Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2008, Article ID 458687, 12 pages
doi:10.1155/2008/458687

Research Article

Iterated Oscillation Criteria for Delay Dynamic Equations of First Order

M. Bohner,[1] B. Karpuz,[2] and Ö. Öcalan[2]

[1] *Department of Economics and Finance, Missouri University of Science and Technology, Rolla, MO 65409-0020, USA*
[2] *Department of Mathematics, Faculty of Science and Arts, Afyon Kocatepe University, ANS Campus, 03200 Afyonkarahisar, Turkey*

Correspondence should be addressed to B. Karpuz, bkarpuz@aku.edu.tr

Received 9 June 2008; Accepted 4 December 2008

Recommended by John Graef

We obtain new sufficient conditions for the oscillation of all solutions of first-order delay dynamic equations on arbitrary time scales, hence combining and extending results for corresponding differential and difference equations. Examples, some of which coincide with well-known results on particular time scales, are provided to illustrate the applicability of our results.

1. Introduction

Oscillation theory on $\mathbb{Z}$ and $\mathbb{R}$ has drawn extensive attention in recent years. Most of the results on $\mathbb{Z}$ have corresponding results on $\mathbb{R}$ and vice versa because there is a very close relation between $\mathbb{Z}$ and $\mathbb{R}$. This relation has been revealed by Hilger in [1], which unifies discrete and continuous analysis by a new theory called *time scale theory*.

As is well known, a first-order delay differential equation of the form

$$x'(t) + p(t)x(t-\tau) = 0, \tag{1.1}$$

where $t \in \mathbb{R}$ and $\tau \in \mathbb{R}^+ := [0, \infty)$, is oscillatory if

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} p(\eta)d\eta > \frac{1}{\mathrm{e}} \tag{1.2}$$

holds [2, Theorem 2.3.1]. Also the corresponding result for the difference equation

$$\Delta x(t) + p(t)x(t-\tau) = 0, \tag{1.3}$$

where $t \in \mathbb{Z}$, $\Delta x(t) = x(t+1) - x(t)$ and $\tau \in \mathbb{N}$, is

$$\liminf_{t\to\infty} \sum_{\eta=t-\tau}^{t-1} p(\eta) > \left(\frac{\tau}{\tau+1}\right)^{\tau+1} \tag{1.4}$$

[2, Theorem 7.5.1]. Li [3] and Shen and Tang [4, 5] improved (1.2) for (1.1) to

$$\liminf_{t\to\infty} p_n(t) > \frac{1}{e^n}, \tag{1.5}$$

where

$$p_n(t) = \begin{cases} 1, & n = 0, \\ \displaystyle\int_{t-\tau}^{t} p(\eta)p_{n-1}(\eta)d\eta, & n \in \mathbb{N}. \end{cases} \tag{1.6}$$

Note that (1.2) is a particular case of (1.5) with $n = 1$. Also a corresponding result of (1.4) for (1.3) has been given in [6, Corollary 1], which coincides in the discrete case with our main result as

$$\liminf_{t\to\infty} p_n(t) > \left(\frac{\tau}{\tau+1}\right)^{n(\tau+1)}, \tag{1.7}$$

where p_n is defined by a similar recursion in [6], as

$$p_n(t) = \begin{cases} 1, & n = 0, \\ \displaystyle\sum_{\eta=t-\tau}^{t-1} p(\eta)p_{n-1}(\eta), & n \in \mathbb{N}. \end{cases} \tag{1.8}$$

Our results improve and extend the known results in [7, 8] to arbitrary time scales. We refer the readers to [9, 10] for some new results on the oscillation of delay dynamic equations.

Now, we consider the first-order delay dynamic equation

$$x^{\Delta}(t) + p(t)x(\tau(t)) = 0, \tag{1.9}$$

where $t \in \mathbb{T}$, $\mathbb{T}$ is a time scale (i.e., any nonempty closed subset of $\mathbb{R}$) with $\sup \mathbb{T} = \infty$, $p \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^+)$, the delay function $\tau : \mathbb{T} \to \mathbb{T}$ satisfies $\lim_{t\to\infty} \tau(t) = \infty$ and $\tau(t) \leq t$ for all $t \in \mathbb{T}$. If $\mathbb{T} = \mathbb{R}$, then $x^{\Delta} = x'$ (the usual derivative), while if $\mathbb{T} = \mathbb{Z}$, then $x^{\Delta} = \Delta x$ (the usual forward difference). On a time scale, the *forward jump operator* and the *graininess function* are defined by

$$\sigma(t) := \inf (t,\infty)_{\mathbb{T}}, \qquad \mu(t) := \sigma(t) - t, \tag{1.10}$$

where $(t,\infty)_{\mathbb{T}} := (t,\infty) \cap \mathbb{T}$ and $t \in \mathbb{T}$. We refer the readers to [11, 12] for further results on time scale calculus.

A function $f : \mathbb{T} \to \mathbb{R}$ is called *positively regressive* if $f \in C_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$ and $1 + \mu(t)f(t) > 0$ for all $t \in \mathbb{T}$, and we write $f \in \mathcal{R}^+(\mathbb{T})$. It is well known that if $f \in \mathcal{R}^+([t_0,\infty)_{\mathbb{T}})$, then there exists a positive function x satisfying the initial value problem

$$x^{\Delta}(t) = f(t)x(t), \quad x(t_0) = 1, \tag{1.11}$$

where $t_0 \in \mathbb{T}$ and $t \in [t_0,\infty)_{\mathbb{T}}$, and it is called the *exponential function* and denoted by $\mathrm{e}_f(\cdot, t_0)$. Some useful properties of the exponential function can be found in [11, Theorem 2.36].

The setup of this paper is as follows: while we state and prove our main result in Section 2, we consider special cases of particular time scales in Section 3.

2. Main results

We state the following lemma, which is an extension of [3, Lemma 2] and improvement of [10, Lemma 2].

Lemma 2.1. *Let x be a nonoscillatory solution of* (1.9). *If*

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(\eta)\Delta\eta > 0, \tag{2.1}$$

then

$$\liminf_{t\to\infty} y_x(t) < \infty, \tag{2.2}$$

where

$$y_x(t) := \frac{x(\tau(t))}{x(t)} \quad \textit{for } t \in [t_0,\infty)_{\mathbb{T}}. \tag{2.3}$$

Proof. Since (1.9) is linear, we may assume that x is an eventually positive solution. Then, x is eventually nonincreasing. Let $x(t),\ x(\tau(t)) > 0$ for all $t \in [t_1,\infty)_{\mathbb{T}}$, where $t_1 \in [t_0,\infty)_{\mathbb{T}}$. In view of (2.1), there exists $\varepsilon > 0$ and an increasing divergent sequence $\{\xi_n\}_{n\in\mathbb{N}} \subset [t_1,\infty)_{\mathbb{T}}$ such that

$$\int_{\tau(\xi_n)}^{\sigma(\xi_n)} p(\eta)\Delta\eta \ge \int_{\tau(\xi_n)}^{\xi_n} p(\eta)\Delta\eta \ge \varepsilon \quad \forall n \in \mathbb{N}_0. \tag{2.4}$$

Now, consider the function $\Gamma_n : [\tau(\xi_n), \sigma(\xi_n))_{\mathbb{T}} \to \mathbb{R}$ defined by

$$\Gamma_n(t) := \int_{\tau(\xi_n)}^{t} p(\eta)\Delta\eta - \frac{\varepsilon}{2}. \tag{2.5}$$

We see that $\Gamma_n(\tau(\xi_n)) < 0$ and $\Gamma_n(\xi_n) > 0$ for all $n \in \mathbb{N}$. Therefore, there exists $\zeta_n \in [\tau(\xi_n), \xi_n)_{\mathbb{T}}$ such that $\Gamma_n(\zeta_n) \leq 0$ and $\Gamma_n(\sigma(\zeta_n)) \geq 0$ for all $n \in \mathbb{N}$. Clearly, $\{\zeta_n\}_{n\in\mathbb{N}} \subset [t_1, \infty)_{\mathbb{T}}$ is a nondecreasing divergent sequence. Then, for all $n \in \mathbb{N}$, we have

$$\int_{\tau(\xi_n)}^{\sigma(\zeta_n)} p(\eta)\Delta\eta \overset{(2.5)}{=} \frac{\varepsilon}{2} + \Gamma_n(\sigma(\zeta_n)) \geq \frac{\varepsilon}{2} \tag{2.6}$$

and

$$\int_{\zeta_n}^{\sigma(\xi_n)} p(\eta)\Delta\eta \overset{(2.5)}{=} \int_{\tau(\xi_n)}^{\sigma(\xi_n)} p(\eta)\Delta\eta - \left(\Gamma_n(\zeta_n) + \frac{\varepsilon}{2}\right) \geq \frac{\varepsilon}{2} - \Gamma_n(\zeta_n) \geq \frac{\varepsilon}{2}. \tag{2.7}$$

Thus, for all $n \in \mathbb{N}$, we can calculate

$$\begin{aligned} x(\zeta_n) &\geq x(\zeta_n) - x(\sigma(\xi_n)) \overset{(1.9)}{=} \int_{\zeta_n}^{\sigma(\xi_n)} p(\eta)x(\tau(\eta))\Delta\eta \geq x(\tau(\xi_n))\int_{\zeta_n}^{\sigma(\xi_n)} p(\eta)\Delta\eta \\ &\overset{(2.7)}{\geq} \frac{\varepsilon}{2}x(\tau(\xi_n)) \geq \frac{\varepsilon}{2}\left[x(\tau(\xi_n)) - x(\sigma(\zeta_n))\right] \overset{(1.9)}{=} \frac{\varepsilon}{2}\int_{\tau(\xi_n)}^{\sigma(\zeta_n)} p(\eta)x(\tau(\eta))\Delta\eta \\ &\geq \frac{\varepsilon}{2}x(\tau(\zeta_n))\int_{\tau(\xi_n)}^{\sigma(\zeta_n)} p(\eta)\Delta\eta \overset{(2.6)}{\geq} \left(\frac{\varepsilon}{2}\right)^2 x(\tau(\zeta_n)), \end{aligned} \tag{2.8}$$

and using (2.3),

$$y_x(\zeta_n) \leq \left(\frac{2}{\varepsilon}\right)^2. \tag{2.9}$$

Letting n tend to infinity, we see that (2.2) holds. □

For the statement of our main results, we introduce

$$\alpha_n(t) := \begin{cases} 1, & n = 0, \\ \inf\limits_{\substack{\lambda>0 \\ -\lambda p\alpha_{n-1}\in\mathcal{R}^+([\tau(t),t)_{\mathbb{T}})}} \left\{\dfrac{1}{\lambda e_{-\lambda p\alpha_{n-1}}(t, \tau(t))}\right\}, & n \in \mathbb{N}, \end{cases} \tag{2.10}$$

for $t \in [s, \infty)_{\mathbb{T}}$, where $\tau^n(s) \in [t_0, \infty)_{\mathbb{T}}$.

Lemma 2.2. *Let x be a nonoscillatory solution of* (1.9). *If there exists $n_0 \in \mathbb{N}$ such that*

$$\liminf_{t\to\infty} \alpha_{n_0}(t) > 1, \tag{2.11}$$

then

$$\lim_{t\to\infty} y_x(t) = \infty, \tag{2.12}$$

where y_x is defined in (2.3).

Proof. Since (1.9) is linear, we may assume that x is an eventually positive solution. Then, x is eventually nonincreasing. There exists $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t)$, $x(\tau(t)) > 0$ for all $t \in [t_1,\infty)_{\mathbb{T}}$. Thus, $y_x(t) \geq 1$ for all $t \in [t_1,\infty)_{\mathbb{T}}$. We rewrite (1.9) in the form

$$x^{\Delta}(t) + y_x(t)p(t)x(t) = 0 \tag{2.13}$$

for $t \in [t_1,\infty)_{\mathbb{T}}$. Integrating (2.13) from t to $\sigma(t)$, where $t \in [t_1,\infty)_{\mathbb{T}}$, we get

$$0 = x(\sigma(t)) - x(t) + \mu(t)y_x(t)p(t)x(t) > -x(t)\left[1 - \mu(t)y_x(t)p(t)\right], \tag{2.14}$$

which implies $-y_x p \in \mathcal{R}^+([t_1,\infty)_{\mathbb{T}})$. From (2.13), we see that

$$x(t) = x(t_1)\mathrm{e}_{-y_x p}(t,t_1) \quad \forall t \in [t_1,\infty)_{\mathbb{T}}, \tag{2.15}$$

and thus

$$y_x(t) = \frac{1}{\mathrm{e}_{-y_x p}(t,\tau(t))} \quad \forall t \in [t_2,\infty)_{\mathbb{T}}, \tag{2.16}$$

where $\tau(t_2) \in [t_1,\infty)_{\mathbb{T}}$. Note $\mathcal{R}^+([t_1,\infty)_{\mathbb{T}}) \subset \mathcal{R}^+([\tau(t),\infty)_{\mathbb{T}}) \subset \mathcal{R}^+([\tau(t),t)_{\mathbb{T}})$ for $t \in [t_2,\infty)_{\mathbb{T}}$. Now define

$$z_n(t) := \begin{cases} y_x(t), & n = 0, \\ \inf\left\{z_{n-1}(\eta) : \eta \in [\tau(t),t)_{\mathbb{T}}\right\}, & n \in \mathbb{N}. \end{cases} \tag{2.17}$$

By the definition (2.17), we have $y_x(\eta) \geq z_1(t)$ for all $\eta \in [\tau(t),t)_{\mathbb{T}}$ and all $t \in [t_2,\infty)_{\mathbb{T}}$, which yields $-z_1(t)p \in \mathcal{R}^+([\tau(t),t)_{\mathbb{T}})$ for all $t \in [t_2,\infty)_{\mathbb{T}}$. Then, we see that

$$y_x(t) \overset{(2.16)}{=} \frac{1}{\mathrm{e}_{-y_x p}(t,\tau(t))} \overset{(2.17)}{\geq} \frac{1}{\mathrm{e}_{-z_1(t)p}(t,\tau(t))} = \frac{z_1(t)}{z_1(t)\mathrm{e}_{-z_1(t)p}(t,\tau(t))} \overset{(2.10)}{\geq} \alpha_1(t)z_1(t) \tag{2.18}$$

holds for all $t \in [t_2, \infty)_{\mathbb{T}}$ (see also [13, Corollary 2.11]). Therefore, from (2.13), we have

$$x^{\Delta}(t) + z_1(t)p(t)\alpha_1(t)x(t) \le 0 \tag{2.19}$$

for $t \in [t_2, \infty)_{\mathbb{T}}$. Integrating (2.19) from t to $\sigma(t)$, where $t \in [t_2, \infty)_{\mathbb{T}}$, we get

$$0 \ge x(\sigma(t)) - x(t) + \mu(t)z_1(t)p(t)\alpha_1(t)x(t) > -x(t)\left[1 - \mu(t)z_1(t)p(t)\alpha_1(t)\right], \tag{2.20}$$

which implies that $-z_1 p\alpha_1 \in \mathcal{R}^+([t_2, \infty)_{\mathbb{T}})$. Thus, $-z_2(t)p\alpha_1 \in \mathcal{R}^+([\tau(t), t)_{\mathbb{T}})$ for all $t \in [t_3, \infty)_{\mathbb{T}}$, where $\tau(t_3) \in [t_2, \infty)_{\mathbb{T}}$, and we see that

$$y_x(t) \overset{(2.16),(2.17)}{\ge} \frac{1}{\mathrm{e}_{-z_1 p\alpha_1}(t, \tau(t))} \overset{(2.17)}{\ge} \frac{1}{\mathrm{e}_{-z_2(t)p\alpha_1}(t, \tau(t))} = \frac{z_2(t)}{z_2(t)\mathrm{e}_{-z_2(t)p\alpha_1}(t, \tau(t))} \overset{(2.10)}{\ge} \alpha_2(t)z_2(t) \tag{2.21}$$

for all $t \in [t_3, \infty)_{\mathbb{T}}$. By induction, there exists $t_{n_0+1} \in [t_{n_0}, \infty)_{\mathbb{T}}$ with $\tau(t_{n_0+1}) \in [t_{n_0}, \infty)_{\mathbb{T}}$ and

$$y_x(t) \ge z_{n_0}(t)\alpha_{n_0}(t) \tag{2.22}$$

for all $t \in \left[t_{n_0+1}, \infty\right)_{\mathbb{T}}$. To prove now (2.12), we assume on the contrary that $\liminf_{t\to\infty} y_x(t) < \infty$. Taking lim inf on both sides of (2.22), we get

$$\begin{aligned} \liminf_{t\to\infty} y_x(t) &\ge \liminf_{t\to\infty}\left[z_{n_0}(t)\alpha_{n_0}(t)\right] \\ &\ge \liminf_{t\to\infty} z_{n_0}(t) \liminf_{t\to\infty} \alpha_{n_0}(t) \\ &\overset{(2.17)}{=} \liminf_{t\to\infty} y_x(t) \liminf_{t\to\infty} \alpha_{n_0}(t), \end{aligned} \tag{2.23}$$

which implies that $\liminf_{t\to\infty} \alpha_{n_0}(t) \le 1$, contradicting (2.11). Therefore, (2.12) holds. □

Theorem 2.3. *Assume* (2.1). *If there exists* $n_0 \in \mathbb{N}$ *such that* (2.11) *holds, then every solution of* (1.9) *oscillates on* $[t_0, \infty)_{\mathbb{T}}$.

Proof. The proof is an immediate consequence of Lemmas 2.1 and 2.2. □

We need the following lemmas in the sequel.

Lemma 2.4 (see [7, Lemma 2]). *For nonnegative* p *with* $-p \in \mathcal{R}^+([s, t)_{\mathbb{T}})$, *one has*

$$1 - \int_s^t p(\eta)\Delta\eta \le \mathrm{e}_{-p}(t, s) \le \exp\left\{-\int_s^t p(\eta)\Delta\eta\right\}. \tag{2.24}$$

Now, we introduce

$$\beta_n(t) := \sup\left\{\alpha_{n-1}(\eta) : \eta \in [\tau(t), t)_{\mathbb{T}}\right\} \tag{2.25}$$

for $n \in \mathbb{N}$ and $t \in [s, \infty)_{\mathbb{T}}$, where $\tau^n(s) \in [t_0, \infty)_{\mathbb{T}}$.

Lemma 2.5. *If there exists $n_0 \in \mathbb{N}$ such that*

$$\limsup_{t\to\infty} \frac{1}{\beta_{n_0}(t)}\left(1 - \frac{1}{\alpha_{n_0}(t)}\right) > 0 \tag{2.26}$$

holds, then (2.1) *is true.*

Proof. There exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $-p\alpha_{n_0-1} \in \mathcal{R}^+([t_1, \infty)_{\mathbb{T}})$ (see the proof of Lemma 2.2). Then, Lemma 2.4 implies

$$\alpha_{n_0}(t) \overset{(2.10)}{\leq} \frac{1}{e_{-p\alpha_{n_0-1}}(t, \tau(t))} \leq \frac{1}{1 - \int_{\tau(t)}^{t} p(\eta)\alpha_{n_0-1}(\eta)\Delta\eta} \overset{(2.25)}{\leq} \frac{1}{1 - \beta_{n_0}(t)\int_{\tau(t)}^{t} p(\eta)\Delta\eta}, \tag{2.27}$$

which yields

$$\int_{\tau(t)}^{t} p(\eta)\Delta\eta \geq \frac{1}{\beta_{n_0}(t)}\left(1 - \frac{1}{\alpha_{n_0}(t)}\right) \quad \forall t \in [t_1, \infty)_{\mathbb{T}}. \tag{2.28}$$

In view of (2.26), taking lim sup on both sides of the above inequality, we see that (2.1) holds. Hence, the proof is done. □

Theorem 2.6. *Assume that there exists $n_0 \in \mathbb{N}$ such that* (2.26) *and* (2.11) *hold. Then, every solution of* (1.9) *is oscillatory on $[t_0, \infty)_{\mathbb{T}}$.*

Proof. The proof follows from Lemmas 2.1, 2.2, and 2.5. □

Remark 2.7. We obtain the main results of [7, 8] by letting $n_0 = 1$ in Theorem 2.6. In this case, we have $\beta_1(t) \equiv 1$ for all $t \in [t_0, \infty)_{\mathbb{T}}$. Note that (2.1) and (2.26), respectively, reduce tos

$$\liminf_{t\to\infty} \alpha_1(t) > 1, \qquad \limsup_{t\to\infty} \alpha_1(t) > 1, \tag{2.29}$$

which indicates that (2.26) is implied by (2.1).

3. Particular time scales

This section is dedicated to the calculation of α_n on some particular time scales. For convenience, we set

$$p_n(t) := \begin{cases} 1, & n = 0, \\ \displaystyle\int_{\tau(t)}^{t} p_{n-1}(\eta)p(\eta)\Delta\eta, & n \in \mathbb{N}. \end{cases} \tag{3.1}$$

Example 3.1. Clearly, if $\mathbb{T} = \mathbb{R}$ and $\tau(t) = t - \tau$, then (3.1) reduces to (1.6) and thus we have

$$\begin{aligned} \alpha_1(t) &= \inf_{\lambda>0}\left\{\frac{1}{\lambda\exp\{-\lambda p_1(t)\}}\right\} = \mathrm{e}p_1(t), \\ \alpha_2(t) &= \inf_{\lambda>0}\left\{\frac{1}{\lambda\exp\{-\mathrm{e}\lambda p_2(t)\}}\right\} = \mathrm{e}^2 p_2(t) \end{aligned} \tag{3.2}$$

by evaluating (2.10). For the general case, it is easy to see that

$$\alpha_n(t) = \mathrm{e}^n p_n(t) \tag{3.3}$$

for $n \in \mathbb{N}$. Thus if there exists $n_0 \in \mathbb{N}$ such that

$$\liminf_{t\to\infty} p_{n_0}(t) > \frac{1}{\mathrm{e}^{n_0}}, \tag{3.4}$$

then every solution of (1.1) is oscillatory on $[t_0, \infty)_{\mathbb{R}}$. Note that (3.4) implies $\limsup_{t\to\infty} p_1(t) \geq 1/\mathrm{e} > 0$. Otherwise, we have $\limsup_{t\to\infty} p_n(t) < 1/\mathrm{e}^n$ for $n = 2, 3, \ldots, n_0$. This result for the differential equation (1.1) is a special case of Theorem 2.3 given in Section 2, and it is presented in [3, Theorem 1], [4, Corollary 1], and [5, Corollary 1].

Example 3.2. Let $\mathbb{T} = \mathbb{Z}$ and $\tau(t) = t - \tau$, where $\tau \in \mathbb{N}$. Then (3.1) reduces to (1.8). From (2.10), we have

$$\begin{aligned} \alpha_1(t) &= \inf_{\substack{\lambda>0 \\ 1-\lambda p(\eta)>0 \\ \eta\in[t-\tau,t-1]_{\mathbb{Z}}}} \left\{\frac{1}{\lambda}\left(\prod_{\eta=t-\tau}^{t-1}[1-\lambda p(\eta)]\right)^{-1}\right\} \\ &\geq \inf_{\substack{\lambda>0 \\ 1-\lambda p(\eta)>0 \\ \eta\in[t-\tau,t-1]_{\mathbb{Z}}}} \left\{\frac{1}{\lambda}\left(\frac{1}{\tau}\sum_{\eta=t-\tau}^{t-1}[1-\lambda p(\eta)]\right)^{-\tau}\right\} \\ &\geq \inf_{\lambda>0}\left\{\frac{1}{\lambda}\left(1 - \frac{\lambda}{\tau}p_1(t)\right)^{-\tau}\right\} = \left(\frac{\tau+1}{\tau}\right)^{\tau+1} p_1(t). \end{aligned} \tag{3.5}$$

In the second line above, the well-known inequality between the arithmetic and the geometric mean is used. In the next step, we see that

$$\begin{aligned}
\alpha_2(t) &= \inf_{\substack{\lambda>0\\ 1-\lambda p(\eta)\alpha_1(\eta)>0\\ \eta\in[t-\tau,t-1]_{\mathbb{Z}}}} \left\{ \frac{1}{\lambda}\left(\prod_{\eta=t-\tau}^{t-1} [1-\lambda\alpha_1(\eta)p(\eta)] \right)^{-1} \right\} \\
&\geq \inf_{\substack{\lambda>0\\ 1-\lambda((\tau+1)/\tau)^{\tau+1}p_1(\eta)p(\eta)>0\\ \eta\in[t-\tau,t-1]_{\mathbb{Z}}}} \left\{ \frac{1}{\lambda}\left(\prod_{\eta=t-\tau}^{t-1} \left(1-\lambda\left(\frac{\tau+1}{\tau}\right)^{\tau+1} p_1(\eta)p(\eta)\right) \right)^{-1} \right\} \\
&\geq \inf_{\substack{\lambda>0\\ 1-\lambda((\tau+1)/\tau)^{\tau+1}p_1(\eta)p(\eta)>0\\ \eta\in[t-\tau,t-1]_{\mathbb{Z}}}} \left\{ \frac{1}{\lambda}\left(\frac{1}{\tau}\sum_{\eta=t-\tau}^{t-1} \left(1-\lambda\left(\frac{\tau+1}{\tau}\right)^{\tau+1} p_1(\eta)p(\eta)\right) \right)^{-\tau} \right\} \\
&\geq \inf_{\lambda>0} \left\{ \frac{1}{\lambda}\left(1-\frac{\lambda}{\tau}\left(\frac{\tau+1}{\tau}\right)^{\tau+1} p_2(t) \right)^{-\tau} \right\} = \left(\frac{\tau+1}{\tau}\right)^{2(\tau+1)} p_2(t).
\end{aligned} \tag{3.6}$$

By induction, we get

$$\alpha_n(t) \geq \left(\frac{\tau+1}{\tau}\right)^{n(\tau+1)} p_n(t) \tag{3.7}$$

for $n \in \mathbb{N}$. Therefore, every solution of (1.3) is oscillatory on $[t_0,\infty)_{\mathbb{Z}}$ provided that there exists $n_0 \in \mathbb{N}$ satisfying

$$\liminf_{t\to\infty} p_{n_0}(t) > \left(\frac{\tau}{\tau+1}\right)^{n_0(\tau+1)}. \tag{3.8}$$

Note that (3.8) implies that $\limsup_{t\to\infty} p_1(t) \geq (\tau/(\tau+1))^{\tau+1} > 0$. Otherwise, we would have $\limsup_{t\to\infty} p_n(t) < (\tau/(\tau+1))^{n(\tau+1)}$ for $n = 2, 3, \ldots, n_0$. This result for the difference equation (1.3) is a special case of Theorem 2.3 given in Section 2, and a similar result has been presented in [6, Corollary 1].

Example 3.3. Let $\mathbb{T} = q^{\mathbb{N}_0} := \{q^n : n \in \mathbb{N}_0\}$ and $\tau(t) = t/q^\tau$, where $q > 1$ and $\tau \in \mathbb{N}$. This time scale is different than the well-known time scales $\mathbb{R}$ and $\mathbb{Z}$ since $t + s \notin \mathbb{T}$ for $t, s \in \mathbb{T}$. In the present case, (3.1) reduces to

$$p_n(t) = \begin{cases} 1, & n = 0, \\ (q-1)\displaystyle\sum_{\eta=1}^{\tau} \frac{t}{q^\eta} p\left(\frac{t}{q^\eta}\right) p_{n-1}\left(\frac{t}{q^\eta}\right), & n \in \mathbb{N}, \end{cases} \tag{3.9}$$

and the exponential function takes the form

$$e_{-p}(t, q^{-\tau}t) = \prod_{\eta=1}^{\tau}\left[1 - (q-1)p\left(\frac{t}{q^{\eta}}\right)\frac{t}{q^{\eta}}\right]. \tag{3.10}$$

Therefore, one can show

$$\begin{aligned}\lambda e_{-\lambda p}(t, q^{-\tau}t) &= \lambda\prod_{\eta=1}^{\tau}\left[1 - \lambda(q-1)p\left(\frac{t}{q^{\eta}}\right)\frac{t}{q^{\eta}}\right] \\ &\le \lambda\left(1 - \frac{\lambda(q-1)}{\tau}\sum_{\eta=1}^{\tau}p\left(\frac{t}{q^{\eta}}\right)\frac{t}{q^{\eta}}\right)^{\tau} \le \left(\frac{\tau}{\tau+1}\right)^{\tau+1}\frac{1}{p_1(t)}\end{aligned} \tag{3.11}$$

and

$$\alpha_1(t) \ge \left(\frac{\tau+1}{\tau}\right)^{\tau+1} p_1(t). \tag{3.12}$$

For the general case, for $n \in \mathbb{N}$, it is easy to see that

$$\alpha_n(t) \ge \left(\frac{\tau+1}{\tau}\right)^{n(\tau+1)} p_n(t). \tag{3.13}$$

Therefore, if there exists $n_0 \in \mathbb{N}$ such that

$$\liminf_{t\to\infty} p_{n_0}(t) > \left(\frac{\tau}{\tau+1}\right)^{n_0(\tau+1)}, \tag{3.14}$$

then every solution of

$$x^{\Delta}(t) + p(t)x\left(\frac{t}{q^{\tau}}\right) = 0, \quad \text{where } x^{\Delta}(t) = \frac{x(qt) - x(t)}{(q-1)t}, \tag{3.15}$$

is oscillatory on $[t_0, \infty)_{q^{\mathbb{N}_0}}$. Clearly, (3.14) ensures $\limsup_{t\to\infty} p_1(t) \ge (\tau/(\tau+1))^{\tau+1} > 0$. This result for the q-difference equation (3.15) is a special case of Theorem 2.3 given in Section 2, and it has not been presented in the literature thus far.

Example 3.4. Let $\mathbb{T} = \{\xi_m : m \in \mathbb{N}\}$ and $\tau(\xi_m) = \xi_{m-\tau}$, where $\{\xi_m\}_{m\in\mathbb{N}}$ is an increasing divergent sequence and $\tau \in \mathbb{N}$. Then, the exponential function takes the form

$$\lambda e_{-\lambda p}(\xi_m, \xi_{m-\tau}) = \lambda \prod_{\eta=m-\tau}^{m-1}\left[1 - \lambda(\xi_{\eta+1} - \xi_{\eta})p(\xi_{\eta})\right]. \tag{3.16}$$

One can show that (2.10) satisfies

$$\alpha_n(\xi_m) \geq \left(\frac{\tau}{\tau+1}\right)^{n(\tau+1)} p_n(\xi_m), \tag{3.17}$$

where (3.1) has the form

$$p_n(\xi_m) = \begin{cases} 1, & n = 0, \\ \displaystyle\sum_{\eta=m-\tau}^{m-1} (\xi_{\eta+1} - \xi_\eta) p(\xi_\eta) p_{n-1}(\xi_\eta), & n \in \mathbb{N}. \end{cases} \tag{3.18}$$

Therefore, existence of $n_0 \in \mathbb{N}$ satisfying

$$\liminf_{m\to\infty} p_{n_0}(\xi_m) > \left(\frac{\tau}{\tau+1}\right)^{n_0(\tau+1)} \tag{3.19}$$

ensures by Theorem 2.3 that every solution of

$$x^\Delta(\xi_m) + p(\xi_m)x(\xi_{m-\tau}) = 0, \quad \text{where } x^\Delta(\xi_m) = \frac{x(\xi_{m+1}) - x(\xi_m)}{\xi_{m+1} - \xi_m}, \tag{3.20}$$

is oscillatory on $[\xi_\tau, \infty)_{\mathbb{T}}$. We note again that $\limsup_{m\to\infty} p_1(\xi_m) \geq (\tau/(\tau+1))^{\tau+1} > 0$ follows from (3.19).

References

[1] S. Hilger, *Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten*, Ph. D. thesis, Universität Würzburg, Würzburg, Germany, 1988.

[2] I. Győri and G. Ladas, *Oscillation Theory of Delay Differential Equations with Applications*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, NY, USA, 1991.

[3] B. Li, "Multiple integral average conditions for oscillation of delay differential equations," *Journal of Mathematical Analysis and Applications*, vol. 219, no. 1, pp. 165–178, 1998.

[4] J. Shen and X. Tang, "New oscillation criteria for linear delay differential equations," *Computers & Mathematics with Applications*, vol. 36, no. 6, pp. 53–61, 1998.

[5] X. Tang and J. Shen, "Oscillations of delay differential equations with variable coefficients," *Journal of Mathematical Analysis and Applications*, vol. 217, no. 1, pp. 32–42, 1998.

[6] X. H. Tang and J. S. Yu, "Oscillation of delay difference equation," *Computers & Mathematics with Applications*, vol. 37, no. 7, pp. 11–20, 1999.

[7] M. Bohner, "Some oscillation criteria for first order delay dynamic equations," *Far East Journal of Applied Mathematics*, vol. 18, no. 3, pp. 289–304, 2005.

[8] B. G. Zhang and X. Deng, "Oscillation of delay differential equations on time scales," *Mathematical and Computer Modelling*, vol. 36, no. 11-13, pp. 1307–1318, 2002.

[9] R. Agarwal and M. Bohner, "An oscillation criterion for first order dynamic equations," to appear in *Functional Differential Equations*.

[10] Y. Şahiner and I. P. Stavroulakis, "Oscillations of first order delay dynamic equations," *Dynamic Systems and Applications*, vol. 15, no. 3-4, pp. 645–655, 2006.

[11] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Application*, Birkhäuser, Boston, Mass, USA, 2001.

[12] M. Bohner and A. Peterso, Eds., *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, Mass, USA, 2003.

[13] E. Akin-Bohner, M. Bohner, and F. Akın, "Pachpatte inequalities on time scales," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 6, no. 1, article 6, pp. 1–23, 2005.