii), from the proof in (i), we know that any $\mathbf{v} = \{v_i^{(t)}\}$ of the form (5.7) is a solution we want and the converse is also true by Lemma 4.2. The proof is complete. □

Now we consider the case $\Upsilon = \Delta = 2$. In this case, Υ and Δ are specific integers. Hence it is relatively easy to find the $(2, 2)$-periodic traveling wave solutions of (4.4) with velocity $-\delta/\tau$ for any τ, δ satisfying (2.16). Depending on the parity of τ, we have two results.

Theorem 5.4. *Let $\Upsilon = \Delta = 2, \alpha, \kappa \in \mathbb{R}$ with $\kappa \neq 0$ and τ, δ satisfy* (2.16) *with even τ. Then* (4.4) *has no* $(2, 2)$*-periodic traveling wave solutions with velocity $-\delta/\tau$.*

Proof. Since τ is even, by Lemma 4.4(i), a necessary condition for the existence of $(2, 2)$-periodic traveling wave solutions with velocity $-\delta/\tau$ is that Υ is odd. This is contrary to our assumption that $\Upsilon = 2$. □

Theorem 5.5. *Let $\Upsilon = \Delta = 2, \alpha, \kappa \in \mathbb{R}$ with $\kappa \neq 0$ and τ, δ satisfy* (2.16) *with odd τ. Then the following results hold.*

(i) *If δ is even, then* (4.4) *has no* $(2, 2)$*-periodic traveling wave solutions with velocity $-\delta/\tau$.*

(ii) *If δ is odd, $\kappa = -2$ and $\alpha = -1/4$, then* (4.4) *has no* $(2, 2)$*-periodic traveling wave solutions with velocity $-\delta/\tau$.*

(iii) *If δ is odd, $\kappa = -2$ and $\alpha \neq -1/4$, then* (4.4) *has no* $(2, 2)$*-periodic traveling wave solutions with velocity $-\delta/\tau$.*

(iv) *If δ is odd, $\kappa \neq -2$ and $\kappa - 4\alpha = -2$, then any $\mathbf{v} = \{v_i^{(t)}\}$ of the form*

$$\left\{v_i^{(t)}\right\}_{i\in\mathbb{Z}} = \begin{cases} \widehat{u}, & \text{if } t \text{ is even}, \\ \widehat{u'}, & \text{if } t \text{ is odd}. \end{cases} \tag{5.12}$$

where $u = a(-1,1)^\dagger$ and $u' = -a(-1,1)^\dagger$ with $a \in \mathbb{R} \setminus \{0\}$, is a $(2,2)$-periodic traveling wave solution with velocity $-\delta/\tau$, and the converse is true.

(v) *If δ is odd, $\kappa \neq -2$ and $\kappa - 4\alpha \neq -2$, then (4.4) has no $(2,2)$-periodic traveling wave solutions with velocity $-\delta/\tau$.*

Proof. To see (i), suppose δ is even. By Lemma 4.4(ii), a necessary condition for the existence of such solutions is $\delta = T_1\Upsilon/2$ for some odd integer T_1. Hence the fact that $\Upsilon = 2$ implies δ is odd. This leads to a contradiction.

For (ii), let $\kappa = -2, \alpha = -1/4$ and δ is odd. By direct computation, we have -1 and 1 are eigenvalues of $(1+\kappa)I_2 - \alpha A_2$ with corresponding eigenvectors $(1,1)^\dagger$ and $(-1,1)^\dagger$, respectively. Suppose $\mathbf{v} = \{v_i^{(t)}\}$ is a $(2,2)$-periodic traveling wave solution with velocity $-\delta/\tau$. By Lemma 4.3(ii), we have

$$\left(v_1^{(0)}, v_2^{(0)}\right)^\dagger = a(1,1)^\dagger + b(-1,1)^\dagger, \tag{5.13}$$

where $a, b \in \mathbb{R}$ with $a \neq 0$. By Lemma 4.3(i), we have

$$\left(v_1^{(t+1)}, v_2^{(t+1)}\right)^\dagger = [(1+\kappa)I_2 - \alpha A_2]^{t+1}\left(v_1^{(0)}, v_2^{(0)}\right)^\dagger \quad \forall t \in N. \tag{5.14}$$

From (5.13) and (5.14), it is clear that

$$\left(v_1^{(t)}, v_2^{(t)}\right)^\dagger = \begin{cases} (a-b, a+b)^\dagger, & \text{if } t \text{ is even}, \\ (-a-b, -a+b)^\dagger, & \text{if } t \text{ is odd}. \end{cases} \tag{5.15}$$

Since $\mathbf{v}$ is spatial 2-periodic, we see that

$$\left\{v_i^{(t)}\right\}_{i\in\mathbb{Z}} = \begin{cases} \widehat{u}, & \text{if } t \text{ is even}, \\ \widehat{u'}, & \text{if } t \text{ is odd}. \end{cases} \tag{5.16}$$

where $u = (a-b, a+b)^\dagger$ and $u' = (-a-b, -a+b)^\dagger$. From our assumption on $\mathbf{v}$, we have

$$v_i^{(t+\tau)} = v_{i+\delta}^{(t)}, \quad v_i^{(t+2)} = v_i^{(t)}, \quad v_i^{(t)} = v_{i+2}^{(t)} \quad \forall t \in N,\ i \in \mathbb{Z}. \tag{5.17}$$

Since δ and τ are both odd, by (5.17), we have

$$v_i^{(t+1)} = \cdots = v_i^{(t+\tau)} = v_{i+\delta}^{(t)} = \cdots = v_{i+1}^{(t)} \quad \forall i, t. \tag{5.18}$$

Since $\mathbf{v}$ is of form (5.16) and satisfies (5.18), we have $a+b=a-b=0$, that is, $a=b=0$. This is contrary to $a \neq 0$. The proof is complete. □

For (iii), suppose that δ is odd, $\kappa = -2$ and $\alpha \neq -1/4$. Then we have that -1 is an eigenvalue of $(1+\kappa)I_2 - \alpha A_2$ with corresponding eigenvector $(1,1)^\dagger$ and another eigenvalue $1+\kappa-4\alpha \neq 1$. Suppose that $\mathbf{v} = \{v_i^{(t)}\}$ is a $(2,2)$-periodic traveling wave solution with velocity $-\delta/\tau$. By Lemma 4.3(ii), we have

$$\left(v_1^{(0)}, v_2^{(0)}\right)^\dagger = a(1,1)^\dagger \quad \text{for some } a \in \mathbb{R} \setminus \{0\}. \tag{5.19}$$

Since 2 is a spatial period of $\mathbf{v}$, by (5.19), it is easy to see that 1 is the least spatial period. This leads to a contradiction. Hence (4.4) has no $(2,2)$-periodic traveling wave solutions with velocity $-\delta/\tau$.

The assertion (iv) is proved by the same method used in (ii).

For (v), suppose $\kappa \neq -2$ and $\kappa - 4\alpha \neq -2$. Then we know that -1 is not an eigenvalue of $(1+\kappa)I_2 - \alpha A_2$. By Lemma 4.3(iii), $(2,2)$-periodic traveling wave solutions with velocity $-\delta/\tau$ do not exist.

Finally, we consider the case where $\Upsilon \geq 3$ and $\Delta = 2$. Let τ, δ satisfy (2.16), and $\Upsilon \geq 3$, $\Delta = 2$, $\alpha, \kappa \in \mathbb{R}$ with $\kappa \neq 0$. Depending on the parity of the number τ, we have the following two subcases:

(C-1) $\Upsilon \geq 3$, $\Delta = 2$, $\alpha, \kappa \in \mathbb{R}$ with $\kappa \neq 0$ and τ, δ satisfy (2.16) with odd τ;

(C-2) $\Upsilon \geq 3$, $\Delta = 2$, $\alpha, \kappa \in \mathbb{R}$with $\kappa \neq 0$ and τ, δ satisfy (2.16) with even τ.

Here the facts in Lemma 3.2 will be used to check the spatial period of a double sequence $\mathbf{v} = \{v_i^{(t)}\}$. Furthermore, when τ is odd, the conclusions in Lemma 3.3 will be used to check whether a double sequence $\mathbf{v} = \{v_i^{(t)}\}$ is a traveling wave.

Now we focus on case (C-1). Note that $1+\kappa-\alpha\lambda^{(0,\Upsilon)} = 1+\kappa \neq 1$ since $\kappa \neq 0$. Depending on whether $1+\kappa-\alpha\lambda^{(k,\Upsilon)} = 1$ for some even $k \in \{1,\ldots,[\Upsilon/2]\}$, we have the following two theorems.

Theorem 5.6. *Let $\Upsilon, \Delta, \alpha, \kappa, \tau$, and δ satisfy* (C-1) *above and let $\lambda^{(i,\xi)}$ and $u^{(i,\xi)}$ be defined by* (3.2) *and* (3.4), *respectively. Suppose $1+\kappa-\alpha\lambda^{(k,\Upsilon)} = 1$ for some even $k \in \{1,\ldots,[\Upsilon/2]\}$. Then*

(i) (4.4) *has a (Δ,Υ)-periodic traveling wave solution with velocity $-\delta/\tau$ if, and only if, Υ is even, $\delta = T_1\Upsilon/2$ for some odd integer T_1, and there exists some $j \in \{1,\ldots,[\Upsilon/2]\}$ such that $1+\kappa-\alpha\lambda^{(j,\Upsilon)} = -1$, and either (a) $(j,\Upsilon) = 1$ or (b) $(j,\Upsilon) \neq 1$, j is odd and for any $\eta \in \{1,\ldots,\Upsilon-1\}$ with $\eta \mid \Upsilon$, one has either $\eta k/\Upsilon \notin \mathbb{Z}^+$ or $\eta j/\Upsilon \notin \mathbb{Z}^+$;*

(ii) *furthermore, if $(j,\Upsilon) = 1$, every such solution $\mathbf{v} = \{v_i^{(t)}\}$ is of the form*

$$\left\{v_i^{(t)}\right\}_{i\in\mathbb{Z}} = \begin{cases} \widehat{u}, & \text{if } t \text{ is even}, \\ \widehat{u}', & \text{if } t \text{ is odd}. \end{cases} \tag{5.20}$$

where

$$\widehat{u} = au^{(j,\Upsilon)} + bu^{(\Upsilon-j,\Upsilon)} + cu^{(k,\Upsilon)} + du^{(\Upsilon-k,\Upsilon)},$$
$$\widehat{u'} = -au^{(j,\Upsilon)} - bu^{(\Upsilon-j,\Upsilon)} + cu^{(k,\Upsilon)} + du^{(\Upsilon-k,\Upsilon)} \tag{5.21}$$

for some $a, b, c, d \in \mathbb{R}$ *such that* $au^{(j,\Upsilon)} + bu^{(\Upsilon-j,\Upsilon)}$ *is a nonzero vector, and the converse is true; while if* $(j, \Upsilon) \neq 1$*, every such solution* $\mathbf{v} = \{v_i^{(t)}\}$ *is of the form*

$$\left\{v_i^{(t)}\right\}_{i\in\mathbb{Z}} = \begin{cases} \widehat{u}, & \text{if } t \text{ is even}, \\ \widehat{u'}, & \text{if } t \text{ is odd}. \end{cases} \tag{5.22}$$

where

$$\widehat{u} = au^{(j,\Upsilon)} + bu^{(\Upsilon-j,\Upsilon)} + cu^{(k,\Upsilon)} + du^{(\Upsilon-k,\Upsilon)},$$
$$\widehat{u'} = -au^{(j,\Upsilon)} - bu^{(\Upsilon-j,\Upsilon)} + cu^{(k,\Upsilon)} + du^{(\Upsilon-k,\Upsilon)}, \tag{5.23}$$
$$au^{(j,\Upsilon)} + bu^{(\Upsilon-j,\Upsilon)} \neq (0, \dots, 0)^{\dagger}$$

for some $a, b, c, d \in \mathbb{R}$ *such that* $cu^{(k,\Upsilon)} + du^{(\Upsilon-k,\Upsilon)}$ *is a nonzero vector, and the converse is true.*

Proof. Let $\mathbf{v} = \{v_i^{(t)}\}$ be a (Δ, Υ)-periodic traveling wave solution with velocity $-\delta/\tau$. Since τ is odd, by Lemma 4.4(ii), we have that Υ is even and $\delta = T_1\Upsilon/2$ for some odd integer T_1. From Lemma 4.3(iii), we also have

$$1 + \kappa - \alpha\lambda^{(j,\Upsilon)} = -1 \quad \text{for some } j \in \left\{0, 1, \dots, \left[\frac{\Upsilon}{2}\right]\right\}. \tag{5.24}$$

In view of $1 + \kappa - \alpha\lambda^{(k,\Upsilon)} = 1$ and (5.24), we know that $\operatorname{span}\{u^{(j,\Upsilon)}, u^{(\Upsilon-j,\Upsilon)}\}$ and $\operatorname{span}\{u^{(k,\Upsilon)}, u^{(\Upsilon-k,\Upsilon)}\}$ are eigenspaces of $(1+\kappa)I_\Upsilon - \alpha A_\Upsilon$ corresponding to the eigenvalues -1 and 1, respectively. By Lemma 4.3 (ii), we have

$$\left(v_1^{(0)}, \dots, v_\Upsilon^{(0)}\right)^{\dagger} = au^{(j,\Upsilon)} + bu^{(\Upsilon-j,\Upsilon)} + cu^{(k,\Upsilon)} + du^{(\Upsilon-k,\Upsilon)}, \tag{5.25}$$

where $a, b, c, d \in \mathbb{R}$ and $au^{(j,\Upsilon)} + bu^{(\Upsilon-j,\Upsilon)}$ is a nonzero vector. By Lemma 4.3(i), we also see that

$$\left(v_1^{(t+1)}, \dots, v_\Upsilon^{(t+1)}\right)^{\dagger} = [(1+\kappa)I_\Upsilon - \alpha A_\Upsilon]^{t+1}\left(v_1^{(0)}, \dots, v_\Upsilon^{(0)}\right)^{\dagger} \quad \forall t \in N. \tag{5.26}$$

Hence it is clear that

$$\left(v_1^{(t)}, \dots, v_\Upsilon^{(t)}\right)^{\dagger} = \begin{cases} au^{(j,\Upsilon)} + bu^{(\Upsilon-j,\Upsilon)} + cu^{(k,\Upsilon)} + du^{(\Upsilon-k,\Upsilon)}, & \text{if } t \text{ is even}, \\ -au^{(j,\Upsilon)} - bu^{(\Upsilon-j,\Upsilon)} + cu^{(k,\Upsilon)} + du^{(\Upsilon-k,\Upsilon)}, & \text{if } t \text{ is odd}. \end{cases} \tag{5.27}$$

Now we want to show that $j \neq 0$ and j satisfies condition (a) or (b). First, we may assume that $j = 0$. By (5.27), we have

$$\left(v_1^{(t)}, \ldots, v_\Upsilon^{(t)}\right)^\dagger = \begin{cases} (e, \ldots, e)^\dagger + cu^{(k,\Upsilon)} + du^{(\Upsilon-k,\Upsilon)}, & \text{if } t \text{ is even}, \\ (-e, \ldots, -e)^\dagger + cu^{(k,\Upsilon)} + du^{(\Upsilon-k,\Upsilon)}, & \text{if } t \text{ is odd}. \end{cases} \tag{5.28}$$

where $e = a+b$. Under the assumption $j = 0$, we also have that $cu^{(k,\Upsilon)} + du^{(\Upsilon-k,\Upsilon)}$ is a nonzero vector. Otherwise, 1 is the least spatial period and this is contrary to $\Upsilon > 1$. Recall that Υ is a spatial period of $\mathbf{v}$. Hence by (5.28), we have

$$\left\{v_i^{(t)}\right\}_{i\in\mathbb{Z}} = \begin{cases} \widehat{u}, & \text{if } t \text{ is odd}, \\ \widehat{u'}, & \text{if } t \text{ is even}. \end{cases} \tag{5.29}$$

where

$$\begin{aligned} u &= (e, \ldots, e)^\dagger + cu^{(k,\Upsilon)} + du^{(\Upsilon-k,\Upsilon)}, \\ u' &= (-e, \ldots, -e)^\dagger + cu^{(k,\Upsilon)} + du^{(\Upsilon-k,\Upsilon)}. \end{aligned} \tag{5.30}$$

By Lemma 3.2(ii), $\mathbf{v}$ is spatial Υ-periodic if, and only if, $(k, \Upsilon) = 1$. Note that Υ and k are both even. This leads to a contradiction. In other words, we have $j \neq 0$, that is, $j \in \{1, \ldots, [\Upsilon/2]\}$. Next, we prove that j satisfies condition (a) or (b). We may assume that the result is not true. In other words, we have either $(j, \Upsilon) \neq 1$ and j is even or $(j, \Upsilon) \neq 1$ and $\eta j/\Upsilon, \eta k/\Upsilon \in \mathbb{Z}^+$ for some $\eta \in \{1, \ldots, \Upsilon - 1\}$ with $\eta \mid \Upsilon$. Under this assumption, we have $cu^{(k,\Upsilon)} + du^{(\Upsilon-k,\Upsilon)} \neq 0$. Otherwise, by (5.27), Lemma 3.2(iii), and the fact that $(j, \Upsilon) \neq 1$, we know that $\mathbf{v}$ is not spatial Υ-periodic. This leads to a contradiction. Note that τ is odd, $\delta = T_1\Upsilon/2$ for some odd T_1, and $\mathbf{v}$ is a $(2, \Upsilon)$-periodic traveling wave. These facts imply that $\mathbf{v}$ has the following property:

$$v_i^{(t+1)} = \cdots = v_i^{(t+\tau)} = v_{i+\delta}^{(t)} = \cdots = v_{i+\Upsilon/2}^{(t)} \quad \text{for } t \in N,\ i \in \mathbb{Z}. \tag{5.31}$$

If $(j, \Upsilon) \neq 1$ and j is even, by Lemma 3.3(i), $\mathbf{v}$ does not satisfy (5.31). This leads to a contradiction. If $(j, \Upsilon) \neq 1$ and $\eta j/\Upsilon, \eta k/\Upsilon, \in \mathbb{Z}^+$ for some $\eta \in \{1, \ldots, \Upsilon - 1\}$ with $\eta \mid \Upsilon$, by Lemma 3.2(i), we see that $\mathbf{v}$ is not spatial Υ-periodic. This leads to a contradiction again. In conclusion, we have that j satisfies condition (a) or (b).

For the converse, suppose that Υ is even, $\delta = T_1\Upsilon/2$ for some odd integer T_1 and $1 + \kappa - \alpha\lambda^{(j,\Upsilon)} = -1$ for some $j \in \{1, \ldots, [\Upsilon/2]\}$. We further suppose that j satisfies (a) and let $\mathbf{v} = \{v_i^{(t)}\}$ be defined by (5.20). Recall that $\operatorname{span}\{u^{(j,\Upsilon)}, u^{(\Upsilon-j,\Upsilon)}\}$ and $\operatorname{span}\{u^{(k,\Upsilon)}, u^{(\Upsilon-k,\Upsilon)}\}$ are eigenspaces of $(1 + \kappa)I_\Upsilon - \alpha A_\Upsilon$ corresponding to the eigenvalues -1 and 1, respectively. Hence by direct computation, we have that $\mathbf{v}$ is a solution of (4.4). Since $au^{(j,\Upsilon)} + bu^{(\Upsilon-j,\Upsilon)} \neq 0$, we also have that $\{v_i^{(t)}\}_{i\in\mathbb{Z}}$ is temporal 2-periodic. Since $(j, \Upsilon) = 1$, we have $\eta j/\Upsilon \notin \mathbb{Z}^+$ for any $\eta \in \{1, \ldots, \Upsilon - 1\}$ with $\eta \mid \Upsilon$. By (i) and (iii) of Lemma 3.2, it is easy to check that $\mathbf{v}$ is spatial

Υ-periodic. The fact $(j,\Upsilon)=1$ implies that j is odd. From (i) and (ii) of Lemma 3.3, we have that

$$v_i^{(t+1)} = v_{i+\Upsilon/2}^{(t)} \quad \forall i\in\mathbb{Z},\ t\in N. \tag{5.32}$$

Since $\delta = T_1\Upsilon/2$ for some odd integer T_1 and τ is odd, by (5.32), we have

$$v_i^{(t+\tau)} = \cdots = v_i^{(t+1)} = v_{i+\Upsilon/2}^{(t)} = \cdots = v_{i+\delta}^{(t)} \quad \forall i\in\mathbb{Z},\ t\in N. \tag{5.33}$$

In other words, $\mathbf{v}$ is a (Δ,Υ)-periodic traveling wave solution with velocity $-\delta/\tau$. If j satisfies (b), we simply let $\mathbf{v}=\{v_i^{(t)}\}$ be defined by (5.22) and then the desired result may be proved by similar arguments.

To see (ii), suppose that $\mathbf{v}=\{v_i^{(t)}\}$ is a (Δ,Υ)-periodic traveling wave solution with velocity $-\delta/\tau$. From the proof in (i), we have shown that

$$\left(v_1^{(t)},\ldots,v_\Upsilon^{(t)}\right)^\dagger = \begin{cases} au^{(j,\Upsilon)} + bu^{(\Upsilon-j,\Upsilon)} + cu^{(k,\Upsilon)} + du^{(\Upsilon-k,\Upsilon)}, & \text{if } t \text{ is even},\\ -au^{(j,\Upsilon)} - bu^{(\Upsilon-j,\Upsilon)} + cu^{(k,\Upsilon)} + du^{(\Upsilon-k,\Upsilon)}, & \text{if } t \text{ is odd}. \end{cases} \tag{5.34}$$

where $a,b,c,d\in\mathbb{R}$ and $au^{(j,\Upsilon)}+bu^{(\Upsilon-j,\Upsilon)}$ is a nonzero vector. Since Υ is a spatial period of $\mathbf{v}$, we have that $\mathbf{v}=\{v_i^{(t)}\}$ is of the form (5.20). Now we just need to show that if $(j,\Upsilon)\neq 1$, then we have $cu^{(k,\Upsilon)}+du^{(\Upsilon-k,\Upsilon)}\neq 0$. Suppose to the contrary that $cu^{(k,\Upsilon)}+du^{(\Upsilon-k,\Upsilon)}$ is a zero vector, and $(j,\Upsilon)\neq 1$. By Lemma 3.2 (iii), $\mathbf{v}$ is not spatial Υ-periodic. This leads to a contradiction. The converse has been shown in the second part of the proof of (i). □

Theorem 5.7. *Let $\Upsilon,\Delta,\alpha,\kappa,\tau$, and δ satisfy* (C-1) *above and let $\lambda^{(i,\xi)}$ and $u^{(i,\xi)}$ be defined by* (3.2) *and* (3.4)*, respectively. Suppose $1+\kappa-\alpha\lambda^{(k,\Upsilon)}\neq 1$ for all even $k\in\{1,\ldots,[\Upsilon/2]\}$. Then*

(i) (4.4) *has a (Δ,Υ)-periodic traveling wave solution with velocity $-\delta/\tau$ if, and only if, Υ is even, $\delta=T_1\Upsilon/2$ for some odd integer T_1 and there exists some $j\in\{1,\ldots,[\Upsilon/2]\}$ with $(j,\Upsilon)=1$ such that $1+\kappa-\alpha\lambda^{(j,\Upsilon)}=-1$;*

(ii) *furthermore, every such solution $\mathbf{v}=\{v_i^{(t)}\}$ is of the form*

$$\left\{v_i^{(t)}\right\}_{i\in\mathbb{Z}} = \begin{cases} \widehat{u}, & \text{if } t \text{ is even},\\ \widehat{u'}, & \text{if } t \text{ is odd}. \end{cases} \tag{5.35}$$

where $\widehat{u}=au^{(j,\Upsilon)}+bu^{(\Upsilon-j,\Upsilon)}$ and $\widehat{u'}=-au^{(j,\Upsilon)}-bu^{(\Upsilon-j,\Upsilon)}$ for some a,b such that $|a|+|b|\neq 0$, and the converse is true.

Next, we focus on case (C-2) and recall that $1+\kappa-\alpha\lambda^{(0,\Upsilon)}\neq 1$. Depending on whether $1+\kappa-\alpha\lambda^{(k,\Upsilon)}=1$ for some $k\in\{1,\ldots,[\Upsilon/2]\}$, we also have the following theorems.

Theorem 5.8. *Let $\Upsilon, \Delta, \alpha, \kappa, \tau$, and δ satisfy* (C-2) *above and let $\lambda^{(i,\xi)}$ and $u^{(i,\xi)}$ be defined by* (3.2) *and* (3.4)*, respectively. Suppose $1+\kappa-\alpha\lambda^{(k,\Upsilon)}=1$ for some $k \in \{1,\dots,[\Upsilon/2]\}$. Then*

(i) (4.4) *has a (Δ,Υ)- periodic traveling wave solution with velocity $-\delta/\tau$ if, and only if, Υ is odd, $\delta = T_1\Upsilon$ for some odd integer T_1, and $1+\kappa-\alpha\lambda^{(j,\Upsilon)}=-1$ for some $j \in \{0,1,\dots,[\Upsilon/2]\}$ such that either (a) $j=0$ and $(k,\Upsilon)=1$ or (b) $j \neq 0$ with $(j,\Upsilon)=1$ or (c) $j\neq 0$ with $(j,\Upsilon)\neq 1$ and for any $\eta \in \{1,\dots,\Upsilon-1\}$ with $\eta \mid \Upsilon$, one has either $\eta k/\Upsilon \notin \mathbb{Z}^+$ or $\eta j/\Upsilon \notin \mathbb{Z}^+$;*

(ii) *furthermore, if j satisfies condition* (i)–(a) *above, every such solution $\mathbf{v}=\{v_i^{(t)}\}$ is of the form* (5.20)*, and the converse is true; while if j satisfies condition* (i)–(b) *above, every such solution $\mathbf{v}=\{v_i^{(t)}\}$ is of the form* (5.22)*, and the converse is true.*

Theorem 5.9. *Let $\Upsilon, \Delta, \alpha, \kappa, \tau$, and δ satisfy (C-2) above and $\lambda^{(i,\xi)}$ and let $u^{(i,\xi)}$ be defined by* (3.2) *and* (3.4) *respectively. Suppose $1+\kappa-\alpha\lambda^{(k,\Upsilon)} \neq 1$ for all $k \in \{1,\dots,[\Upsilon/2]\}$. Then*

(i) (4.4) *has a (Δ,Υ)-periodic traveling wave solution with velocity $-\delta/\tau$ if, and only if, Υ is odd, $\delta = T_1\Upsilon$ for some odd integer T_1, and there exists some $j \in \{1,\dots,[\Upsilon/2]\}$ with $(j,\Upsilon)=1$ such that $1+\kappa-\alpha\lambda^{(j,\Upsilon)}=-1$; and*

(ii) *furthermore, every such solution $\mathbf{v}=\{v_i^{(t)}\}$ is of the form* (5.35)*, and the converse is true.*

6. Concluding Remarks and Examples

Recall that one of our main concerns is whether mathematical models can be built that supports doubly periodic traveling patterns (with a priori unknown velocities and periodicities). In the previous discussions, we have found necessary and sufficient conditions for the existence of traveling waves with arbitrarily given least spatial periods and least temporal periods and traveling speeds. Therefore, we may now answer our original question as follows. Suppose that we are given the parameters α and κ, where $\alpha,\kappa \in \mathbb{R}$ with $\kappa \neq 0$, and the reaction-diffusion network:

$$v_i^{(t+1)} - v_i^{(t)} = \alpha\left(v_{i+1}^{(t)} - 2v_i^{(t)} + v_{i-1}^{(t)}\right) + \kappa v_i^{(t)}, \quad t \in N,\ i \in \mathbb{Z}. \tag{6.1}$$

For any $\Upsilon \geq 2$ and $j \in \{1,\dots,[\Upsilon/2]\}$, we define

$$\begin{aligned}\Gamma_j^{1,\Upsilon} &= \left\{(\alpha,\kappa) \in \mathbb{R}^2 \mid 1+\kappa-\alpha\lambda^{(j,\Upsilon)} = 1\right\},\\ \Gamma_j^{-1,\Upsilon} &= \left\{(\alpha,\kappa) \in \mathbb{R}^2 \mid 1+\kappa-\alpha\lambda^{(j,\Upsilon)} = -1\right\},\end{aligned} \tag{6.2}$$

where $\lambda^{(j,\Upsilon)}$ is defined by (3.2). By theorems in Section 5, it is then easy to see the following result.

Corollary 6.1. *Let α and $\kappa \in \mathbb{R}$ with $\kappa \neq 0$.*

(1) *The double sequence $\mathbf{v}=\{v_i^{(t)}=0\}$ is the unique $(1,1)$-periodic traveling wave solution of* (6.1) *with velocity $-\delta/\tau$ for arbitrary δ and τ satisfying* (2.16).

(2) *Suppose $(\alpha,\kappa) \in \Gamma_j^{1,\Upsilon}$ where $j \in \{1,\dots,[\Upsilon/2]\}$ with $(\Upsilon,j)=1$. Then* (6.1) *has at least one $(1,\Upsilon)$-periodic traveling wave solution with velocity $-\delta/\tau$ for arbitrary δ and τ which satisfy* (2.16) *and $\delta = 0 \bmod \Upsilon$.*

(3) *Suppose* $\kappa = -2$. *Then* (6.1) *has at least one* $(2,1)$*-periodic traveling wave solution with velocity* $-\delta/\tau$ *for arbitrary* δ *and* τ *satisfying* (2.16).

(4) *Suppose* $(\alpha,\kappa) = (-1/4,-2)$ *or* $\kappa \neq -2$ *and* $\kappa - 4\alpha = -2$. *Then* (6.1) *has at least one* $(2,2)$*-periodic traveling wave solution with velocity* $-\delta/\tau$ *for arbitrary* δ *and* τ *which are both odd and satisfy* (2.16).

(5) *Suppose* (i) $(\alpha,\kappa) \in \Gamma_j^{-1,\Upsilon}$ *where* Υ *is even,* $j \in \{1,\ldots,[\Upsilon/2]\}$ *with* $(j,\Upsilon) = 1$ *or* (ii) $(\alpha,\kappa) \in \Gamma_j^{-1,\Upsilon} \cap \Gamma_k^{1,\Upsilon}$ *where* Υ *is even,* $j,k \in \{1,\ldots,[\Upsilon/2]\}$ *with* j *odd and* k *even and for any* $\eta \in \{1,\ldots,\Upsilon-1\}$ *with* $\eta \mid \Upsilon$, *one has either* $\eta k/\Upsilon \notin \mathbb{Z}^+$ *or* $\eta j/\Upsilon \notin \mathbb{Z}^+$. *Then* (6.1) *has at least one* $(2,\Upsilon)$*-periodic traveling wave solution with velocity* $-\delta/\tau$ *for arbitrary* δ *and* τ *which satisfy* (2.16), τ *is odd, and* $\delta = T_1\Upsilon/2$ *for some odd integer* T_1.

(6) *Suppose* (i) $(\alpha,\kappa) \in \Gamma_j^{-1,\Upsilon}$ *where* Υ *is odd,* $j \in \{1,\ldots,[\Upsilon/2]\}$ *with* $(j,\Upsilon) = 1$ *or* (ii) $(\alpha,\kappa) \in \Gamma_j^{-1,\Upsilon}\cap\Gamma_k^{1,\Upsilon}$ *where* Υ *is odd,* $j,k \in \{1,\ldots,[\Upsilon/2]\}$ *and for any* $\eta \in \{1,\ldots,\Upsilon-1\}$ *with* $\eta \mid \Upsilon$, *one has either* $\eta k/\Upsilon \notin \mathbb{Z}^+$ *or* $\eta j/\Upsilon \notin \mathbb{Z}^+$. *Then* (6.1) *has at least one* $(2,\Upsilon)$*-periodic traveling wave solution with velocity* $-\delta/\tau$ *for arbitrary* δ *and* τ *which satisfy* (2.16), τ *is even, and* $\delta = 0 \mod \Upsilon$.

Finally, we provide some examples to illustrate the conclusions in the previous sections.

Example 6.2. Let $\kappa = \sqrt{2}$, $\alpha = 1$, $r = 0$, $\tau = 5$, $\delta = 4$, $\Upsilon = 8$, and $\Delta = 2$. Consider the equation

$$v_i^{(t+1)} - v_i^{(t)} = v_{i+1}^{(t)} - 2v_i^{(t)} + v_{i-1}^{(t)} + \sqrt{2}v_i^{(t)}, \quad i \in \mathbb{Z},\ t \in N. \tag{6.3}$$

We want to find all $(2,8)$-periodic traveling wave solutions of (6.3) with velocity $-4/5$.

By direct computation,

$$1 + \kappa - \alpha\lambda^{(i,8)} = 1 + \sqrt{2} - \lambda^{(i,8)} \neq 1 \quad \text{for } i = 0,1,2,4. \tag{6.4}$$

It is also clear that Υ is even, $\delta = T_1(\Upsilon/2)$ for some odd integer $T_1, 1 + \kappa - \alpha\lambda^{(3,8)} = 1$, and $(3,8) = 1$. By Theorem 5.7(i), (6.3) has $(2,8)$-periodic traveling wave solution with velocity $-4/5$. By Theorem 5.7(ii), any such solution $\mathbf{v} = \{v_i^{(t)}\}$ of (6.3) is of the form

$$\left\{v_i^{(t)}\right\}_{i\in\mathbb{Z}} = \begin{cases} \widehat{u}, & \text{if } t \text{ is even}, \\ \widehat{u'}, & \text{if } t \text{ is odd}. \end{cases} \tag{6.5}$$

where $u = au^{(3,8)} + bu^{(5,8)}$ as well as $u' = -au^{(3,8)} - bu^{(5,8)}$ for some $a,b \in \mathbb{R}$ with $|a| + |b| \neq 0$, and the converse is true. Recall that

$$\begin{aligned} u^{(3,8)} &= \left(u_1^{(3,8)},\ldots,u_8^{(3,8)}\right)^{\dagger}, \\ u^{(5,8)} &= \left(u_1^{(5,8)},\ldots,u_8^{(5,8)}\right)^{\dagger}, \end{aligned} \tag{6.6}$$

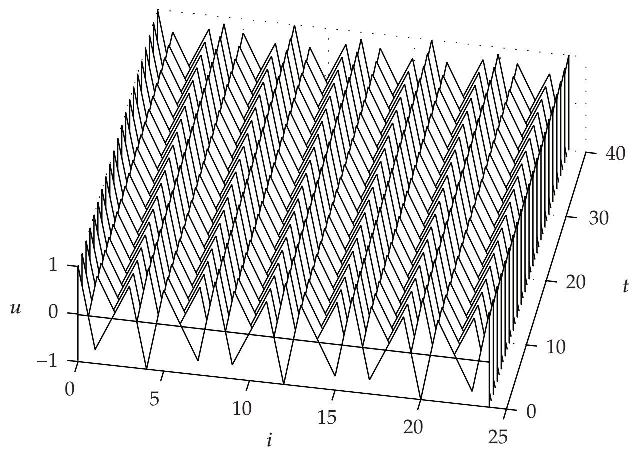

Figure 2: A $(2,8)$-periodic traveling wave solution with velocity $-4/5$.

where

$$u_m^{(i,8)} = \frac{1}{\sqrt{8}(\cos(2mi\pi/8) + \sin(2mi\pi/8))} \quad \text{for } i = 3,5, \ m \in \{1,\dots,8\}. \tag{6.7}$$

In Figure 2, we take $a = b = \sqrt{2}$ for illustration.

Example 6.3. Let $r = 0$, $\tau = 4$, $\delta = 27$, $\Upsilon = 9$ and $\Delta = 2$. Set

$$\kappa = -\frac{8\sin^2(4/9)\pi}{-3 + 4\sin^2(4/9)\pi}, \qquad \alpha = -\frac{2}{-3 + 4\sin^2(4/9)\pi}. \tag{6.8}$$

Consider the equation

$$v_i^{(t+1)} - v_i^{(t)} = \alpha\left(v_{i+1}^{(t)} - 2v_i^{(t)} + v_{i-1}^{(t)}\right) + \kappa v_i^{(t)}, \quad i \in \mathbb{Z}, \ t \in N. \tag{6.9}$$

We want to find all (Δ, Υ)-periodic traveling wave solutions of (6.9) with velocity $-27/4$. By direct computation, we have $1 + \kappa - \alpha\lambda^{(4,9)} = 1$. From our assumption, we also have $\delta = 0 \mod \Upsilon$. Note that $1 + \kappa - \alpha\lambda^{(3,9)} = -1$, $(3,9) \neq 1$ and $4\eta/9 \notin \mathbb{Z}^+$ for any $\eta < 9$ with $\eta \mid 9$. By Theorem 5.8(i), (6.9) has doubly periodic traveling wave solutions. By Theorem 5.8(ii), any solution $\mathbf{v} = \{v_i^{(t)}\}$ is of the form

$$\left\{v_i^{(t)}\right\}_{i\in\mathbb{Z}} = \begin{cases} \widehat{u}, & \text{if } t \text{ is even}, \\ \widehat{u'}, & \text{if } t \text{ is odd}. \end{cases} \tag{6.10}$$

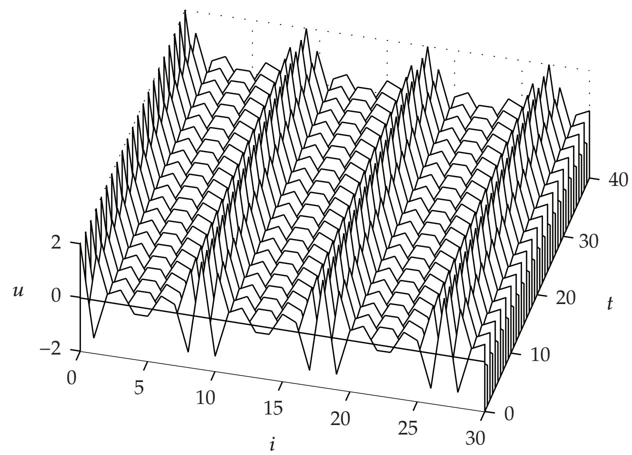

Figure 3: A $(2,9)$-periodic traveling wave solution with velocity $-27/4$.

where

$$\begin{aligned} \widehat{u} &= au^{(3,9)} + bu^{(6,9)} + cu^{(4,9)} + du^{(5,9)}, \\ \widehat{u'} &= -au^{(3,9)} - bu^{(6,9)} + cu^{(4,9)} + du^{(5,9)} \end{aligned} \tag{6.11}$$

for some $a, b, c, d \in \mathbb{R}$ such that $au^{(3,9)} + bu^{(6,9)}$ and $cu^{(4,9)} + du^{(5,9)}$ are both nonzero, and the converse is true. Recall that $u^{(i,9)} = \left(u_1^{(i,9)}, \ldots, u_{10}^{(i,9)}\right)^{\dagger}$ where

$$u_m^{(i,9)} = \frac{1}{3}\left(\cos\left(\frac{2mi\pi}{9}\right) + \sin\left(\frac{2mi\pi}{9}\right)\right) \tag{6.12}$$

for $i \in \{3,4,5,6\}$ and $m \in \{1,\ldots,9\}$. In Figure 3, we take $a = b = c = d = 3/2$ for illustration.

Example 6.4. Let $\tau = 7$, $\delta = 15$, $r = 0$, $\kappa = -1$, $\Upsilon = 10$ and $\Delta = 2$. Consider the equation

$$v_i^{(t+1)} - v_i^{(t)} = \alpha\left(v_{i+1}^{(t)} - 2v_i^{(t)} + v_{i-1}^{(t)}\right) - v_i^{(t)}, \quad i \in \mathbb{Z},\ t \in N, \tag{6.13}$$

where $\alpha \in \mathbb{R}$. We want to find all $(2,10)$-periodic traveling wave solutions of (6.9) with velocity $-15/7$.

By direct computation, we have

$$1 + \kappa - \alpha\lambda^{(i,10)} \neq -1 \quad \forall i \in \{1,\ldots,5\}, \tag{6.14}$$

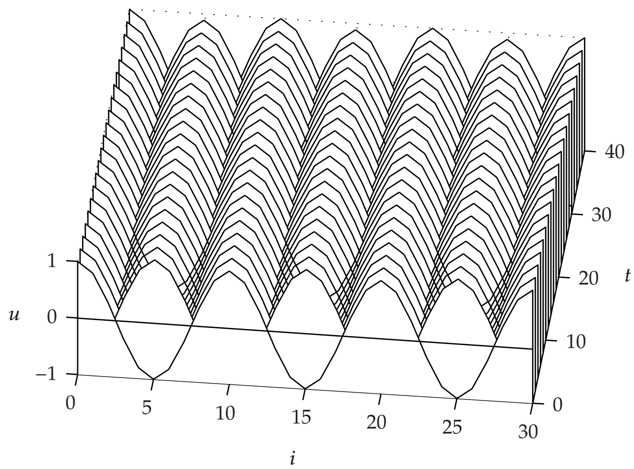

Figure 4: A $(2, 10)$-periodic traveling wave solution with velocity $-15/7$.

where $\alpha \notin \{1/\lambda^{(i,10)} \mid i \in \{1, \ldots, 5\}\}$. By direct computation again, we also know that

$$1 + \kappa - \alpha\lambda^{(i,10)} \neq 1 \quad \forall i \in \{1, \ldots, 5\}, \tag{6.15}$$

where $\alpha \in \{1/\lambda^{(i,10)} \mid i \in \{1, \ldots, 5\}\}$.

First, let $\alpha \in \mathbb{R}$ with $\alpha \notin \{1/\lambda^{(i,10)} \mid i \in \{1, \ldots, 5\}\}$. By Theorem 5.7(i), the fact that

$$1 + \kappa - \alpha\lambda^{(i,10)} = -1 \quad \text{for some } i \in \{1, \ldots, 5\} \tag{6.16}$$

is necessary for the existence of doubly periodic traveling wave solutions. From (6.14), one has that (6.13) has no $(2, 10)$-periodic traveling wave solutions of (6.13) with velocity $-15/7$.

Secondly, let $\alpha = 1/\lambda^{(j,10)}$, where $j \in \{1, 3\}$. Recall (6.15), we see that $1+\kappa-\alpha\lambda^{(i,10)} \neq 1$ for all $i \in \{1, \ldots, 5\}$. By our assumption, it is easy to check that Υ is even and $\delta = T_1(\Upsilon/2)$ for some odd integer T_1. We also have $1 + \kappa - \alpha\lambda^{(j,10)} = -1$ and note that $(j, 10) = 1$ because of $j \in \{1, 3\}$. By Theorem 5.7(i), (6.13) has doubly periodic traveling wave solutions. By Theorem 5.7(ii), any solution $\mathbf{v} = \{v_i^{(t)}\}$ is of the form

$$\left\{v_i^{(t)}\right\}_{i\in\mathbb{Z}} = \begin{cases} \widehat{u}, & \text{if } t \text{ is even}, \\ \widehat{u'}, & \text{if } t \text{ is odd}. \end{cases} \tag{6.17}$$

where $u = au^{(j,10)} + bu^{(10-j,10)}$ and $u' = -au^{(j,10)} - bu^{(10-j,10)}$ for some $a, b \in \mathbb{R}$ with $|a| + |b| \neq 0$, and the converse is true. Recall that $u^{(i,10)} = \left(u_1^{(i,10)}, \ldots, u_{10}^{(i,10)}\right)^{\dagger}$ where

$$u_m^{(i,10)} = \frac{1}{\sqrt{10}}\left(\cos\left(\frac{2mi\pi}{10}\right) + \sin\left(\frac{2mi\pi}{10}\right)\right) \tag{6.18}$$

for $i \in \{1,3,7,9\}$ and $m \in \{1,\ldots,10\}$. In Figure 4, we take $\alpha = \lambda^{(1,10)}$ and $a = b = \sqrt{10}/2$ for illustration.

Finally, let $\alpha = 1/\lambda^{(j,10)}$, where $j \in \{2,4,5\}$. We also have $1 + \kappa - \alpha\lambda^{(i,10)} \neq 1$ for all $i \in \{1,\ldots,5\}$, Υ is even, $\delta = T_1(\Upsilon/2)$ for some odd integer, and $1 + \kappa - \alpha\lambda^{(j,10)} = -1$. However, it is clear that $(j,10) > 1$. By Theorem 5.7(i), (6.13) has no doubly periodic traveling wave solutions.

We have given a complete account for the existence of (Δ,Υ)-periodic traveling wave solutions with velocity $-\delta/\tau$ for either

$$v_i^{(t+1)} - v_i^{(t)} = \alpha\left(v_{i-1}^{(t)} - 2v_i^{(t)} + v_{i+1}^{(t)}\right) + \kappa, \quad i \in \mathbb{Z},\ t \in N, \kappa \neq 0 \tag{6.19}$$

or

$$v_i^{(t+1)} - v_i^{(t)} = \alpha\left(v_{i-1}^{(t)} - 2v_i^{(t)} + v_{i+1}^{(t)}\right) + \kappa v_i^{(t)}, \quad i \in \mathbb{Z},\ t \in N, \kappa \neq 0. \tag{6.20}$$

In particular, the former equation does not have any such solutions, while the latter may, but only when $\Delta = 1$ or 2. We are then able to pinpoint the exact conditions on $\Upsilon, \delta, \tau, \alpha$, and κ such that the desired solutions exist. Although we are concerned with the case where the reaction term is linear, the number of parameters involved, however, leads us to a relatively difficult problem as can be seen in our previous discussions.

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