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Fractional q-symmetric calculus on a time scale

Mingzhe Sun and Chengmin Hou*

*Correspondence:
cmhou@foxmail.com
Department of Mathematics,
Yanbian University, Yanji, 133002,
P.R. China

Abstract

In this paper, the definitions of q-symmetric exponential function and q-symmetric gamma function are presented. By a q-symmetric exponential function, we shall illustrate the Laplace transform method and define and solve several families of linear fractional q-symmetric difference equations with constant coefficients. We also introduce a q-symmetric analogue Mittag-Leffler function and study q-symmetric Caputo fractional initial value problems. It is hoped that our work will provide foundation and motivation for further studying of fractional q-symmetric difference systems.

MSC: 92B20; 68T05; 39A11; 34K13

Keywords: fractional q-symmetric exponential function; fractional q-symmetric gamma function; Laplace transform; initial value problem

1 Introduction

The q-calculus is not of recent appearance. It was initiated in the twenties of the last century. However, it has gained considerable popularity and importance during the last three decades or so. This is due to its distinguished applications in numerous diverse fields of physics such as cosmic strings and black holes [1], conformal quantum mechanics [2], nuclear and high energy physics [3], just to name a few. Early developments for q-fractional calculus can be found in the work of Al-Salam and co-authors [4, 5] or Agarwal [6]. A q-Laplace transform method has been developed by Abdi [7] and applied to q-difference equations [8, 9]. Moreover, there is currently much activity to reexamine and further develop the q-special functions. Notable early work includes the work of Jackson [10–14], Hahn [15, 16] and Agarwal [17]. We also refer the reader to more recent articles [18–26] and [27, 28].

The q-symmetric quantum calculus has proven to be useful in several fields, in particular in quantum mechanics [29]. As noticed in [30], consistently with the q-deformed theory, the standard q-symmetric integral must be generalized to the basic integral defined. Recently, we introduced basic concepts of fractional q-symmetric integral and derivative operators in [31]. The basic theory of q-symmetric quantum calculus operators need be explored. The object of this paper is to further develop the theory of fractional q-symmetric calculus. First, the definitions of q-symmetric exponential function and q-symmetric gamma function are presented. Second, we give a fractional q-symmetric transform method. We then define some q-symmetric difference equations and apply the q-symmetric transform method to obtain solutions. Finally, we study a Caputo q-fractional initial value problem and give a q-analogue Mittag-Leffler function.

2 The q-symmetric gamma and the q-symmetric exponential functions

Let $t_0 \in \mathbb{R}$ and define

$$\mathbb{T}_{t_0} = \{t : t = t_0 q^n, n \in \mathbb{N}_0\} \cup \{t : t = t_0 q^{-n}, n \in \mathbb{N}_0\} \cup \{0\}, \quad 0 < q < 1.$$

If there is no confusion concerning t_0, we shall denote $\mathbb{T}_{t_0}$ by $\mathbb{T}$.

The basic q-symmetric integrals of $f : \mathbb{T} \longrightarrow \mathbb{R}$ are defined through the relations

$$(\widetilde{I}_{q,0} f)(t) = \int_0^x f(t)\,\widetilde{d}_q t = x(1-q^2) \sum_{k=0}^{\infty} q^{2k} f(xq^{2k+1}). \tag{2.1}$$

For $a \in \mathbb{T}$,

$$(\widetilde{I}_{q,a} f)(t) = \int_a^x f(t)\,\widetilde{d}_q t = \int_0^x f(t)\,\widetilde{d}_q t - \int_0^a f(t)\,\widetilde{d}_q t.$$

For a function $f : \mathbb{T} \longrightarrow \mathbb{R}$, the q-symmetric derivative is defined by

$$(\widetilde{D}_q f)(x) = \frac{f(qx) - f(q^{-1}x)}{x(q-q^{-1})}, \qquad (\widetilde{D}_q f)(0) := f'(0), \tag{2.2}$$

and the q-symmetric derivatives of higher order are defined by

$$(\widetilde{D}_q^0 f)(x) = f(x), \qquad (\widetilde{D}_q^n f)(x) = (\widetilde{D}_q \widetilde{D}_q^{n-1} f)(x), \quad n \in \mathbb{N}^+.$$

As for q-symmetric derivatives, we can define an operator $\widetilde{I}_q^n$ by

$$(\widetilde{I}_{q,a}^0 f)(x) = f(x), \qquad (\widetilde{I}_{q,a}^n f)(x) = (\widetilde{I}_{q,a} \widetilde{I}_{q,a}^{n-1} f)(x), \quad n \in \mathbb{N}^+. \tag{2.3}$$

For operators defined in this manner, the following is valid:

$$(\widetilde{D}_q \widetilde{I}_{q,0}) f(x) = f(x), \qquad (\widetilde{I}_{q,0} \widetilde{D}_q f)(x) = f(x) - f(0). \tag{2.4}$$

The formula for q-symmetric integration by parts is

$$\int_a^b u(qx)(\widetilde{D}_q v)(x)\,\widetilde{d}_q x = [u(x)v(x)]\big|_a^b - \int_a^b v(q^{-1}x)(\widetilde{D}_q u)(x)\,\widetilde{d}_q x. \tag{2.5}$$

Definition 2.1 The q-symmetric factorial is defined in the following way. If n is a non-negative integer, then

$$\overline{(a-b)}^{(0)} = 1, \qquad \overline{(a-b)}^{(k)} = \prod_{i=0}^{k-1} (a - bq^{2i+1}) \quad (k \in \mathbb{N}, a, b \in \mathbb{R}).$$

Their natural expansions to reals are

$$\overline{(a-b)}^{(\alpha)} = a^{\alpha}\frac{\prod_{i=0}^{\infty}(1-\frac{b}{a}q^{2i+1})}{\prod_{i=0}^{\infty}(1-\frac{b}{a}q^{2(i+\alpha)+1})} \quad (\alpha \in \mathbb{R}, a \neq 0). \tag{2.6}$$

Next, for a real parameter $q \in \mathbb{R}^{+} \setminus \{1\}$, we introduce a q-real number $\overline{[a]}_q$ by

$$\overline{[a]}_q = \frac{1-q^{2a}}{1-q^2} \quad (a \in \mathbb{R}).$$

We shall state several properties of the q-symmetric factorial function, each property is verified using the definition and a straightforward calculation.

Theorem 2.1 *For a q-symmetric factorial function, we have*

(i) $\overline{(t-s)}^{(\beta+\nu)} = \overline{(t-s)}^{(\beta)}\overline{(t-q^{2\beta}s)}^{(\nu)}$,
(ii) $\overline{(at-as)}^{(\beta)} = a^{\beta}\overline{(t-s)}^{(\beta)}$,
(iii) ${}_t\widetilde{D}_q\overline{(t-s)}^{(\beta)} = \overline{[\beta]}_q\overline{(q^{-1}t-s)}^{(\beta-1)}$,
(iv) ${}_s\widetilde{D}_q\overline{(t-q^{-1}s)}^{(\beta)} = -\overline{[\beta]}_q\overline{(t-s)}^{(\beta-1)}$.

Definition 2.2 The q-symmetric exponential function is defined as

$$\overline{e}_q(t) = \prod_{n=0}^{\infty}(1-q^{2n+1}t), \qquad \overline{e}_q(0) = 1.$$

Note that $\overline{e}_q(q^{-1}) = 0$ and

$$\widetilde{D}_q\overline{e}_q(q^{-1}t) = -\frac{1}{1-q^2}\overline{e}_q(t). \tag{2.7}$$

We are now in a position to give the integral representation of the q-symmetric gamma function. Let $\alpha \in \mathbb{R} \setminus \{\ldots,-2,-1,0\}$. Define the q-symmetric gamma function by

$$\widetilde{\Gamma}_q(\alpha) = \frac{1}{1-q^2}\int_0^1\left(\frac{q^{-1}t}{1-q^2}\right)^{\alpha-1}\overline{e}_q(t)\,\widetilde{d}_qt. \tag{2.8}$$

For an integer n, we denote

$$\overline{[0]}_q! = 1, \qquad \overline{[n]}_q! = \overline{[n]}_q\overline{[n-1]}_q\cdots\overline{[1]}_q = \frac{(1-q^{2n})}{(1-q^2)}\frac{(1-q^{2n-2})}{(1-q^2)}\cdots\frac{(1-q^2)}{(1-q^2)}.$$

Lemma 2.1 *For* $\alpha \in \mathbb{R}$,

$$\widetilde{\Gamma}_q(\alpha+1) = \overline{[\alpha]}_q\widetilde{\Gamma}_q(\alpha), \qquad \widetilde{\Gamma}_q(1) = 1.$$

For any positive integer k,

$$\widetilde{\Gamma}_q(k+1) = \overline{[k]}_q\widetilde{\Gamma}_q(k) = \overline{[k]}_q!.$$

Proof By (2.5), (2.7), (2.8), we can get

$$\begin{aligned}
\widetilde{\Gamma}_q(\alpha+1) &= \frac{1}{1-q^2}\int_0^1 \left(\frac{q^{-1}t}{1-q^2}\right)^{\alpha} \overline{e}_q(t)\,\widetilde{d}_q t \\
&= \frac{1}{1-q^2}\int_0^1 \left(\frac{q^{-1}t}{1-q^2}\right)^{\alpha} \left[-\widetilde{D}_q(1-q^2)\overline{e}_q\left(q^{-1}t\right)\right]\widetilde{d}_q t \\
&= -\frac{1}{1-q^2}\overline{e}_q\left(q^{-1}t\right)\frac{t^{\alpha}}{(1-q^2)^{\alpha-1}}\bigg|_0^1 \\
&\quad + \frac{1}{1-q^2}\int_0^1 \overline{[\alpha]}_q \overline{e}_q(t) q^{1-\alpha}\frac{t^{\alpha-1}}{(1-q^2)^{\alpha-1}}\,\widetilde{d}_q t \\
&= \frac{\overline{[\alpha]}_q}{1-q^2}\int_0^1 \left(\frac{q^{-1}t}{(1-q^2)}\right)^{\alpha-1} \overline{e}_q(t)\,\widetilde{d}_q t \\
&= \overline{[\alpha]}_q \widetilde{\Gamma}_q(\alpha), \\
\widetilde{\Gamma}_q(1) &= \frac{1}{1-q^2}\int_0^1 \overline{e}_q(t)\,\widetilde{d}_q t \\
&= \frac{1}{1-q^2}\int_0^1 \left[-\widetilde{D}_q(1-q^2)\overline{e}_q\left(q^{-1}t\right)\right]\widetilde{d}_q t \\
&= -\overline{e}_q\left(q^{-1}t\right)\Big|_0^1 \\
&= 1.
\end{aligned}$$

□

Remark 2.1 In [31], authors introduced the q-symmetric gamma function as

$$\begin{aligned}
\widetilde{\Gamma^*}_q(\alpha) &= \frac{\prod_{i=0}^{\infty}(1-q^{2i+2})}{\prod_{i=0}^{\infty}(1-q^{2(i+\alpha-1)+2})}\left(1-q^2\right)^{1-\alpha} \\
&= \overline{(1-q)}^{(\alpha-1)}\left(1-q^2\right)^{1-\alpha} \quad \left(x \in \mathbb{R}\backslash\{0,-1,-2,\ldots\}\right).
\end{aligned}$$

To see that $\widetilde{\Gamma}_q(\alpha) = \widetilde{\Gamma^*}_q(\alpha)$, we use the following formula given by Atici and Eloe in [9]:

$$\int_0^1 s^{\alpha-1}e_q(qs)\,d_q s = (1-q)\prod_{i=0}^{\infty}\frac{1-q^{i+1}}{1-q^{i+\alpha}},$$

where $e_q(t) = \prod_{n=0}^{\infty}(1-q^n t)$.

In fact

$$\begin{aligned}
\int_0^1 \left(q^{-1}s\right)^{\alpha-1}\overline{e}_q(s)\,\widetilde{d}_q s &= \left(1-q^2\right)\sum_{k=0}^{\infty} q^{2k}\left(q^{-1}q^{2k+1}\right)^{\alpha-1}\overline{e}_q\left(q^{2k+1}\right) \\
&= \left(1-q^2\right)\sum_{k=0}^{\infty} q^{2k\alpha}\prod_{n=0}^{\infty}\left(1-q^{2(k+n+1)}\right) \\
&= \left(1-q^2\right)\sum_{k=0}^{\infty} \left(q^2\right)^{k\alpha}\prod_{n=0}^{\infty}\left(1-\left(q^2\right)^{k+n+1}\right) \\
&= \int_0^1 s^{\alpha-1}e_{q^2}\left(q^2 s\right)d_{q^2}s = \left(1-q^2\right)\prod_{i=0}^{\infty}\frac{(1-q^{2(i+1)})}{(1-q^{2(i+\alpha)})},
\end{aligned}$$

hence

$$\begin{aligned}\widetilde{\Gamma}_q(\alpha) &= \frac{1}{(1-q^2)^{\alpha-1}}\prod_{i=0}^{\infty}\frac{(1-q^{2(i+1)})}{(1-q^{2(i+\alpha)})}\\ &= \bigl(1-q^2\bigr)^{1-\alpha}\overline{(1-q)}^{(\alpha-1)}.\end{aligned}$$

3 Fractional *q*-symmetric integral and derivative

We now introduce the fractional q-symmetric integral operator (see [31])

$$\bigl(\widetilde{I}_{q,0}^{\alpha}f\bigr)(x) = \frac{1}{\widetilde{\Gamma}_q(\alpha)}q^{\binom{\alpha}{2}}\int_0^x \overline{(x-\tau)}^{(\alpha-1)}f\bigl(q^{\alpha-1}\tau\bigr)\,\widetilde{d}_q\tau \quad \bigl(\alpha\in\mathbb{R}^+\bigr), \tag{3.1}$$

where

$$\binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)} \quad (k\in\mathbb{N}).$$

Lemma 3.1 ([31]) *Let $\alpha,\beta\in\mathbb{R}^+$. The fractional q-symmetric integration has the following semigroup property*:

$$\bigl(\widetilde{I}_{q,0}^{\alpha}\widetilde{I}_{q,0}^{\beta}f\bigr)(x) = \bigl(\widetilde{I}_{q,0}^{\alpha+\beta}f\bigr)(x).$$

Lemma 3.2 ([31]) *For $\alpha\in\mathbb{R}^+$, the following identity is valid*:

$$\bigl(\widetilde{I}_{q,0}^{\alpha}f\bigr)(x) = \bigl(\widetilde{I}_{q,0}^{\alpha+1}\widetilde{D}_qf\bigr)(x) + \frac{f(0)}{\widetilde{\Gamma}_q(\alpha+1)}q^{\binom{\alpha}{2}}x^{\alpha}.$$

We define the fractional q-symmetric derivative of Riemann-Liouville type of a function $f(x)$ by

$$\bigl(\widetilde{D}_{q,0}^{\alpha}f\bigr)(x) = \begin{cases}(\widetilde{I}_{q,0}^{-\alpha}f)(x), & \alpha<0,\\ f(x), & \alpha=0,\\ (D_q^{[\alpha]}\widetilde{I}_{q,0}^{[\alpha]-\alpha}f)(x), & \alpha>0,\end{cases} \tag{3.2}$$

where $[\alpha]$ denotes the smallest integer greater or equal to α.

Lemma 3.3 ([31]) *For $\alpha\in\mathbb{R}\setminus\mathbb{N}_0$, the following is valid*:

$$\bigl(\widetilde{D}_q\widetilde{D}_{q,0}^{\alpha}f\bigr)(x) = \bigl(\widetilde{D}_{q,0}^{\alpha+1}f\bigr)(x).$$

Lemma 3.4 ([31]) *For $\alpha\in\mathbb{R}\setminus\mathbb{N}_0$, the following is valid*:

$$\bigl(\widetilde{D}_q\widetilde{D}_{q,0}^{\alpha}f\bigr)(x) = \bigl(\widetilde{D}_{q,0}^{\alpha}\widetilde{D}_qf\bigr)(x) + \frac{f(0)}{\widetilde{\Gamma}_q(-\alpha)}q^{\binom{-(\alpha+1)}{2}}x^{-\alpha-1}.$$

Lemma 3.5 ([31]) *For $\alpha\in\mathbb{R}\setminus\mathbb{N}_0$, the following is valid*:

$$\bigl(\widetilde{D}_{q,0}^{\alpha}\widetilde{I}_{q,0}^{\alpha}f\bigr)(x) = f(x).$$

Lemma 3.6 ([31]) *Let $\alpha \in (N-1,N]$. Then, for some constants $c_i \in \mathbb{R}$, $i = 1,2,\ldots,N$, the following equality holds*:

$$\bigl(\widetilde{I}_{q,0}^{\alpha}\widetilde{D}_{q,0}^{\alpha}f\bigr)(x) = f(x) + c_1x^{\alpha-1} + c_2x^{\alpha-2} + \cdots + c_Nx^{\alpha-N}.$$

If we change the order of operators, we can introduce another type of fractional q-derivative.

The fractional q-symmetric derivative of Caputo type is

$$\bigl({}^c\widetilde{D}_{q,0}^{\alpha}f\bigr)(x) = \begin{cases} (\widetilde{I}_{q,0}^{-\alpha}f)(x), & \alpha < 0, \\ f(x), & \alpha = 0, \\ (\widetilde{I}_{q,0}^{[\alpha]-\alpha}D_q^{[\alpha]}f)(x), & \alpha > 0, \end{cases} \tag{3.3}$$

where $[\alpha]$ denotes the smallest integer greater or equal to α.

Lemma 3.7 ([31]) *For $\alpha \in \mathbb{R} \setminus \mathbb{N}_0$, and $x > 0$, the following is valid*:

$$\begin{aligned} &\bigl({}^c\widetilde{D}_{q,0}^{\alpha+1}f\bigr)(x) - \bigl({}^c\widetilde{D}_{q,0}^{\alpha}\widetilde{D}_qf\bigr)(x) \\ &\quad = \begin{cases} \frac{f(0)}{\Gamma_q(-\alpha)}q^{\binom{-(\alpha+1)}{2}}x^{-\alpha-1}, & \alpha < -1, \\ 0, & \alpha \geq -1. \end{cases} \end{aligned} \tag{3.4}$$

Lemma 3.8 ([31]) *For $\alpha \in \mathbb{R} \setminus \mathbb{N}_0$ and $x > 0$, the following is valid*:

$$\begin{aligned} &\bigl(\widetilde{D}_q^c\widetilde{D}_{q,0}^{\alpha}f\bigr)(x) - \bigl({}^c\widetilde{D}_{q,0}^{\alpha+1}f\bigr)(x) \\ &\quad = \begin{cases} 0, & \alpha \leq -1, \\ \frac{f(0)}{\Gamma_q(-\alpha)}q^{\binom{-(\alpha+1)}{2}}x^{-\alpha-1}, & \alpha > -1. \end{cases} \end{aligned} \tag{3.5}$$

Lemma 3.9 ([31]) *Let $\alpha \in (N-1,N]$. Then, for some constants $c_i \in \mathbb{R}$, $i = 0,1,\ldots,N-1$, the following equality holds*:

$$\bigl(\widetilde{I}_{q,0}^{\alpha}\,{}^c\widetilde{D}_{q,0}^{\alpha}f\bigr)(x) = f(x) + c_0 + c_1x + c_2x^2 + \cdots + c_{N-1}x^{N-1}.$$

Lemma 3.10 *Let $f : \mathbb{T} \times \mathbb{T} \longrightarrow \mathbb{R}$, then the following identity holds*:

$$\widetilde{D}_q\int_0^t f(t,s)\,\widetilde{d}_qs = \int_0^{q^{-1}t} {}_t\widetilde{D}_qf(t,s)\,\widetilde{d}_qs + f(qt,t).$$

Proof

$$\begin{aligned} {}_t\widetilde{D}_q\int_0^t f(t,s)\,\widetilde{d}_qs &= {}_t\widetilde{D}_q\left(t\bigl(1-q^2\bigr)\sum_{k=0}^{\infty}q^{2k}f\bigl(t,q^{2k+1}t\bigr)\right) \\ &= \frac{1}{(q-q^{-1})t}\left(qt\bigl(1-q^2\bigr)\sum_{k=0}^{\infty}q^{2k}f\bigl(qt,q^{2k+2}t\bigr)\right. \end{aligned}$$

$$
\begin{aligned}
&\quad - q^{-1}t(1-q^2)\sum_{k=0}^{\infty} q^{2k} f(q^{-1}t, q^{2k}t)\Bigg) \\
&= -\sum_{k=0}^{\infty} q^{2k+2} f(qt, q^{2k+2}t) + \sum_{k=0}^{\infty} q^{2k} f(q^{-1}t, q^{2k}t) \\
&= -\sum_{k=0}^{\infty} q^{2k} f(qt, q^{2k}t) + \sum_{k=0}^{\infty} q^{2k} f(q^{-1}t, q^{2k}t) + f(qt,t) \\
&= q^{-1}t(1-q^2)\sum_{k=0}^{\infty} q^{2k} \frac{[f(qt,q^{2k}t) - f(q^{-1}t, q^{2k}t)]}{(q-q^{-1})t} + f(qt,t) \\
&= \int_0^{q^{-1}t} {}_t\widetilde{D}_q f(t,s)\,\widetilde{d}_q s + f(qt,t). \qquad \square
\end{aligned}
$$

Theorem 3.1 *If $f(t)$ is defined and finite, then for $\nu > 0$ with $N-1 < \nu \le N$,*

$$
(\widetilde{D}_q^{\nu} f)(t) = \frac{q^{\binom{-\nu}{2}}}{\widetilde{\Gamma}_q(-\nu)} \int_0^t \overline{(t-s)}^{(-\nu-1)} f(q^{-\nu}s)\,\widetilde{d}_q s.
$$

Proof Using Lemma 3.10 and Theorem 2.1, we have

$$
\begin{aligned}
(\widetilde{D}_q^{\nu} f)(t) &= \widetilde{D}_q^{N} \widetilde{I}_{q,0}^{N-\nu} f(t) \\
&= \widetilde{D}^{N}\left(\frac{q^{\binom{N-\nu}{2}}}{\widetilde{\Gamma}_q(N-\nu)} \int_0^t \overline{(t-s)}^{(N-\nu-1)} f(q^{N-\nu-1}s)\,\widetilde{d}_q s\right) \\
&= \widetilde{D}_q^{N-1}\widetilde{D}_q\left(\frac{q^{\binom{N-\nu}{2}}}{\widetilde{\Gamma}_q(N-\nu)} \int_0^t \overline{(t-s)}^{(N-\nu-1)} f(q^{N-\nu-1}s)\,\widetilde{d}_q s\right) \\
&= \widetilde{D}_q^{N-1}\left(\frac{q^{\binom{N-\nu}{2}}}{\widetilde{\Gamma}_q(N-\nu)} \int_0^{q^{-1}t} \widetilde{D}_q\overline{(t-s)}^{(N-\nu-1)} f(q^{N-\nu-1}s)\,\widetilde{d}_q s \right.\\
&\quad \left. + \overline{(qt-t)}^{(N-\nu-1)} f(q^{N-\nu-1}t)\right) \\
&= \widetilde{D}_q^{N-1}\left(\frac{q^{\binom{N-\nu}{2}}\overline{[N-\nu-1]}_q}{\widetilde{\Gamma}_q(N-\nu)} \int_0^{q^{-1}t} \overline{(q^{-1}t-s)}^{(N-\nu-2)} f(q^{N-\nu-1}s)\,\widetilde{d}_q s\right) \\
&= \widetilde{D}_q^{N-1}\left(\frac{q^{\binom{N-\nu}{2}}q^{-N+\nu+1}}{\widetilde{\Gamma}_q(N-\nu-1)} \int_0^{t} \overline{(t-s)}^{(N-\nu-2)} f(q^{N-\nu-2}s)\,\widetilde{d}_q s\right) \\
&= \widetilde{D}_q^{N-1}\left(\frac{q^{\binom{N-\nu-1}{2}}}{\widetilde{\Gamma}_q(N-\nu-1)} \int_0^{t} \overline{(t-s)}^{(N-\nu-2)} f(q^{N-\nu-2}s)\,\widetilde{d}_q s\right) \\
&= \widetilde{D}_q^{N-2}\left(\frac{q^{\binom{N-\nu-2}{2}}}{\widetilde{\Gamma}_q(N-\nu-2)} \int_0^{t} \overline{(t-s)}^{(N-\nu-3)} f(q^{N-\nu-3}s)\,\widetilde{d}_q s\right) \\
&= \cdots \\
&= \frac{q^{\binom{-\nu}{2}}}{\widetilde{\Gamma}_q(-\nu)} \int_0^t \overline{(t-s)}^{(-\nu-1)} f(q^{-\nu-1}s)\,\widetilde{d}_q s. \qquad \square
\end{aligned}
$$

4 The q-symmetric Laplace transform

For convenience, we need some preliminaries.

Let $q^2 = \overline{q}$, then we have

$$\begin{aligned}\widetilde{\Gamma}_q(x) &= \frac{\prod_{i=0}^{\infty}(1-q^{2i+2})}{\prod_{i=0}^{\infty}(1-q^{2(i+x-1)+2})}\left(1-q^2\right)^{1-x} \\ &= (1-\overline{q})^{(x-1)}(1-\overline{q})^{1-x} = \Gamma_{\overline{q}}(x) \quad \bigl(x \in \mathbb{R}\backslash\{0,-1,-2,\ldots\}\bigr). \end{aligned} \tag{4.1}$$

The basic q-integrals are defined by

$$\begin{aligned}(I_{q,0}f)(t) &= \int_0^x f(t)\,d_qt = x(1-q)\sum_{k=0}^{\infty} q^k f\bigl(xq^k\bigr), \\ (I_{q,a}f)(t) &= \int_a^x f(t)\,d_qt = \int_0^x f(t)\,d_qt - \int_0^a f(t)\,d_qt. \end{aligned} \tag{4.2}$$

Definition 4.1 ([9] q-beta function) For any $x, y > 0$, $B_q(x,y) = \int_0^1 t^{x-1}(1-qt)^{(y-1)}\,d_qt$.

Recall that

$$B_q(x,y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}.$$

Therefore,

$$B_{\overline{q}}(x,y) = \frac{\Gamma_{\overline{q}}(x)\Gamma_{\overline{q}}(y)}{\Gamma_{\overline{q}}(x+y)} = \frac{\widetilde{\Gamma}_q(x)\widetilde{\Gamma}_q(y)}{\widetilde{\Gamma}_q(x+y)}. \tag{4.3}$$

Similar to q-beta function, we shall define the q-symmetric beta function.

Definition 4.2 (q-symmetric beta function) For any $x, y > 0$, $\widetilde{B}_q(x,y) = \int_0^1 (q^{-1}t)^{x-1} \times \overline{(1-t)}^{(y-1)}\,\widetilde{d}_qt$.

Lemma 4.1 *For any $x, y > 0$, the following equality is valid*:

$$\widetilde{B}_q(x,y) = \frac{\widetilde{\Gamma}_q(x)\widetilde{\Gamma}_q(y)}{\widetilde{\Gamma}_q(x+y)}.$$

Proof By (2.1), (4.2), (4.3) and Definition 4.2, we have

$$\begin{aligned}\widetilde{B}_q(x,y) &= \int_0^1 \bigl(q^{-1}t\bigr)^{x-1}\overline{(1-t)}^{(y-1)}\,\widetilde{d}_qt \\ &= \bigl(1-q^2\bigr)\sum_{k=0}^{\infty} q^{2k}\bigl(q^{2k}\bigr)^{x-1}\overline{\bigl(1-q^{2k+1}\bigr)}^{(y-1)} \\ &= \bigl(1-q^2\bigr)\sum_{k=0}^{\infty} q^{2k}\bigl(q^{2k}\bigr)^{x-1}\prod_{n=0}^{\infty}\frac{1-q^{2k+2n+2}}{1-q^{2k+2n+2y}} \\ &= \int_0^1 t^{x-1}\bigl(1-q^2t\bigr)^{(y-1)}\,d_{q^2}t = B_{q^2}(x,y) = \frac{\widetilde{\Gamma}_q(x)\widetilde{\Gamma}_q(y)}{\widetilde{\Gamma}_q(x+y)}. \end{aligned}$$

□

Lemma 4.2 ([31]) *For* $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}_0$, $\lambda \in (-1, \infty)$, *the following is valid*:

(i) $\widetilde{I}_{q,0}^{\alpha} x^{\lambda} = \frac{\widetilde{\Gamma}_q(\lambda+1)}{\widetilde{\Gamma}_q(\lambda+\alpha+1)} q^{\binom{\alpha}{2}+\lambda\alpha} x^{\lambda+\alpha}$,

(ii) $\widetilde{D}_{q,0}^{\alpha} x^{\lambda} = \frac{\widetilde{\Gamma}_q(\lambda+1)}{\widetilde{\Gamma}_q(\lambda-\alpha+1)} q^{\binom{\alpha}{2}-\lambda\alpha} x^{\lambda-\alpha}$.

We shall define a q-symmetric Laplace transform as follows:

$$\widetilde{L}_q\{f(t)\}(s) = \frac{1}{1-q^2} \int_0^{\frac{1}{s}} f(q^{-1}t) \overline{e}_q(st) \widetilde{d}_q t.$$

Lemma 4.3 *For any* $\alpha \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\}$,

$$\widetilde{L}_q\left\{\frac{t^{\alpha-1}}{(1-q^2)^{\alpha-1}}\right\}(s) = \frac{\widetilde{\Gamma}_q(\alpha)}{s^{\alpha}}.$$

Proof

$$\begin{aligned}
\widetilde{L}_q\left\{\frac{t^{\alpha-1}}{(1-q^2)^{\alpha-1}}\right\}(s) &= \frac{1}{1-q^2} \int_0^{\frac{1}{s}} \left(\frac{q^{-1}t}{1-q^2}\right)^{\alpha-1} \overline{e}_q(st) \widetilde{d}_q t \\
&= \frac{1}{(1-q^2)s^{\alpha}} \int_0^{1} \left(\frac{q^{-1}t}{1-q^2}\right)^{\alpha-1} \overline{e}_q(t) \widetilde{d}_q t \\
&= \frac{1}{s^{\alpha}} \widetilde{\Gamma}_q(\alpha).
\end{aligned}$$

□

We now turn our attention to a shift theorem for the q-symmetric Laplace transform. First note the following identity.

$$\begin{aligned}
\widetilde{D}_q \overline{e}_q^{-1}(t) &= \widetilde{D}_q \frac{1}{\prod_{n=0}^{\infty}(1-q^{2n+1}t)} \\
&= \frac{1}{(q-q^{-1})t}\left(\frac{1}{\prod_{n=0}^{\infty}(1-q^{2n+2}t)} - \frac{1}{\prod_{n=0}^{\infty}(1-q^{2n}t)}\right) \\
&= \frac{1}{(q-q^{-1})t} \frac{1}{\prod_{n=0}^{\infty}(1-q^{2n+2}t)}\left(1 - \frac{1}{1-t}\right) \\
&= \frac{q}{(1-q^2)} \overline{e}_q^{-1}(q^{-1}t).
\end{aligned}$$

Lemma 4.4 *Let a be any real number. Then*

$$\widetilde{L}_q\{\overline{e}_q^{-1}(at)\}(s) = \frac{1}{s-aq}.$$

Let n denote a positive integer. Then

$$\begin{aligned}
\widetilde{L}_q\{t^n \overline{e}_q^{-1}(at)\}(s) &= \frac{1}{s-aq} \prod_{j=1}^{n} \frac{1-q^{2j}}{s-q^{2j+1}a} \\
&= \frac{(1-q^2)^n \widetilde{[n]}_q!}{\widetilde{(s-a)}^{(n+1)}}.
\end{aligned}$$

Proof Note that

$$\begin{aligned}\widetilde{D}_q\left(\overline{e}_q^{-1}(at)\overline{e}_q\left(sq^{-1}t\right)\right) &= \overline{e}_q(st)\widetilde{D}_q\overline{e}_q^{-1}(at) + \overline{e}_q^{-1}\left(aq^{-1}t\right)\widetilde{D}_q\overline{e}_q\left(sq^{-1}t\right)\\ &= \overline{e}_q(st)\frac{aq}{1-q^2}\overline{e}_q^{-1}\left(aq^{-1}t\right) - \overline{e}_q(st)\frac{s}{1-q^2}\overline{e}_q^{-1}\left(aq^{-1}t\right)\\ &= -\frac{s-aq}{1-q^2}\overline{e}_q(st)\overline{e}_q^{-1}\left(aq^{-1}t\right),\end{aligned}$$

and

$$\begin{aligned}\int_0^{\frac{1}{s}}\widetilde{D}_q\left(\overline{e}_q^{-1}(at)\overline{e}_q\left(sq^{-1}t\right)\right)\widetilde{d}_qt &= -\frac{s-aq}{1-q^2}\int_0^{\frac{1}{s}}\overline{e}_q^{-1}\left(aq^{-1}t\right)\overline{e}_q(st)\,\widetilde{d}_qt\\ &= \overline{e}_q^{-1}(at)\overline{e}_q\left(sq^{-1}t\right)\Big|_0^{\frac{1}{s}} = -1,\end{aligned}$$

so for $n = 0$,

$$\widetilde{L}_q\left\{\overline{e}_q^{-1}(at)\right\}(s) = \frac{1}{s-aq}.$$

The proof proceeds by induction. Let $n \geq 1$ be an integer. Note that

$$\begin{aligned}\widetilde{D}_q\left(t^n\overline{e}_q^{-1}(at)\overline{e}_q\left(sq^{-1}t\right)\right) &= \overline{e}_q(st)\widetilde{D}_q\left(t^n\overline{e}_q^{-1}(at)\right) + \left(\left(q^{-1}t\right)^n\overline{e}_q^{-1}\left(aq^{-1}t\right)\right)\widetilde{D}_q\overline{e}q\left(sq^{-1}t\right)\\ &= \overline{e}_q(st)\left[(qt)^n\widetilde{D}_q\left(\overline{e}_q^{-1}(at)\right) + \overline{e}_q^{-1}\left(aq^{-1}t\right)\widetilde{D}_q\left(t^n\right)\right]\\ &\quad - \frac{s}{1-q^2}\overline{e}_q(st)\left(q^{-1}t\right)^n\overline{e}_q^{-1}\left(aq^{-1}t\right)\\ &= \overline{e}_q(st)\left[\frac{qa}{1-q^2}(qt)^n\overline{e}_q^{-1}\left(aq^{-1}t\right) + \overline{[n]}_q\left(q^{-1}t\right)^{n-1}\overline{e}_q^{-1}\left(aq^{-1}t\right)\right]\\ &\quad - \frac{s}{1-q^2}\overline{e}_q(st)\left(q^{-1}t\right)^n\overline{e}_q^{-1}\left(aq^{-1}t\right)\\ &= \overline{e}_q(st)\overline{e}_q^{-1}\left(aq^{-1}t\right)\left[\frac{aq^{n+1}t^n - q^{-n}t^ns}{1-q^2} + \frac{(1-q^{2n})(q^{-1}t)^{n-1}}{1-q^2}\right],\end{aligned}$$

we have

$$\begin{aligned}&\frac{1}{1-q^2}\int_0^{\frac{1}{s}}\left(aq^{2n+1}-s\right)\left(q^{-1}t\right)^n\overline{e}_q^{-1}\left(aq^{-1}t\right)\overline{e}_q(st)\,\widetilde{d}_qt\\ &\quad + \frac{1-q^{2n}}{1-q^2}\int_0^{\frac{1}{s}}\overline{e}_q^{-1}\left(aq^{-1}t\right)\left(q^{-1}t\right)^{n-1}\overline{e}_q(st)\,\widetilde{d}_qt = 0,\end{aligned}$$

i.e.,

$$\begin{aligned}&\frac{s-aq^{2n+1}}{1-q^2}\int_0^{\frac{1}{s}}\left(q^{-1}t\right)^n\overline{e}_q^{-1}\left(aq^{-1}t\right)\overline{e}_q(st)\,\widetilde{d}_qt\\ &\quad = \frac{1-q^{2n}}{1-q^2}\int_0^{\frac{1}{s}}\overline{e}_q^{-1}\left(aq^{-1}t\right)\left(q^{-1}t\right)^{n-1}\overline{e}_q(st)\,\widetilde{d}_qt,\end{aligned}$$

so

$$\begin{aligned}
\widetilde{L}_q\{t^n\overline{e}_q^{-1}(at)\}(s) &= \frac{1-q^{2n}}{s-aq^{2n+1}}\widetilde{L}_q\{t^{n-1}\overline{e}_q^{-1}(at)\}(s) \\
&= \frac{1-q^{2n}}{s-aq^{2n+1}}\frac{1-q^{2n-2}}{s-aq^{2n-1}}\widetilde{L}_q\{t^{n-2}\overline{e}_q^{-1}(at)\}(s) \\
&= \cdots \\
&= \prod_{j=0}^{n}\frac{1-q^{2j}}{s-aq^{2j+1}} = \frac{(1-q^2)^n\overline{[n]}_q!}{\overline{(s-a)}^{(n+1)}}.
\end{aligned}$$

□

Next, let $F_1(t) = t^\mu$, $F_2(t) = t^{\nu-1}$. Define $F_2[t] = \overline{(t-rt)}^{(\nu-1)}$ and the convolution

$$(F_1 * F_2)(t) = \frac{t}{1-q^2}\int_0^1 \overline{(t-rt)}^{(\nu-1)}F_1(rq^{-1}t)\,\widetilde{d}_q r,$$

where $\nu \in \mathbb{R}\setminus\{\ldots,-2,-1,0\}$, $\mu \in (-1,+\infty)$.

By the power rule, we have

$$(F_1 * F_2)(t) = \frac{\widetilde{\Gamma}_q(\nu)\widetilde{\Gamma}_q(\mu+1)}{(1-q^2)\widetilde{\Gamma}_q(\mu+\nu+1)}t^{\mu+\nu}.$$

In fact, by (3.1), Lemma 4.2, we have

$$\begin{aligned}
(F_1 * F_2)(t) &= \frac{t}{1-q^2}\int_0^1 \overline{(t-rt)}^{(\nu-1)}F_1(rq^{-1}t)\,\widetilde{d}_q r \\
&= \frac{1}{1-q^2}\int_0^t \overline{(t-s)}^{(\nu-1)}F_1(q^{-1}s)\,\widetilde{d}_q s \\
&= \frac{1}{1-q^2}\int_0^t \overline{(t-s)}^{(\nu-1)}(q^{-1}s)^\mu\,\widetilde{d}_q s \\
&= \frac{1}{1-q^2}\int_0^t \overline{(t-s)}^{(\nu-1)}(q^{\nu-1}s)^\mu q^{-\mu\nu}\,\widetilde{d}_q s \\
&= \frac{q^{-\mu\nu}\widetilde{\Gamma}_q(\nu)}{(1-q^2)}q^{-\binom{\nu}{2}}\widetilde{I}_{q,0}s^\mu \\
&= \frac{\widetilde{\Gamma}_q(\nu)\widetilde{\Gamma}_q(\mu+1)}{(1-q^2)\widetilde{\Gamma}_q(\mu+\nu+1)}t^{\mu+\nu}.
\end{aligned}$$

Now we simply apply Lemma 4.3 to each of F_1, F_2, $F_1 * F_2$ and obtain a convolution theorem.

Theorem 4.1

$$\widetilde{L}_q\{(F_1 * F_2)(t)\}(s) = \widetilde{L}_q\{(F_1)(t)\}(s)\widetilde{L}_q\{(F_2)(t)\}(s). \tag{4.4}$$

Proof

$$\begin{aligned}\widetilde{L}_q\{(F_1 * F_2)(t)\}(s) &= \frac{\widetilde{\Gamma}_q(\nu)\widetilde{\Gamma}_q(\mu+1)}{(1-q^2)^2\widetilde{\Gamma}_q(\mu+\nu+1)}\int_0^{\frac{1}{s}} (q^{-1}t)^{\mu+\nu}\overline{e}_q(st)\,\widetilde{d}_qt \\ &= \frac{\widetilde{\Gamma}_q(\nu)\widetilde{\Gamma}_q(\mu+1)(1-q^2)^{\mu+\nu}}{(1-q^2)^2\widetilde{\Gamma}_q(\mu+\nu+1)}\int_0^{\frac{1}{s}} \left(\frac{q^{-1}t}{1-q^2}\right)^{\mu+\nu}\overline{e}_q(st)\,\widetilde{d}_qt \\ &= \frac{\widetilde{\Gamma}_q(\nu)\widetilde{\Gamma}_q(\mu+1)(1-q^2)^{\mu+\nu}}{(1-q^2)\widetilde{\Gamma}_q(\mu+\nu+1)}\frac{\widetilde{\Gamma}_q(\mu+\nu+1)}{s^{\mu+\nu+1}} \\ &= \frac{\overline{\Gamma}_q(\mu+1)\widetilde{\Gamma}_q(\nu)(1-q^2)^{\mu+\nu-1}}{s^{\mu+\nu+1}}, \\ \widetilde{L}_q\{(F_1)(t)\}(s) &= \frac{(1-q^2)^{\mu}}{1-q^2}\int_0^{\frac{1}{s}} \left(\frac{q^{-1}t}{1-q^2}\right)^{\mu}\overline{e}_q(st)\,\widetilde{d}_qt \\ &= (1-q^2)^{\mu}\frac{\widetilde{\Gamma}_q(\mu+1)}{s^{\mu+1}}.\end{aligned}$$

Similarly,

$$\widetilde{L}_q\{(F_2)(t)\}(s) = (1-q^2)^{\nu-1}\frac{\widetilde{\Gamma}_q(\nu)}{s^{\nu}}.$$

Thus (4.4) holds. □

The convolution theorem will be valid for functions F_1 representing linear sums of functions of the form t^{μ}. Clearly, μ is not necessary an integer.

Corollary 4.1 *Let F_1 be an analytic function, and let $F_2 = t^{\nu-1}$ on $\mathbb{T}\setminus\{0\}$. Then Theorem* 4.1 *holds.*

We now obtain some of the standard properties for the $\widetilde{L}_q$-transform.

Lemma 4.5 *Assume $f = F_1$ is of the type such that* (4.4) *is valid. Then*
(i) $\widetilde{L}_q\{\widetilde{I}_q^{\nu} f(q^{-\nu}t)\}(s) = \frac{q^{\binom{-\nu}{2}}(1-q^2)^{\nu}}{s^{\nu}}\widetilde{L}_q\{f(t)\}(s)$.
(ii) $\widetilde{L}_q\{\widetilde{D}_q^{\nu} f(q^{\nu}t)\}(s) = \frac{q^{\binom{\nu}{2}}(1-q^2)^{\nu}}{s^{\nu}}\widetilde{L}_q\{f(t)\}(s)$.

Proof (i) Note that

$$\begin{aligned}(\widetilde{I}_q^{\nu} f)(q^{-\nu}t) &= \frac{1}{\widetilde{\Gamma}_q(\nu)}q^{\binom{\nu}{2}}\int_0^{q^{-\nu}t}\overline{(q^{-\nu}t-s)}^{(\nu-1)}f(q^{\nu-1}s)\,\widetilde{d}_qs \\ &= \frac{1}{\widetilde{\Gamma}_q(\nu)}q^{\binom{\nu}{2}-\nu^2}\int_0^{t}\overline{(t-s)}^{(\nu-1)}f(q^{-1}s)\,\widetilde{d}_qs \\ &= \frac{t}{\widetilde{\Gamma}_q(\nu)}q^{\binom{-\nu}{2}}\int_0^{1}\overline{(t-ts)}^{(\nu-1)}f(q^{-1}ts)\,\widetilde{d}_qs \\ &= \frac{1-q^2}{\widetilde{\Gamma}_q(\nu)}q^{\binom{-\nu}{2}}\{(f * F_2)(t)\},\end{aligned}$$

we have

$$\widetilde{L}_q\{\widetilde{I}_q^{\nu} f(q^{-\nu}t)\}(s) = \frac{1-q^2}{\widetilde{\Gamma}_q(\nu)} q^{\binom{-\nu}{2}} \widetilde{L}_q\{f(t)\}(s)\widetilde{L}_q\{F_2(t)\}(s)$$
$$= \frac{(1-q^2)^{\nu}}{s^{\nu}} q^{\binom{-\nu}{2}} \widetilde{L}_q\{f(t)\}(s).$$

Similarly, by Theorem 3.1, we may easily see that (ii) holds. □

For the next set of properties, first note that

$$\widetilde{D}_q(\overline{e}_q(sq^{-1}t)f(t)) = \overline{e}_q(st)\widetilde{D}_q f(t) + f(q^{-1}t)\overline{e}_q(sq^{-1}t)$$
$$= \overline{e}_q(st)\widetilde{D}_q f(t) + f(q^{-1}t)\left(-\frac{s}{1-q^2}\overline{e}_q(st)\right)$$

and

$$\int_0^{\frac{1}{s}} \widetilde{D}_q(\overline{e}_q(sq^{-1}t)f(t))\,\widetilde{d}_q t = \overline{e}_q(sq^{-1}t)f(t)\big|_0^{\frac{1}{s}} = -f(0).$$

Thus

$$\widetilde{L}_q\{\widetilde{D}_q f(qt)\}(s) = \frac{s}{1-q^2}\widetilde{L}_q\{f(t)\}(s) - f(0).$$

It follows by induction that if m denotes a positive integer, then

$$\widetilde{L}_q\{\widetilde{D}_q^m f(q^m t)\}(s) = \frac{s^m}{(1-q^2)^m}\widetilde{L}_q\{f(t)\}(s) - \sum_{i=0}^{m-1}\frac{s^{m-1-i}}{(1-q^2)^{m-i}}\left[D_q^i f(q^{m-i-1}t)\right]\big|_{t=0}. \tag{4.5}$$

By Lemma 4.5 and (4.5), we may easily obtain the following Theorem 4.2.

Theorem 4.2 *If f is an analytic function on $\mathbb{T}\setminus\{0\}$, $\nu \in (N-1,N]$, then we have*

$$\widetilde{L}_q\{\widetilde{D}_q^m \widetilde{D}_q^{\nu} f(q^{m+\nu}t)\}(s)$$
$$= \frac{s^{m+\nu} q^{\binom{\nu}{2}}}{(1-q^2)^{m+\nu}}\widetilde{L}_q\{f(t)\}(s) - \sum_{i=0}^{m-1}\frac{s^{m-i-1}}{(1-q^2)^{m-i}}\widetilde{D}_q^i \widetilde{D}_q^{\nu} f(q^{m+\nu-i-1}t)\big|_{t=0}.$$

Theorem 4.3 *If $\widetilde{L}_q\{f(t)\}(s) = F(s)$, then $\widetilde{L}_q\{f(at)\}(s) = \frac{1}{a}F(\frac{s}{a})$.*

Proof

$$\widetilde{L}_q\{f(at)\}(s) = \frac{1}{1-q^2}\int_0^{\frac{1}{s}} f(q^{-1}at)\overline{e}_q(st)\,\widetilde{d}_q t$$
$$= \frac{1}{1-q^2}\int_0^{\frac{a}{s}} f(q^{-1}t)\overline{e}_q\left(\frac{st}{a}\right)\widetilde{d}_q t$$
$$= \frac{1}{a}F\left(\frac{s}{a}\right).$$

□

Example 4.1 Consider the following fractional q-symmetric difference equations:

(a) $\widetilde{D}_q^{3/2} y(q^{3/2}t) = 0$ for $\mathbb{T} \setminus \{0\}$.

(b) $\widetilde{D}_q \widetilde{D}_q^{1/2} y(q^{3/2}t) = 0$ for $\mathbb{T} \setminus \{0\}$. Assume that $\widetilde{D}_q^{1/2} y(0)$ is defined and finite.

(c) $\widetilde{D}_q^2 \widetilde{I}_q^{1/2} y(q^{3/2}t) = 0$ for $\mathbb{T} \setminus \{0\}$. Assume that $\widetilde{I}_{q,0}^{1/2} y(0)$, $\widetilde{D}_q \widetilde{I}_{q,0}^{1/2} y(0)$ is defined and finite.

Note that each equation given in each of (a), (b) and (c) is not equivalent since our method requires knowledge of the fractional derivatives or integrals of the solution defined at zero as well.

We search for analytic solutions on $\mathbb{T} \setminus \{0\}$ for each equation by using a q-symmetric Laplace transform.

For part (a), if we take the Laplace transform of each side of the equation, then by Theorem 4.2 we have $y(t) = 0$.

For part (b), if we take the Laplace transform of each side of the equation and use the properties of the q-symmetric Laplace transform, we obtain

$$\widetilde{L}_q\{y(t)\}(s) = \frac{(1-q^2)^{1/2} q^{-\binom{1/2}{2}}}{s^{3/2}} \widetilde{D}_q^{\frac{1}{2}} y\bigl(q^{1/2}t\bigr)\big|_{t=0}.$$

By using Theorem 4.2, we have

$$y(t) = \frac{q^{-\binom{1/2}{2}}}{\widetilde{\Gamma}_q(3/2)} t^{\frac{1}{2}} \widetilde{D}_q^{\frac{1}{2}} y\bigl(q^{1/2}t\bigr)\big|_{t=0}.$$

For part (c), if we take the Laplace transform of each side of the equation and use the properties of the q-symmetric Laplace transform, we obtain

$$\widetilde{L}_q\{y(t)\}(s) = \frac{(1-q^2)^{-1/2} q^{-\binom{-1/2}{2}}}{s^{1/2}} \widetilde{I}_q^{\frac{1}{2}} y(qt)|_{t=0} + \frac{(1-q^2)^{1/2} q^{-\binom{-1/2}{2}}}{s^{3/2}} \widetilde{D}_q \widetilde{I}_q^{\frac{1}{2}} y(t)|_{t=0}.$$

By using Lemma 4.3, we can get

$$y(t) = \frac{q^{-\binom{-1/2}{2}}}{\widetilde{\Gamma}_q(1/2)} \widetilde{I}_q^{\frac{1}{2}} y(qt)|_{t=0} t^{-\frac{1}{2}} + \frac{q^{-\binom{-1/2}{2}}}{\widetilde{\Gamma}_q(3/2)} \widetilde{D}_q \widetilde{I}_q^{\frac{1}{2}} y(t)|_{t=0} t^{\frac{1}{2}}.$$

Example 4.2 Consider the following fractional q-symmetric difference equation:

$$\widetilde{D}_q \widetilde{I}_q^{2/3} y\bigl(q^{1/3}t\bigr) = t^{\mu} \quad \text{for } \mathbb{T} \setminus \{0\}.$$

Applying the Laplace transform to each side of the equation, we can get

$$\begin{aligned}
&\frac{s^{\frac{1}{3}} q^{\binom{-2/3}{2}}}{(1-q^2)^{\frac{1}{3}}} \widetilde{L}_q\{y(t)\}(s) - \frac{1}{1-q^2} \widetilde{I}_q^{2/3} y\bigl(q^{-2/3}t\bigr)\big|_{t=0} = \bigl(1-q^2\bigr)^{\mu} \frac{\widetilde{\Gamma}_q(\mu+1)}{s^{\mu+1}}, \\
&y(t) = \frac{q^{-\binom{-2/3}{2}}}{\widetilde{\Gamma}_q(1/3)} \widetilde{I}_q^{\frac{2}{3}} y\bigl(q^{-\frac{2}{3}}t\bigr)\big|_{t=0} t^{-\frac{2}{3}} + \frac{\widetilde{\Gamma}_q(\mu+1) q^{-\binom{-2/3}{2}}}{\widetilde{\Gamma}_q(\mu+4/3)} t^{\mu+\frac{1}{3}}.
\end{aligned}$$

Example 4.3 Consider the problem

$$\widetilde{D}_q^2 \widetilde{I}_q^{1/3} y\bigl(q^{5/3}t\bigr) + \alpha \widetilde{D}_q \widetilde{I}_q^{1/3} y\bigl(q^{2/3}t\bigr) = t^2 \bar{e}_q\bigl(\bigl(q^2-1\bigr)\alpha t\bigr).$$

Applying the Laplace transform to each side of the equation, we can get

$$\begin{aligned}
&\frac{s^{\frac{5}{3}}q^{\binom{-1/3}{2}}}{(1-q^2)^{\frac{5}{3}}}\widetilde{L}_q\{y(t)\}(s)-\frac{s}{(1-q^2)^2}\widetilde{I}_q^{1/3}y(q^{-1/3}t)\big|_{t=0}\\
&\quad-\frac{1}{1-q^2}\widetilde{D}_q\widetilde{I}_q^{1/3}y(q^{2/3}t)\big|_{t=0}+\alpha\frac{s^{\frac{2}{3}}q^{\binom{-1/3}{2}}}{(1-q^2)^{\frac{2}{3}}}\widetilde{L}_q\{y(t)\}(s)\\
&\quad-\alpha\frac{1}{1-q^2}\widetilde{I}_q^{1/3}y(q^{-1/3}t)\big|_{t=0}=\frac{(1-q^2)^2\overline{[2]}_q!}{\overline{(s+(1-q^2)\alpha)}^{(3)}},
\end{aligned}$$

$$\begin{aligned}
\widetilde{L}_q\{y(t)\}(s)&=\frac{s^{\frac{1}{3}}q^{-\binom{-1/3}{2}}}{(1-q^2)^{\frac{1}{3}}(s+(1-q^2)\alpha)}\widetilde{I}_q^{\frac{1}{3}}y(q^{-\frac{1}{3}}t)\big|_{t=0}\\
&\quad+\frac{(1-q^2)^{\frac{2}{3}}q^{-\binom{-1/3}{2}}}{s^{\frac{5}{3}}(s+(1-q^2)\alpha)}\big[\widetilde{D}_q\widetilde{I}_q^{\frac{1}{3}}y(q^{\frac{2}{3}}t)\big|_{t=0}+\alpha\widetilde{I}_q^{\frac{1}{3}}y(q^{-\frac{1}{3}}t)\big|_{t=0}\big]\\
&\quad+\frac{(1-q^2)^{\frac{11}{3}}\overline{[2]}_q!q^{-\binom{-1/3}{2}}}{s^{\frac{2}{3}}(s+(1-q^2)\alpha)^4},
\end{aligned}$$

$$\begin{aligned}
y(t)&=\frac{(1-q^2)q^{-\binom{-1/3}{2}}}{\widetilde{\Gamma}_q(-1/3)}\overline{e}_q^{-1}\big((q^2-1)\alpha t\big)*t^{-\frac{4}{3}}\\
&\quad+\frac{(1-q^2)q^{-\binom{-1/3}{2}}}{\widetilde{\Gamma}_q(2/3)}\\
&\quad\times\big[\widetilde{D}_q\widetilde{I}_q^{\frac{1}{3}}y(q^{\frac{2}{3}}t)\big|_{t=0}+\alpha\widetilde{I}_q^{\frac{1}{3}}y(q^{-\frac{1}{3}}t)\big|_{t=0}\big]\overline{e}_q^{-1}\big((q^2-1)\alpha t\big)*t^{\frac{2}{3}}\\
&\quad+\frac{(1-q^2)\overline{[2]}_q!q^{-\binom{-1/3}{2}}}{\widetilde{\Gamma}_q(2/3)}\big(t^3\overline{e}_q^{-1}\big((q^2-1)\alpha t\big)\big)*t^{-\frac{1}{3}}.
\end{aligned}$$

5 A q-symmetric fractional initial problem and q-symmetric Mittag-Leffler function

The following identity, which is useful for transforming q-symmetric fractional difference equations into q-symmetric fractional integrals, will be our key in this section.

Example 5.1 Let $0<\alpha\leq 1$ and consider the q-symmetric fractional difference equation

$$^{c}\widetilde{D}_q^{\alpha}y(q^{\alpha}t)=\lambda y(t)+f(t),\qquad y(0)=a_0,\quad t\in\mathbb{T}.$$

If we apply $\widetilde{I}_{q,0}^{\alpha}$ to the equation, then by Lemma 3.9 we see that

$$y(t)=a_0+\lambda\widetilde{I}_{q,0}^{\alpha}y(q^{-\alpha}t)+\widetilde{I}_{q,0}^{\alpha}g(t),$$

where $g(t)=f(q^{-\alpha}t)$.

To obtain an explicit clear solution, we apply the method of successive approximation. Set $y_0(t)=a_0$ and

$$y_m(t)=a_0+\lambda I_{q,0}^{\alpha}y_{m-1}(q^{-\alpha}t)+\widetilde{I}_{q,0}^{\alpha}g(t),\quad m=1,2,\ldots.$$

For $m = 1$, we have by the power formula Theorem 2.1

$$
\begin{aligned}
y_1(t) &= a_0\left(1+\lambda\frac{q^{\binom{\alpha}{2}}}{\widetilde{\Gamma}_q(\alpha+1)}t^{\alpha}\right)+\widetilde{I}^{\alpha}_{q,0}g(t),\\
y_2(t) &= a_0+\lambda\widetilde{I}^{\alpha}_{q,0}\left(a_0\left(1+\lambda\frac{q^{\binom{\alpha}{2}}}{\widetilde{\Gamma}_q(\alpha+1)}\left(q^{-\alpha}t\right)^{\alpha}\right)+\widetilde{I}^{\alpha}_{q,0}g\left(q^{-\alpha}t\right)\right)+I^{\alpha}_{q,0}g(t)\\
&= a_0+a_0\lambda\frac{q^{\binom{\alpha}{2}}}{\widetilde{\Gamma}_q(\alpha+1)}t^{\alpha}+a_0\lambda^2\frac{q^{\binom{\alpha}{2}}}{\widetilde{\Gamma}_q(2\alpha+1)}\widetilde{I}^{\alpha}_{q,0}\left(q^{-\alpha}t\right)^{\alpha}+\lambda\widetilde{I}^{\alpha}_{q,0}\left(\widetilde{I}^{\alpha}_{q,0}g\right)\left(q^{-\alpha}t\right)+\widetilde{I}^{\alpha}_{q,0}g(t)\\
&= a_0\left(1+\lambda\frac{q^{\binom{\alpha}{2}}}{\widetilde{\Gamma}_q(\alpha+1)}t^{\alpha}+\lambda^2\frac{q^{2\binom{\alpha}{2}}}{\widetilde{\Gamma}_q(2\alpha+1)}t^{2\alpha}\right)+\lambda\widetilde{I}^{\alpha}_{q,0}\left(\widetilde{I}^{\alpha}_{q,0}g\right)\left(q^{-\alpha}t\right)+\widetilde{I}^{\alpha}_{q,0}g(t).
\end{aligned}
$$

Since

$$
\left(\widetilde{I}^{\alpha}_{q,0}g\right)\left(q^{-\alpha}t\right)=\frac{q^{2\binom{\alpha}{2}}}{\widetilde{\Gamma}_q(\alpha)}\int_0^{q^{-\alpha}t}\overline{\left(q^{-\alpha}t-s\right)}^{(\alpha-1)}g\left(q^{\alpha-1}s\right)\widetilde{d}_qs=q^{-\alpha^2}\widetilde{I}^{\alpha}_{q,0}g\left(q^{-\alpha}t\right),
$$

we have

$$
y_2(t)=a_0\left(1+\lambda\frac{q^{\binom{\alpha}{2}}}{\widetilde{\Gamma}_q(\alpha+1)}t^{\alpha}+\lambda^2\frac{q^{2\binom{\alpha}{2}}}{\widetilde{\Gamma}_q(2\alpha+1)}t^{2\alpha}\right)+\lambda q^{-\alpha^2}\widetilde{I}^{2\alpha}_{q,0}g\left(q^{-\alpha}t\right)+\widetilde{I}^{\alpha}_{q,0}g(t).
$$

If we proceed inductively and let $m\longrightarrow\infty$, we obtain the solution.

$$
\begin{aligned}
y(t) &= a_0\left(1+\sum_{k=1}^{\infty}\frac{\lambda^k q^{k\binom{\alpha}{2}}}{\widetilde{\Gamma}_q(k\alpha+1)}t^{k\alpha}\right)+\int_0^t\sum_{k=0}^{\infty}\frac{\lambda^k q^{k\binom{\alpha}{2}}\overline{(t-s)}^{(k\alpha+\alpha-1)}}{\widetilde{\Gamma}_q(k\alpha+\alpha)}f\left(q^{-1}s\right)\widetilde{d}_qs\\
&= a_0\left(1+\sum_{k=1}^{\infty}\frac{\lambda^k q^{k\binom{\alpha}{2}}}{\widetilde{\Gamma}_q(k\alpha+1)}t^{k\alpha}\right)\\
&\quad+\int_0^t\overline{(t-s)}^{(\alpha-1)}\sum_{k=0}^{\infty}\frac{\lambda^k q^{k\binom{\alpha}{2}}\overline{\left(t-q^{2(\alpha-1)}s\right)}^{(k\alpha)}}{\widetilde{\Gamma}_q(k\alpha+\alpha)}f\left(q^{-1}s\right)\widetilde{d}_qs.
\end{aligned}
$$

If we set $\alpha = 1$, $\lambda = 1$, $a_0 = 0$, $f(t) = 0$ and note Remark 2.1, we come to the q-symmetric exponential formula

$$
\begin{aligned}
\overline{e}_q^{-1}\left(q^{-1}t\right) &= \frac{1}{\prod_{n=0}^{\infty}(1-q^{2n}t)}=\sum_{k=0}^{\infty}\frac{t^k}{\overline{(q)}_n},\\
\overline{e}_q^{-1}\left(\left(1-q^2\right)q^{-1}t\right) &= \sum_{k=0}^{\infty}\frac{t^k}{\widetilde{\Gamma}_q(k+1)},
\end{aligned}
$$

where $\overline{(q)}_n = (1-q^2)(1-q^4)\cdots(1-q^{2k})$ is the q-symmetric Pochhammer symbol.

If compared with the classical case, the above example suggests the following q-symmetric analogue of the Mittag-Leffler function.

Definition 5.1 For $z, z_0 \in \mathbb{C}$ and $\Re(\alpha) > 0$, the q-symmetric Mittag-Leffler function is defined by

$$
{}_q\overline{E}_{\alpha,\beta}(\lambda, z-z_0) = \sum_{k=0}^{\infty} \lambda^k q^{k\binom{\alpha}{2}} \frac{\overline{(z-z_0)}_q^{(\alpha k)}}{\widetilde{\Gamma}_q(\alpha k+\beta)}.
$$

When $\beta = 1$, we simply use ${}_qE_\alpha(\lambda, z-z_0) :=_q E_{\alpha,1}(\lambda, z-z_0)$.

According to Definition 5.1 above, the solution of the q-symmetric difference equation in Example 5.1 is expressed by

$$
y(t) = a_{0q}E_\alpha(\lambda, t) + \int_0^t \overline{(t-s)}^{(\alpha-1)} {}_qE_{\alpha,\alpha}\big(\lambda, t - q^{2(\alpha-1)}s\big) f\big(q^{-1}s\big)\,\widetilde{d}_q s.
$$

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
CH and MS worked together in the derivation of the mathematical results. All authors read and approved the final manuscript.

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