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A second order box-type scheme for fractional sub-diffusion equation with spatially variable coefficient under Neumann boundary conditions

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Abstract

In the present work, a box-type difference scheme with convergence order $O(\tau^2 + h^2)$ is proposed for the fractional sub-diffusion equation with spatially variable coefficient under Neumann boundary conditions. Here h, τ are space and temporal step length, respectively. The method is based on applying the $L2 - 1_\sigma$ formula to approximate the time Caputo fractional derivative and introducing the auxiliary variable. By virtue of the special properties of the $L2 - 1_\sigma$ formula and the mathematical induction method, the unconditional stability and convergence for our scheme are proved by the discrete energy method. Numerical examples are given to verify the theoretical analysis and efficiency of the box-type scheme.

MSC: 65M06; 65M12

Keywords: fractional sub-diffusion equation; box-type difference scheme; $L2 - 1_\sigma$ formula; stability; convergence

1 Introduction

Recently, research interest focused on fractional differential equations has become more and more manifest. This fact reflects the ability of fractional calculation to describe different phenomena in different disciplines such as semiconductor, mechanics, chemistry, porous media, anomalous diffusion, etc. [1–7]. The time fractional sub-diffusion equation (FSDE) is a kind of linear integro-differential equation which can be obtained from the classical diffusion equation by employing fractional derivatives of order α to describe the procedure of anomalous diffusion, where $\alpha \in (0,1)$.

There is much considerable work devoted to the research for numerical methods of FSDE. Langlands and Henry [8] presented an implicit numerical scheme for the homogeneous problem and discussed the accuracy and stability of the scheme. Yuste and Acede [9] developed an explicit scheme whence the stability was strictly proved. Subsequently, Yuste [10] analyzed the weighted average finite difference scheme by the von Neumann method. Zhuang et al. [11] integrated the linear and nonlinear sub-diffusion equations for time variable t, then approximated the resultant equivalent equations with the idea of numerical integrals. Subsequently, an implicit numerical method for this equation with a nonlinear source term in a bounded domain was described and demonstrated in [12]. Heydari [13] proposed a Legendre wavelets Galerkin method to obtain an approximate solution for FSDE. The numerical experiment results revealed that this method is more accurate and efficient in comparison with some compact finite difference methods. Hooshmandasl et al. [14] presented an efficient Galerkin method based on the fractional-order Legendre functions for solving the fractional sub-diffusion equation and time-fractional diffusion-wave equation.

The main way to approximate the fractional derivative is applying the *Grünwald-Letnikov* formula. Cui [15] obtained an implicit scheme by employing the *Grünwald-Letnikov* discretization combined with a compact finite technique in spatial direction. Mohebbi [16] et al. studied a modified anomalous sub-diffusion equation with a nonlinear source term, and a difference scheme with convergence order $O(\tau + h^4)$ was constructed. Some high-order approximation for fractional derivatives was proposed by assembling the shifted *Grünwald-Letnikov* operator with different weights in [17, 18]. Based on this idea, Wang and Vong [19] proposed a second order accuracy formula to approximate the time-fractional derivative and a compact difference scheme was established for solving the modified anomalous fractional sub-diffusion equation.

Another main instrument to handle the time-fractional derivative is the $L1$ formula. Sun and Wu [20] first proposed a fully discrete difference scheme for FSDE by employing the $L1$ approximation, where the truncation error was proved to be of $2 - \alpha$ order in temporal direction. Lin and Xu [21] constructed an effective numerical method by employing the finite difference scheme in time and using the Legendre spectral methods in space. Chen et al. [22] gave an implicit numerical scheme for the problem and proved the unconditional stability and L_2-norm convergence. Gao and Sun [23] applied the $L1$ formula and developed a compact finite difference scheme to promote the spatial accuracy for FSDE. Zhao and Sun [24] proposed a box-type scheme for solving a class of fractional sub-diffusion equations with Neumann boundary conditions. Ren et al. [25] proposed a compact difference scheme for this problem where the convergence order $O(\tau^{2-\alpha} + h^4)$ was obtained.

Considering the nonlocal character and history dependence of the fractional derivative, we need to retain information from all the previous temporal layer when we solve FSDE numerically. Thus, it is meaningful to improve the accuracy of $L1$ formula. Zhang et al. [26] got a second order approximate formula for the Caputo derivative by considering the $L1$ formula on special nonuniform mesh. A difference scheme with $O(\tau^2 + h^4)$ accuracy was proposed, then the stability and convergence were proved. Inspired by the classic Crank-Nicolson method and the construction of $L1$ formula, Zhao and Sun [27] proposed a second order approximation for the variable order fractional derivatives, whence the stability of the scheme was not obtained. Gao and Sun [28] proposed a formula to approximate the Caputo fractional derivative with convergence order $O(\tau^{3-\alpha})$, which was called $L1 - 2$ formula. The stability and convergence of the scheme were not obtained yet. Based on the idea of [28], Alikhanov [29] constructed a new formula (called $L2 - 1_\sigma$ formula) to approximate the Caputo fractional derivative with $O(\tau^{3-\alpha})$ accuracy. The difference scheme of fourth approximation order in space and second order accuracy in time for FSDE was constructed. The stability and convergence for L_2 norm were strictly proved by the energy method.

The works we listed above are mainly focused on FSDE with constant coefficient. However, many practical applications involved variable diffusion coefficients [30–32]. Zhao and Xu [33] considered the Caputo-fractional sub-diffusion equation with spatially variable coefficient, i.e.,

$$ {}_0^C\mathcal{D}_t^\alpha u(x,t) = \frac{\partial}{\partial x}\left(\varphi(x)\frac{\partial u}{\partial x}\right) + f(x,t), $$

where ${}_0^C\mathcal{D}_t^\alpha v(t) \equiv \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{v'(\xi)}{(t-\xi)^\alpha}\,d\xi$ denotes the Caputo fractional derivative. $\Gamma(\cdot)$ means gamma function. By virtue of the $L1$ formula, they constructed a box-type difference scheme with $O(\tau^{2-\alpha}+h^2)$ accuracy to handle the Neumann boundary conditions. Vong et al. [34] considered the same problem, and the global convergence order $O(\tau^{2-\alpha}+h^4)$ was obtained by subtle decomposition of the coefficient matrices.

Be that as it may, we find that there are few reports on finite difference methods of high order accuracy in temporal direction for FSDE with spatially variable coefficient. In this paper, our target is to construct a box-type difference scheme with $O(\tau^2+h^2)$ accuracy for that problem under Neumann boundary conditions. We apply the $L2-1_\sigma$ formula to approximate the Caputo fractional derivative in temporal direction, then give the strict analysis for stability and convergence of the scheme proposed.

The rest of this article is organized as follows. In Section 2, we introduce some necessary notations and preliminary lemmas, then a box-type scheme with the truncation errors of second order in both time and space directions is constructed by introducing the auxiliary variable. The unconditional stability and convergence in maximum norm are strictly proved in Section 3 by the energy method. Two numerical experiment results are listed in Section 4 to testify our theoretical analysis. A brief conclusion ends this paper finally in Section 5.

2 Derivation of the box-type scheme

Consider the following fractional sub-diffusion equation with spatially variable coefficient under Neumann boundary conditions:

$$ {}_0^C\mathcal{D}_t^\alpha u(x,t) = \frac{\partial}{\partial x}\left(\varphi(x)\frac{\partial u}{\partial x}\right) + f(x,t), \quad 0 < x < L, 0 < t \le T, \tag{2.1} $$

$$ u(x,0) = \phi(x), \quad 0 < x < L, \tag{2.2} $$

$$ u_x(0,t) = \lambda_1(t), \qquad u_x(L,t) = \lambda_2(t), \quad 0 \le t \le T, \tag{2.3} $$

where $\alpha \in (0,1)$ is a constant. Furthermore, we suppose that there exist constants C_1 and C_2 such that $0 < C_1 \le \varphi(x) \le C_2$.

For numerical approximation, we give the following mesh partition. Giving two positive integers M and N, then $h = \frac{L}{M}$, $\tau = \frac{T}{N}$ are space and temporal step lengths, respectively. Define $x_i = ih$, $0 \le i \le M$, $t_n = n\tau$, $0 \le n \le N$, $\Omega_h = \{x_i \mid 0 \le i \le M\}$, $\Omega_\tau = \{t_n \mid 0 \le n \le N\}$. In addition, denote $\sigma = 1 - \frac{\alpha}{2}$ and $t_{n-1+\sigma} = (n-1+\sigma)\tau$. Denote $\mathcal{V}_h = \{u \mid u = (u_0, u_1, \ldots, u_M)\}$ and $\mathcal{V}_{0h} = \{u \mid u = (u_0, u_1, \ldots, u_M), u_0 = u_M = 0\}$ as the grid function spaces on Ω_h.

For any grid function $u \in \mathcal{V}_h$, we introduce the notations below.

$$ \delta_x u_{j+\frac{1}{2}} = \frac{1}{h}(u_{j+1} - u_j), \quad 0 \le j \le M-1, $$

$$ \delta_x^2 u_j = \frac{1}{h}(\delta_x u_{j+\frac{1}{2}} - \delta_x u_{j-\frac{1}{2}}), \quad 1 \le j \le M-1. $$

We now introduce some lemmas which will be used in the following analysis.

Alikhanov [29] constructed a new second order difference approximation for the Caputo fractional derivative (called $L2-1_\sigma$ formula). Defining

$$a_0=\sigma^{1-\alpha},\qquad a_l=(l+\sigma)^{1-\alpha}-(l-1+\sigma)^{1-\alpha},\quad l\geq 1,$$
$$b_l=\frac{1}{2-\alpha}\bigl[(l+\sigma)^{2-\alpha}-(l-1+\sigma)^{2-\alpha}\bigr]-\frac{1}{2}\bigl[(l+\sigma)^{1-\alpha}+(l-1+\sigma)^{1-\alpha}\bigr],\quad l\geq 1,$$

when $n=1$, denote

$$C_0^{(n)}=a_0,$$

when $n\geq 2$, denote

$$C_k^{(n)}=\begin{cases}a_0+b_1, & k=0,\\ a_k+b_{k+1}-b_k, & 1\leq k\leq n-2,\\ a_k-b_k, & k=n-1.\end{cases}\tag{2.4}$$

Given a grid function $u=\{u^n\mid 0\leq n\leq N\}$, denote

$$\Delta_{t_{n-1+\sigma}}^{\alpha}u^n=\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\Biggl[C_0^{(n)}u^n-\sum_{j=1}^{n-1}\bigl(C_{n-j-1}^{(n)}-C_{n-j}^{(n)}\bigr)u^j-C_{n-1}^{(n)}u^0\Biggr]\tag{2.5}$$

as the discrete fractional derivative operator, i.e., the $L2-1_\sigma$ formula. Alikhanov analyzed the error of the $L2-1_\sigma$ formula to approximate the Caputo fractional derivative, and got the following lemma.

Lemma 2.1 ([29]) *Suppose $u(t)\in C^3[0,t_n]$, it holds that*

$$\bigl|{}_0^C\mathcal{D}_t^{\alpha}u(t)\big|_{t=t_{n-1+\sigma}}-\Delta_{t_{n-1+\sigma}}^{\alpha}u^n\bigr|=O\bigl(\tau^{3-\alpha}\bigr).$$

Subsequently, the special properties of this difference operator were derived.

Lemma 2.2 ([29]) *Suppose $\alpha\in(0,1)$, $\sigma=1-\frac{\alpha}{2}$, $C_k^{(n)}$ $(0\leq k\leq n-1,\ n\geq 1)$ is defined by* (2.4), *it holds that*

$$C_k^{(n)}>\frac{1-\alpha}{2}(k+\sigma)^{-\alpha},\tag{2.6}$$
$$C_0^{(n)}>C_1^{(n)}>C_2^{(n)}>\cdots>C_{n-2}^{(n)}>C_{n-1}^{(n)}.\tag{2.7}$$

Furthermore, there is an important relation for the second order operator, which will play an irreplaceable role in the analysis of the stability and convergence for our scheme.

Lemma 2.3 ([29]) *Suppose $u=\{u^n\mid 0\leq n\leq N\}$ is a grid function defined on Ω_τ, then it holds that*

$$\bigl(\sigma u^n+(1-\sigma)u^{n-1}\bigr)\Delta_{t_{n-1+\sigma}}^{\alpha}u^n\geq\frac{1}{2}\Delta_{t_{n-1+\sigma}}^{\alpha}\bigl(u^n\bigr)^2.$$

Now we give the derivation of the box-type scheme. Denoting $v(x,t) = \varphi(x)\frac{\partial u}{\partial x}$, then problem (2.1)-(2.3) is equivalent to

$$ {}_0^C\mathcal{D}_t^\alpha u(x,t) = \frac{\partial}{\partial x}v(x,t) + f(x,t), \quad 0 < x < L, 0 < t \leq T, \tag{2.8} $$

$$ v(x,t) = \varphi(x)\frac{\partial u(x,t)}{\partial x}, \quad 0 < x < L, 0 < t \leq T, \tag{2.9} $$

$$ u(x,0) = \phi(x), \quad 0 \leq x \leq L, \tag{2.10} $$

$$ v(0,t) = \varphi(0)\lambda_1(t), \qquad v(L,t) = \varphi(L)\lambda_2(t), \quad 0 \leq t \leq T. \tag{2.11} $$

Define the grid functions

$$ U_j^n = u(x_j, t_n), \qquad V_j^n = v(x_j, t_n), \quad 0 \leq j \leq M, 0 \leq n \leq N, $$

and $f_{j+\frac{1}{2}}^{n-1+\sigma} = f(x_{j+\frac{1}{2}}, t_{n-1+\sigma})$. Suppose $u(x,t) \in C_{x,t}^{(4,3)}([0,L] \times [0,T])$, now we consider equations (2.8) and (2.9) at the grid points $(x_{j+\frac{1}{2}}, t_{n-1+\sigma})$ and $(x_{j+\frac{1}{2}}, t_n)$, respectively. We obtain

$$ {}_0^C\mathcal{D}_t^\alpha u(x_{j+\frac{1}{2}}, t_{n-1+\sigma}) = \frac{\partial v}{\partial x}(x_{j+\frac{1}{2}}, t_{n-1+\sigma}) + f_{j+\frac{1}{2}}^{n-1+\sigma}, \quad 0 \leq j \leq M-1, 1 \leq n \leq N, \tag{2.12} $$

$$ v(x_{j+\frac{1}{2}}, t_n) = \varphi(x_{j+\frac{1}{2}})\frac{\partial u}{\partial x}(x_{j+\frac{1}{2}}, t_n), \quad 0 \leq j \leq M-1, 0 \leq n \leq N. \tag{2.13} $$

Denoting

$$ U_j^{n-1+\sigma} = \sigma U_j^n + (1-\sigma)U_j^{n-1}, \quad 1 \leq n \leq N, $$

and using Taylor expansion, it is not hard to verify that

$$ \begin{aligned} \frac{\partial v}{\partial x}(x_{j+\frac{1}{2}}, t_{n-1+\sigma}) &= \sigma\frac{\partial v}{\partial x}(x_{j+\frac{1}{2}}, t_n) + (1-\sigma)\frac{\partial v}{\partial x}(x_{j+\frac{1}{2}}, t_{n-1}) + O(\tau^2) \\ &= \sigma\delta_x V_{j+\frac{1}{2}}^n + (1-\sigma)\delta_x V_{j+\frac{1}{2}}^{n-1} + O(\tau^2 + h^2) \\ &= \sigma\delta_x V_{j+\frac{1}{2}}^{n-1+\sigma} + O(\tau^2 + h^2), \end{aligned} \tag{2.14} $$

$$ v(x_{j+\frac{1}{2}}, t_n) = V_{j+\frac{1}{2}}^n + O(h^2), \qquad \frac{\partial u}{\partial x}(x_{j+\frac{1}{2}}, t_n) = \delta_x U_{j+\frac{1}{2}}^n + O(h^2). \tag{2.15} $$

From Lemma 2.1 and (2.12)-(2.15), we have

$$ \Delta_{t_{n-1+\sigma}}^\alpha U_{j+\frac{1}{2}}^n = \delta_x V_{j+\frac{1}{2}}^{n-1+\sigma} + f_{j+\frac{1}{2}}^{n-1+\sigma} + (R_1)_{j+\frac{1}{2}}^n, \tag{2.16} $$

$$ V_{j+\frac{1}{2}}^n = \varphi(x_{j+\frac{1}{2}})\delta_x U_{j+\frac{1}{2}}^n + (R_2)_{j+\frac{1}{2}}^n, \tag{2.17} $$

where

$$ \left|(R_1)_{j+\frac{1}{2}}^n\right| + \left|(R_2)_{j+\frac{1}{2}}^n\right| \leq C_R(\tau^2 + h^2), \tag{2.18} $$

here C_R is a constant independent of τ and h. The initial and boundary conditions (2.10)-(2.11) yield

$$V_0^n = \varphi(0)\lambda_1(t_n), \qquad V_M^n = \varphi(L)\lambda_2(t_n), \quad 0 \le n \le N, \tag{2.19}$$

$$U_j^0 = \phi(x_j), \quad 0 \le j \le M. \tag{2.20}$$

Omitting the small terms R_1, R_2 in (2.16) and (2.17), combining with (2.19) and (2.20), we get the following box-type difference scheme for (2.8)-(2.11):

$$\Delta^{\alpha}_{t_{n-1+\sigma}} u^n_{j+\frac{1}{2}} = \delta_x v^{n-1+\sigma}_{j+\frac{1}{2}} + f^{n-1+\sigma}_{j+\frac{1}{2}}, \quad 0 \le j \le M-1, 1 \le n \le N, \tag{2.21}$$

$$v^n_{j+\frac{1}{2}} = \varphi(x_{j+\frac{1}{2}})\delta_x u^n_{j+\frac{1}{2}}, \quad 0 \le j \le M-1, 0 \le n \le N, \tag{2.22}$$

$$v_0^n = \varphi(0)\lambda_1(t_n), \qquad v_M^n = \varphi(L)\lambda_2(t_n), \quad 0 \le n \le N, \tag{2.23}$$

$$u_j^0 = \phi(x_j), \quad 0 \le j \le M. \tag{2.24}$$

Eliminating the auxiliary variable $\{v_j^n\}$, we can get a difference scheme containing only $\{u_j^n\}$ for problem (2.1)-(2.3). It is not hard to prove the following equivalent theorem.

Theorem 2.4 *The difference scheme* (2.21)-(2.24) *is equivalent to*

$$\Delta^{\alpha}_{t_{n-1+\sigma}} u^n_{\frac{1}{2}} = \frac{2}{h}\big[\varphi(x_{\frac{1}{2}})\delta_x u^{n-1+\sigma}_{\frac{1}{2}} - \varphi(0)\lambda_1^{n-1+\sigma}\big] + f^{n-1+\sigma}_{\frac{1}{2}}, \tag{2.25}$$

$$\begin{aligned}&\frac{1}{2}\big(\Delta^{\alpha}_{t_{n-1+\sigma}} u^n_{j-\frac{1}{2}} + \Delta^{\alpha}_{t_{n-1+\sigma}} u^n_{j+\frac{1}{2}}\big) = \delta_x(\varphi\delta_x u)^{n-1+\sigma}_j + \frac{1}{2}\big(f^{n-1+\sigma}_{j-\frac{1}{2}} + f^{n-1+\sigma}_{j+\frac{1}{2}}\big),\\ &\quad 1 \le j \le M-1,\end{aligned} \tag{2.26}$$

$$\Delta^{\alpha}_{t_{n-1+\sigma}} u^n_{M-\frac{1}{2}} = \frac{2}{h}\big[\varphi(L)\lambda_2^{n-1+\sigma} - \varphi(x_{M-\frac{1}{2}})\delta_x u^{n-1+\sigma}_{M-\frac{1}{2}}\big] + f^{n-1+\sigma}_{M-\frac{1}{2}}, \tag{2.27}$$

$$u_j^0 = \phi(x_j), \quad 0 \le j \le M, \tag{2.28}$$

and

$$v^0_{j+\frac{1}{2}} = \varphi(x_{j+\frac{1}{2}})\delta_x u^0_{j+\frac{1}{2}}, \quad 0 \le j \le M-1, \tag{2.29}$$

$$v_0^{n-1+\sigma} = \varphi(x_{\frac{1}{2}})\delta_x u^{n-1+\sigma}_{\frac{1}{2}} - \frac{h}{2}\big(\Delta^{\alpha}_{t_{n-1+\sigma}} u^n_{j-\frac{1}{2}} - f^{n-1+\sigma}_{\frac{1}{2}}\big), \tag{2.30}$$

$$v_j^{n-1+\sigma} = \varphi(x_{j-\frac{1}{2}})\delta_x u^{n-1+\sigma}_{j-\frac{1}{2}} + \frac{h}{2}\big(\Delta^{\alpha}_{t_{n-1+\sigma}} u^n_{j-\frac{1}{2}} - f^{n-1+\sigma}_{j-\frac{1}{2}}\big), \quad 1 \le j \le M, \tag{2.31}$$

where $1 \le n \le N$ *in* (2.25)-(2.31).

Remark 2.5 For the convenience of actual computation, we construct scheme (2.25)-(2.28) for problem (2.1)-(2.3). It follows from Theorem 2.4 that the analysis of the solvability, stability and convergence of the difference scheme (2.25)-(2.28) may be transferred to that of the difference scheme (2.21)-(2.24).

It is clear that at each time level, the difference scheme (2.25)-(2.28) results in a linear system in which the coefficient matrix is tridiagonal and strictly diagonally dominant, thus the difference scheme has a unique solution, and the Thomas algorithm suits. So we have the following.

Theorem 2.6 *The difference scheme* (2.25)-(2.28) *is uniquely solvable.*

3 Analysis of the box-type scheme

We give some essential notations first. Introducing the discrete inner products and the corresponding norms for any $u, v \in \mathcal{V}_h$ as follows

$$\langle u, v\rangle = h\sum_{j=0}^{M-1} u_{i+\frac{1}{2}} v_{i+\frac{1}{2}}, \qquad \langle u, v\rangle_\varphi = h\sum_{j=0}^{M-1} \varphi(x_{i+\frac{1}{2}}) u_{i+\frac{1}{2}} v_{i+\frac{1}{2}},$$
$$\|u\| = \sqrt{\langle u, u\rangle}, \qquad \|u\|_\varphi = \sqrt{\langle u, u\rangle_\varphi}, \qquad \|u\|_\infty = \max_{0\le j\le M} |u_j|,$$

and

$$\|\delta_x u\| = \sqrt{\langle \delta_x u, \delta_x u\rangle}, \qquad \|\delta_x u\|_\varphi = \sqrt{\langle \delta_x u, \delta_x u\rangle_\varphi},$$
$$\|u\|_0 = \sqrt{h\left(\frac{1}{2}u_0^2 + \sum_{j=0}^{M-1} u_j^2 + \frac{1}{2}u_m^2\right)},$$

we now give the following lemmas which will be used in the analysis of the box-type scheme.

Lemma 3.1 ([35, 36]) *For any grid function $u \in \mathcal{V}_{0h}$, it holds that*

$$\|u\|_0^2 \le \frac{L^2}{6}\|\delta_x u\|^2, \tag{3.1}$$
$$\|u\|^2 \le \frac{L^2}{6}\|\delta_x u\|^2. \tag{3.2}$$

Proof One can refer to [35, 36] for (3.1). Considering the following equality

$$\begin{aligned}\left(u_{j+\frac{1}{2}}^n\right)^2 + \frac{h^2}{4}\left(\delta_x u_{j+\frac{1}{2}}^n\right)^2 &= \frac{1}{4}\left[\left(u_{j+1}^n + u_j^n\right)^2 + \left(u_{j+1}^n - u_j^n\right)^2\right] \\ &= \frac{1}{2}\left[\left(u_j^n\right)^2 + \left(u_{j+1}^n\right)^2\right],\end{aligned}$$

summing up j from 0 to $M-1$, we get

$$\|u\|^2 + \frac{h^2}{4}\|\delta_x u\|^2 = \|u\|_0^2.$$

Applying (3.1), the second conclusion is obtained. □

One can easily testify the following.

Lemma 3.2 *For any grid function $v \in \mathcal{V}_h$, it holds that*

$$\sqrt{C_1}\|\delta_x u\| \le \|\delta_x u\|_\varphi \le \sqrt{C_2}\|\delta_x u\|, \tag{3.3}$$
$$\sqrt{C_1}\|u\| \le \|u\|_\varphi \le \sqrt{C_2}\|u\|. \tag{3.4}$$

We have a critical estimation for the maximum norm which will be used for stability and convergence analysis.

Lemma 3.3 ([24]) *Let $u \in \mathcal{V}_h$, then for any positive constant ϵ, it holds that*

$$\|u\|_\infty^2 \le \left(\epsilon + \frac{h^2}{4L}\right)\|\delta_x u\|^2 + \left(\frac{1}{\epsilon} + \frac{1}{L}\right)\|u\|^2. \tag{3.5}$$

We now point out that the box-type difference scheme is unconditionally stable to the initial value and the source term f.

Theorem 3.4 (Stability) *Suppose $\{u_j^n \mid 0 \le j \le M, 0 \le n \le N\}$ is the solution of the following difference scheme*:

$$\Delta_{t_{n-1+\sigma}}^\alpha u_{j+\frac{1}{2}}^n = \delta_x v_{j+\frac{1}{2}}^{n-1+\sigma} + f_{j+\frac{1}{2}}^{n-1+\sigma}, \quad 0 \le j \le M-1, 1 \le n \le N, \tag{3.6}$$

$$v_{j+\frac{1}{2}}^n = \varphi(x_{j+\frac{1}{2}})\delta_x u_{j+\frac{1}{2}}^n, \quad 0 \le j \le M-1, 0 \le n \le N, \tag{3.7}$$

$$v_0^n = 0, \qquad v_M^n = 0, \quad 0 \le n \le N, \tag{3.8}$$

$$u_j^0 = \phi(x_j), \quad 0 \le j \le M, \tag{3.9}$$

then, for every $1 \le n \le N$, we have

$$\|\delta_x u^n\|^2 \le \frac{1}{C_1}\|\delta_x u^0\|_\varphi^2 + \frac{T^\alpha \Gamma(1-\alpha)}{C_1}\max_{1\le n\le N}\|f^{n-1+\sigma}\|^2, \tag{3.10}$$

$$\|u^n\|^2 \le 2\|u^0\|^2 + 4T^\alpha\Gamma(1-\alpha)\|\delta_x u^0\|_\varphi^2 + 12\big[T^\alpha\Gamma(1-\alpha)\big]^2\max_{1\le n\le N}\|f^{n-1+\sigma}\|^2. \tag{3.11}$$

Proof Applying the fractional approximation operator $\Delta_{t_{n-1+\sigma}}^\alpha$ and dividing $\varphi(x_{j+\frac{1}{2}})$ on the both sides of (3.7), we obtain

$$\frac{1}{\varphi(x_{j+\frac{1}{2}})}\Delta_{t_{n-1+\sigma}}^\alpha v_{j+\frac{1}{2}}^n = \Delta_{t_{n-1+\sigma}}^\alpha \delta_x u_{j+\frac{1}{2}}^n.$$

Multiplying the identity above by $v_{j+\frac{1}{2}}^{n-1+\sigma}$ and summing up for j from 0 to $M-1$, we have

$$\big\langle \Delta_{t_{n-1+\sigma}}^\alpha v^n, v^{n-1+\sigma}\big\rangle_{\frac{1}{\varphi}} = \big\langle \Delta_{t_{n-1+\sigma}}^\alpha \delta_x u^n, v^{n-1+\sigma}\big\rangle. \tag{3.12}$$

Multiplying equation (3.6) by $\delta_x v_{j+\frac{1}{2}}^{n-1+\sigma}$ and summing up for j from 0 to $M-1$, we have

$$\big\langle \Delta_{t_{n-1+\sigma}}^\alpha u^n, \delta_x v^{n-1+\sigma}\big\rangle = \|\delta_x v^{n-1+\sigma}\|^2 + \big\langle f^{n-1+\sigma}, \delta_x v^{n-1+\sigma}\big\rangle. \tag{3.13}$$

Adding equalities (3.12) and (3.13) above, we obtain

$$\begin{aligned}&\|\delta_x v^{n-1+\sigma}\|^2 + \big\langle \Delta_{t_{n-1+\sigma}}^\alpha v^n, v^{n-1+\sigma}\big\rangle_{\frac{1}{\varphi}} \\ &\quad = \big\langle \Delta_{t_{n-1+\sigma}}^\alpha \delta_x u^n, v^{n-1+\sigma}\big\rangle + \big\langle \Delta_{t_{n-1+\sigma}}^\alpha u^n, \delta_x v^{n-1+\sigma}\big\rangle - \big\langle f^{n-1+\sigma}, \delta_x v^{n-1+\sigma}\big\rangle.\end{aligned} \tag{3.14}$$

Noticing that $v_0^{n-1+\sigma} = v_M^{n-1+\sigma} = 0$, we have

$$\begin{aligned}
&\left\langle \Delta_{t_{n-1+\sigma}}^{\alpha} \delta_x u^n, v^{n-1+\sigma} \right\rangle + \left\langle \Delta_{t_{n-1+\sigma}}^{\alpha} u^n, \delta_x v^{n-1+\sigma} \right\rangle \\
&\quad = h\sum_{j=0}^{M-1} v_{j+\frac{1}{2}}^{n-1+\sigma} \cdot \Delta_{t_{n-1+\sigma}}^{\alpha} \delta_x u_{j+\frac{1}{2}}^n + h\sum_{j=0}^{M-1} \delta_x v_{j+\frac{1}{2}}^{n-1+\sigma} \cdot \Delta_{t_{n-1+\sigma}}^{\alpha} u_{j+\frac{1}{2}}^n \\
&\quad = \frac{1}{2}\sum_{j=0}^{M-1}\Big[\big(v_{j+1}^{n-1+\sigma} + v_j^{n-1+\sigma}\big)\big(\Delta_{t_{n-1+\sigma}}^{\alpha} u_{j+1}^n - \Delta_{t_{n-1+\sigma}}^{\alpha} u_j^n\big) \\
&\qquad + \big(v_{j+1}^{n-1+\sigma} - v_j^{n-1+\sigma}\big)\big(\Delta_{t_{n-1+\sigma}}^{\alpha} u_{j+1}^n + \Delta_{t_{n-1+\sigma}}^{\alpha} u_j^n\big)\Big] \\
&\quad = \sum_{j=0}^{M-1}\big(v_{j+1}^{n-1+\sigma} \cdot \Delta_{t_{n-1+\sigma}}^{\alpha} u_{j+1}^n - v_j^{n-1+\sigma} \cdot \Delta_{t_{n-1+\sigma}}^{\alpha} u_j^n\big) = 0.
\end{aligned} \tag{3.15}$$

Substituting (3.15) into (3.14), we have

$$\left\| \delta_x v^{n-1+\sigma} \right\|^2 + \left\langle \Delta_{t_{n-1+\sigma}}^{\alpha} v^n, v^{n-1+\sigma} \right\rangle_{\frac{1}{\varphi}} = -\left\langle f^{n-1+\sigma}, \delta_x v^{n-1+\sigma} \right\rangle. \tag{3.16}$$

From Lemma 2.3, we know

$$\begin{aligned}
&\left\langle \Delta_{t_{n-1+\sigma}}^{\alpha} v^n, v^{n-1+\sigma} \right\rangle_{\frac{1}{\varphi}} \\
&\quad = \sum_{j=0}^{M-1} \frac{1}{\varphi(x_{j+\frac{1}{2}})} \cdot \big(\Delta_{t_{n-1+\sigma}}^{\alpha} v_{j+\frac{1}{2}}^n\big) \cdot v_{j+\frac{1}{2}}^{n-1+\sigma} \\
&\quad = \sum_{j=0}^{M-1} \Delta_{t_{n-1+\sigma}}^{\alpha} \left(\frac{v_{j+\frac{1}{2}}^n}{\sqrt{\varphi(x_{j+\frac{1}{2}})}}\right) \cdot \frac{v_{j+\frac{1}{2}}^{n-1+\sigma}}{\sqrt{\varphi(x_{j+\frac{1}{2}})}} \\
&\quad = \sum_{j=0}^{M-1} \Delta_{t_{n-1+\sigma}}^{\alpha} \left(\frac{v_{j+\frac{1}{2}}^n}{\sqrt{\varphi(x_{j+\frac{1}{2}})}}\right) \cdot \left(\sigma \frac{v_{j+\frac{1}{2}}^n}{\sqrt{\varphi(x_{j+\frac{1}{2}})}} + (1-\sigma)\frac{v_{j+\frac{1}{2}}^{n-1}}{\sqrt{\varphi(x_{j+\frac{1}{2}})}}\right) \\
&\quad \geq \frac{1}{2}\sum_{j=0}^{M-1} \Delta_{t_{n-1+\sigma}}^{\alpha} \left(\frac{v_{j+\frac{1}{2}}^n}{\sqrt{\varphi(x_{j+\frac{1}{2}})}}\right)^2 = \frac{1}{2}\Delta_{t_{n-1+\sigma}}^{\alpha} \left\| v^n \right\|_{\frac{1}{\varphi}}^2.
\end{aligned} \tag{3.17}$$

Substituting (3.17) into (3.16), and using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
\left\| \delta_x v^{n-1+\sigma} \right\|^2 + \frac{1}{2}\Delta_{t_{n-1+\sigma}}^{\alpha} \left\| v^n \right\|_{\frac{1}{\varphi}}^2 &\leq -\left\langle f^{n-1+\sigma}, \delta_x v^{n-1+\sigma} \right\rangle \\
&\leq \left\| \delta_x v^{n-1+\sigma} \right\|^2 + \frac{1}{4}\left\| f^{n-1+\sigma} \right\|^2,
\end{aligned}$$

i.e.,

$$\Delta_{t_{n-1+\sigma}}^{\alpha} \left\| v^n \right\|_{\frac{1}{\varphi}}^2 \leq \frac{1}{2}\left\| f^{n-1+\sigma} \right\|^2.$$

That is,

$$C_0^{(n)} \|v^n\|_{\frac{1}{\varphi}}^2 \le \sum_{k=1}^{n-1} \left(C_{n-k-1}^{(n)} - C_{n-k}^{(n)}\right) \|v^k\|_{\frac{1}{\varphi}}^2 + C_{n-1}^{(n)} \|v^0\|_{\frac{1}{\varphi}}^2 + \frac{\mu}{2} \|f^{n-1+\sigma}\|^2, \tag{3.18}$$

where $\mu = \tau^\alpha \cdot \Gamma(2-\alpha)$.

From (2.6) of Lemma 2.2, we know

$$C_{n-1}^{(n)} > \frac{1-\alpha}{2}\left(n-1-\frac{\alpha}{2}\right)^{-\alpha} > \frac{1-\alpha}{2}\left(n-\frac{\alpha}{2}\right)^{-\alpha}, \quad 1 \le n \le N,$$

so that

$$\begin{aligned}\mu &= T^\alpha \cdot \Gamma(1-\alpha)(1-\alpha) \cdot N^{-\alpha} \\ &< T^\alpha \cdot \Gamma(1-\alpha)(1-\alpha)\left(n-\frac{\alpha}{2}\right)^{-\alpha} \\ &< 2C_{n-1}^{(n)} T^\alpha \cdot \Gamma(1-\alpha).\end{aligned} \tag{3.19}$$

Substituting (3.19) into (3.18), we have

$$\begin{aligned}C_0^{(n)} \|v^n\|_{\frac{1}{\varphi}}^2 \le{}& \sum_{k=1}^{n-1} \left(C_{n-k-1}^{(n)} - C_{n-k}^{(n)}\right) \|v^k\|_{\frac{1}{\varphi}}^2 \\ &+ C_{n-1}^{(n)} \left(\|v^0\|_{\frac{1}{\varphi}}^2 + T^\alpha \cdot \Gamma(1-\alpha)\|f^{n-1+\sigma}\|^2\right).\end{aligned} \tag{3.20}$$

Denoting $E = \|v^0\|_{\frac{1}{\varphi}}^2 + T^\alpha \Gamma(1-\alpha) \max_{1\le n\le N} \|f^{n-1+\sigma}\|^2$, now we prove by induction that

$$\|v^n\|_{\frac{1}{\varphi}}^2 \le E, \quad 1 \le n \le N. \tag{3.21}$$

It holds obviously when $n = 1$. Assuming that the conclusion is valid for $n = 1, 2, \ldots, m-1$, i.e.,

$$\|v^n\|_{\frac{1}{\varphi}}^2 \le E, \quad 1 \le n \le m-1,$$

then for $2 \le m \le N$, from (3.20) we have

$$\begin{aligned}C_0^{(m)} \|v^m\|_{\frac{1}{\varphi}}^2 &\le \sum_{k=1}^{m-1} \left(C_{m-k-1}^{(m)} - C_{m-k}^{(m)}\right) \|v^k\|_{\frac{1}{\varphi}}^2 + C_{m-1}^{(m)} E \\ &\le \sum_{k=1}^{m-1} \left(C_{m-k-1}^{(m)} - C_{m-k}^{(m)}\right) E + C_{m-1}^{(m)} E = C_0^{(m)} E.\end{aligned}$$

So (3.21) holds.

From (3.7), we obtain

$$\|v^n\|_{\frac{1}{\varphi}}^2 = \|\delta_x u^n\|_\varphi^2, \quad 0 \le n \le N. \tag{3.22}$$

Substituting (3.22) and (3.3) into (3.21), we obtain (3.10).

Now we estimate $\|u^n\|$.

Multiplying (3.6) and (3.7) by $hu^n_{j+\frac{1}{2}}$ and $hv^{n-1+\sigma}_{j+\frac{1}{2}}$, and summing up for j from 0 to $M-1$, respectively, we have

$$\langle \Delta^\alpha_{t_{n-1+\sigma}} u^n, u^n\rangle = \langle \delta_x v^{n-1+\sigma}, u^n\rangle + \langle f^{n-1+\sigma}, u^n\rangle, \tag{3.23}$$

$$\langle v^n, v^{n-1+\sigma}\rangle_{\frac{1}{\varphi}} = \langle \delta_x u^n, v^{n-1+\sigma}\rangle. \tag{3.24}$$

Adding the two identities above, we have

$$\begin{aligned}&\langle \Delta^\alpha_{t_{n-1+\sigma}} u^n, u^n\rangle + \langle v^n, v^{n-1+\sigma}\rangle_{\frac{1}{\varphi}}\\ &\quad= \langle \delta_x v^{n-1+\sigma}, u^n\rangle + \langle \delta_x u^n, v^{n-1+\sigma}\rangle + \langle f^{n-1+\sigma}, u^n\rangle.\end{aligned} \tag{3.25}$$

Noticing that $v_0^{n-1+\sigma} = v_M^{n-1+\sigma} = 0$, we have

$$\begin{aligned}&\langle \delta_x v^{n-1+\sigma}, u^n\rangle + \langle \delta_x u^n, v^{n-1+\sigma}\rangle\\ &\quad= \sum_{j=0}^{M-1} h\bigl(\delta_x v^{n-1+\sigma}_{j+\frac{1}{2}} \cdot u^n_{j+\frac{1}{2}} + \delta_x u^n_{j+\frac{1}{2}} \cdot u^{n-1+\sigma}_{j+\frac{1}{2}}\bigr)\\ &\quad= \frac{1}{2}\sum_{j=0}^{M-1}\bigl[\bigl(v^{n-1+\sigma}_{j+1} - v^{n-1+\sigma}_j\bigr)\bigl(u^n_{j+1} + u^n_j\bigr) + \bigl(u^n_{j+1} - u^n_j\bigr)\bigl(v^{n-1+\sigma}_{j+1} + v^{n-1+\sigma}_j\bigr)\bigr]\\ &\quad= \sum_{j=0}^{M-1}\bigl(v^{n-1+\sigma}_{j+1} u^n_{j+1} - v^{n-1+\sigma}_j u^n_j\bigr) = v^{n-1+\sigma}_M u^n_M - v^{n-1+\sigma}_0 u^n_0 = 0.\end{aligned}$$

Substituting the result into (3.25) and using the Cauchy-Schwarz inequality, we arrive at

$$\langle \Delta^\alpha_{t_{n-1+\sigma}} u^n, u^n\rangle = -\langle v^n, v^{n-1+\sigma}\rangle_{\frac{1}{\varphi}} + \langle f^{n-1+\sigma}, u^n\rangle \tag{3.26}$$

$$\le \frac{1}{2}\|v^n\|^2_{\frac{1}{\varphi}} + \frac{1}{2}\|v^{n-1+\sigma}\|^2_{\frac{1}{\varphi}} + \langle f^{n-1+\sigma}, u^n\rangle. \tag{3.27}$$

From (3.21) we have

$$\|v^{n-1+\sigma}\|_{\frac{1}{\varphi}} = \|\sigma v^n + (1-\sigma)v^{n-1}\|_{\frac{1}{\varphi}} \le \sigma\|v^n\|_{\frac{1}{\varphi}} + (1-\sigma)\|v^{n-1}\|_{\frac{1}{\varphi}} \le \sqrt{E}. \tag{3.28}$$

Substituting (3.21) and (3.28) into (3.27), we obtain

$$\langle \Delta^\alpha_{t_{n-1+\sigma}} u^n, u^n\rangle \le \langle f^{n-1+\sigma}, u^n\rangle + E,$$

that is,

$$\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\Bigl\langle C_0^{(n)}u^n - \sum_{k=1}^{n-1}\bigl(C^{(n)}_{n-k-1} - C^{(n)}_{n-k}\bigr)u^k - C^{(n)}_{n-1}u^0, u^n\Bigr\rangle \le \langle f^{n-1+\sigma}, u^n\rangle + E,$$

i.e.,

$$C_0^{(n)}\|u^n\|^2 \le \sum_{k=1}^{n-1}\bigl(C^{(n)}_{n-k-1} - C^{(n)}_{n-k}\bigr)\langle u^k, u^n\rangle + C^{(n)}_{n-1}\langle u^0, u^n\rangle + \mu\langle f^{n-1+\sigma}, u^n\rangle + \mu E. \tag{3.29}$$

By the Cauchy-Schwarz inequality we know

$$\langle u^k, u^n\rangle \le \frac{\|u^k\|^2 + \|u^n\|^2}{2}, \qquad \langle u^0, u^n\rangle \le \frac{\|u^n\|^2}{4} + \|u^0\|^2, \tag{3.30}$$

$$\mu\langle f^{n-1+\sigma}, u^n\rangle \le \frac{C_{n-1}^{(n)}}{4}\|u^n\|^2 + \frac{\mu^2}{C_{n-1}^{(n)}}\|f^{n-1+\sigma}\|^2. \tag{3.31}$$

Substituting (3.30) and (3.31) into (3.29), we arrive at

$$\begin{aligned} C_0^{(n)}\|u^n\|^2 \le{}& \sum_{k=1}^{n-1}\left(C_{n-k-1}^{(n)} - C_{n-k}^{(n)}\right)\frac{\|u^k\|^2 + \|u^n\|^2}{2} + \frac{C_{n-1}^{(n)}}{4}\|u^n\|^2 \\ &+ C_{n-1}^{(n)}\|u^0\|^2 + \frac{C_{n-1}^{(n)}}{4}\|u^n\|^2 + \frac{\mu^2}{C_{n-1}^{(n)}}\|f^{n-1+\sigma}\|^2 + \mu E. \end{aligned} \tag{3.32}$$

According to (3.19), we know $\frac{\mu^2}{C_{n-1}^{(n)}} \le 4C_{n-1}^{(n)}[T^\alpha\Gamma(1-\alpha)]^2$. Substituting it into the inequality above, we have

$$\begin{aligned} C_0^{(n)}\|u^n\|^2 \le{}& \sum_{k=1}^{n-1}\left(C_{n-k-1}^{(n)} - C_{n-k}^{(n)}\right)\|u^k\|^2 + 2C_{n-1}^{(n)}\|u^0\|^2 \\ &+ 8C_{n-1}^{(n)}\left[T^\alpha\Gamma(1-\alpha)\right]^2\cdot\|f^{n-1+\sigma}\|^2 + 4C_{n-1}^{(n)}T^\alpha\Gamma(1-\alpha)E. \end{aligned} \tag{3.33}$$

Let

$$G = 2\|u^0\|^2 + 8\left[T^\alpha\Gamma(1-\alpha)\right]^2\cdot\max_{1\le n\le N}\|f^{n-1+\sigma}\|^2 + 4T^\alpha\Gamma(1-\alpha)E \tag{3.34}$$

$$= 2\|u^0\|^2 + 4T^\alpha\Gamma(1-\alpha)\|v^0\|_{\frac{1}{\varphi}}^2 + 12\left[T^\alpha\Gamma(1-\alpha)\right]^2\max_{1\le n\le N}\|f^{n-1+\sigma}\|^2, \tag{3.35}$$

then applying the similar induction process again, we can easily get

$$\|u^n\|^2 \le G.$$

That is (3.11), the proof is completed. □

We have got the estimation of $\|u^n\|^2$ and $\|\delta_x u^n\|^2$, which leads to the estimation of $\|u^n\|_\infty$ by virtue of Lemma 3.3. That means the difference scheme (2.25)-(2.28) is stable to the initial value and the right-hand term.

Next, the convergence of the finite difference scheme (2.25)-(2.28) can be drawn. Denote $e_j^n = U_j^n - u_j^n$, $0 \le j \le M$, $0 \le n \le N$.

Theorem 3.5 (Convergence) *Suppose* $u(x,t) \in C_{x,t}^{(4,3)}([0,L]\times[0,T])$, $\{U_j^n|0 \le j \le M, 0 \le n \le N\}$, $\{u_j^n \mid 0 \le j \le M, 0 \le n \le N\}$ *are the solutions of problem* (2.1)-(2.3) *and the finite difference scheme* (2.25)-(2.28), *respectively. Then there exists a constant C such that*

$$\|e^n\|_\infty \le C\left(\tau^2 + h^2\right), \quad 0 \le n \le N. \tag{3.36}$$

Proof Denote $\xi_j^n = V_j^n - v_j^n$, $0 \le j \le M$, $0 \le n \le N$. Subtracting (2.21)-(2.24) from (2.16)-(2.20), respectively, we obtain the corresponding error equations

$$\Delta_{t_{n-1+\sigma}}^{\alpha} e_{j+\frac{1}{2}}^n = \delta_x \xi_{j+\frac{1}{2}}^{n-1+\sigma} + (R_1)_{j+\frac{1}{2}}^n, \quad 0 \le j \le M-1, 1 \le n \le N, \tag{3.37}$$

$$\xi_{j+\frac{1}{2}}^n = \varphi(x_{j+\frac{1}{2}})\delta_x e_{j+\frac{1}{2}}^n + (R_2)_{j+\frac{1}{2}}^n, \quad 0 \le j \le M-1, 0 \le n \le N, \tag{3.38}$$

$$\xi_0^n = \xi_M^n = 0, \quad 0 \le n \le N, \tag{3.39}$$

$$e_j^0 = 0, \quad 0 \le j \le M. \tag{3.40}$$

Firstly, we estimate $\|\delta_x e^n\|$.

Implementing the fractional derivative operator $\Delta_{t_{n-1+\sigma}}^{\alpha}$ on the both sides of (2.22) leads to

$$\Delta_{t_{n-1+\sigma}}^{\alpha} v_{j+\frac{1}{2}}^n = \Delta_{t_{n-1+\sigma}}^{\alpha} \varphi(x_{j+\frac{1}{2}})\delta_x u_{j+\frac{1}{2}}^n, \quad 0 \le j \le M-1, 0 \le n \le N, \tag{3.41}$$

which can be regarded as the discretion of the equation

$${}_0^C\mathcal{D}_t^{\alpha} v = {}_0^C\mathcal{D}_t^{\alpha} \varphi \frac{\partial u}{\partial x}. \tag{3.42}$$

(3.42) can be obtained by implementing the Caputo derivative on the both sides of (2.9).

Using Taylor expansion and Lemma 2.1, we can easily obtain

$$\Delta_{t_{n-1+\sigma}}^{\alpha} V_{j+\frac{1}{2}}^n = \Delta_{t_{n-1+\sigma}}^{\alpha} \big(\varphi(x_{j+\frac{1}{2}})\delta_x U_{j+\frac{1}{2}}^n\big) + (\hat{R}_2)_{j+\frac{1}{2}}^n, \tag{3.43}$$

and there exists a positive constant $\hat{C}_R$ such that

$$\big|(\hat{R}_2)_{j+\frac{1}{2}}^n\big| \le \hat{C}_R\big(\tau^2 + h^2\big), \quad 0 \le j \le M-1, 0 \le n \le N. \tag{3.44}$$

Subtracting (3.41) from (3.43), we obtain

$$\begin{aligned} &\Delta_{t_{n-1+\sigma}}^{\alpha} \xi_{j+\frac{1}{2}}^n = \Delta_{t_{n-1+\sigma}}^{\alpha} \big(\varphi(x_{j+\frac{1}{2}})\delta_x e_{j+\frac{1}{2}}^n\big) + (\hat{R}_2)_{j+\frac{1}{2}}^n, \\ &\quad 0 \le j \le M-1, 0 \le n \le N. \end{aligned} \tag{3.45}$$

Multiplying (3.37) and (3.45) by $h\delta_x \xi_{j+\frac{1}{2}}^{n+1-\sigma}$ and $h\xi_{j+\frac{1}{2}}^{n-1+\sigma}$, respectively, and summing up for j from 0 to $M-1$, respectively, we have

$$\big\langle \Delta_{t_{n-1+\sigma}}^{\alpha} e^n, \delta_x \xi^{n-1+\sigma}\big\rangle = \big\|\delta_x \xi^{n-1+\sigma}\big\|^2 + \big\langle R_1^n, \delta_x \xi^{n-1+\sigma}\big\rangle, \tag{3.46}$$

$$\big\langle \Delta_{t_{n-1+\sigma}}^{\alpha} \xi^n, \xi^{n-1+\sigma}\big\rangle_{\frac{1}{\varphi}} = \big\langle \Delta_{t_{n-1+\sigma}}^{\alpha} \delta_x e^n, \xi^{n-1+\sigma}\big\rangle + \big\langle \hat{R}_2^n, \xi^{n-1+\sigma}\big\rangle_{\frac{1}{\varphi}}. \tag{3.47}$$

Noticing that $\xi_0^n = \xi_M^n = 0$, we have

$$\begin{aligned} &\big\langle \Delta_{t_{n-1+\sigma}}^{\alpha} e^n, \delta_x \xi^{n-1+\sigma}\big\rangle + \big\langle \Delta_{t_{n-1+\sigma}}^{\alpha} \delta_x e^n, \xi^{n-1+\sigma}\big\rangle \\ &\quad = \Delta_{t_{n-1+\sigma}}^{\alpha} e_M^n \cdot \xi_M^{n-1+\sigma} - \Delta_{t_{n-1+\sigma}}^{\alpha} e_0^n \cdot \xi_0^{n-1+\sigma} = 0. \end{aligned} \tag{3.48}$$

Adding (3.46) and (3.47), then applying (3.48), we obtain

$$\left\|\delta_x\xi^{n-1+\sigma}\right\|^2 + \left\langle\Delta^{\alpha}_{t_{n-1+\sigma}}\xi^n,\xi^{n-1+\sigma}\right\rangle_{\frac{1}{\varphi}} = \left\langle\hat{R}^n_2,\xi^{n-1+\sigma}\right\rangle_{\frac{1}{\varphi}} - \left\langle R^n_1,\delta_x\xi^{n-1+\sigma}\right\rangle. \tag{3.49}$$

By the arguments similar to those given in (3.17) and using Lemma 2.3, we have

$$\left\langle\Delta^{\alpha}_{t_{n-1+\sigma}}\xi^n,\xi^{n-1+\sigma}\right\rangle_{\frac{1}{\varphi}} \ge \frac{1}{2}\Delta^{\alpha}_{t_{n-1+\sigma}}\left\|\xi^n\right\|^2_{\frac{1}{\varphi}},$$

so that

$$\left\|\delta_x\xi^{n-1+\sigma}\right\|^2 + \frac{1}{2}\Delta^{\alpha}_{t_{n-1+\sigma}}\left\|\xi^n\right\|^2_{\frac{1}{\varphi}} \le \left\langle\hat{R}^n_2,\xi^{n-1+\sigma}\right\rangle_{\frac{1}{\varphi}} - \left\langle R^n_1,\delta_x\xi^{n-1+\sigma}\right\rangle. \tag{3.50}$$

Using the Cauchy-Schwarz inequality and according to (3.2) of Lemma 3.1, we arrive at

$$\begin{aligned}&\left\|\delta_x\xi^{n-1+\sigma}\right\|^2 + \frac{1}{2}\Delta^{\alpha}_{t_{n-1+\sigma}}\left\|\xi^n\right\|^2_{\frac{1}{\varphi}}\\ &\quad\le \frac{L^2}{12}\left\|\frac{\hat{R}^n_2}{\varphi}\right\|^2 + \frac{1}{2}\left\|\delta_x\xi^{n-1+\sigma}\right\|^2 + \frac{1}{2}\left\|R^n_1\right\|^2 + \frac{1}{2}\left\|\delta_x\xi^{n-1+\sigma}\right\|^2,\end{aligned}$$

i.e.,

$$C^{(n)}_0\left\|\xi^n\right\|^2_{\frac{1}{\varphi}} \le \sum_{k=1}^{n-1}\left(C^{(n)}_{n-k-1} - C^{(n)}_{n-k}\right)\left\|\xi^k\right\|^2_{\frac{1}{\varphi}} + C^{(n)}_{n-1}\left\|\xi^0\right\|^2_{\frac{1}{\varphi}} \tag{3.51}$$

$$\qquad + \left(\left\|R^n_1\right\|^2 + \frac{L^2}{6}\left\|\frac{\hat{R}^n_2}{\varphi}\right\|^2\right)\mu. \tag{3.52}$$

Taking $n = 0$ in (3.38) and applying (3.40), we know

$$\xi^0_{j+\frac{1}{2}} = (R_2)^0_{j+\frac{1}{2}}. \tag{3.53}$$

Since $0 < C_1 \le \varphi(x) \le C_2$, we know $0 < \frac{1}{C_2} \le \frac{1}{\varphi(x)} \le \frac{1}{C_1}$. Similar to Lemma 3.2, it is not hard to verify

$$\frac{1}{\sqrt{C_2}}\|u\| \le \|u\|_{\frac{1}{\varphi}} \le \frac{1}{\sqrt{C_1}}\|u\|,$$

here $u \in \mathcal{V}_h$. From this and (3.53), we arrive at

$$\left\|\xi^0\right\|^2_{\frac{1}{\varphi}} \le \frac{1}{C_1}\left\|\xi^0\right\|^2 = \frac{1}{C_1}\left\|R^0_2\right\|^2. \tag{3.54}$$

Substituting (3.54), (2.18) and (3.44) into (3.51), we arrive at

$$\begin{aligned}C^{(n)}_0\left\|\xi^n\right\|^2_{\frac{1}{\varphi}} &\le \sum_{k=1}^{n-1}\left(C^{(n)}_{n-k-1} - C^{(n)}_{n-k}\right)\left\|\xi^k\right\|^2_{\frac{1}{\varphi}} + C^{(n)}_{n-1}\cdot\frac{1}{C_1}\cdot LC^2_R\left(\tau^2+h^2\right)^2\\ &\quad + \left(LC^2_R\left(\tau^2+h^2\right)^2 + \frac{L^2}{6}\cdot\frac{1}{C^2_1}\cdot L\hat{C}^2_R\left(\tau^2+h^2\right)^2\right)\mu.\end{aligned} \tag{3.55}$$

Noticing (3.19), we obtain

$$C_0^{(n)}\|\xi^n\|_{\frac{1}{\varphi}}^2 \le \sum_{k=1}^{n-1}\left(C_{n-k-1}^{(n)} - C_{n-k}^{(n)}\right)\|\xi^k\|_{\frac{1}{\varphi}}^2 + C_{n-1}^{(n)}\left[\frac{LC_R^2}{C_1}\right. \tag{3.56}$$

$$\left. + 2\left(LC_R^2 + \frac{L^3\hat{C}_R^2}{6C_1^2}\right)T^\alpha\Gamma(1-\alpha)\right]\left(\tau^2 + h^2\right)^2. \tag{3.57}$$

Let

$$C_3 = \frac{LC_R^2}{C_1} + 2\left(LC_R^2 + \frac{L^3\hat{C}_R^2}{6C_1^2}\right)T^\alpha\Gamma(1-\alpha),$$

and carry out the induction process which is similar to that in Theorem 3.4 again, we can prove that

$$\|\xi^n\|_{\frac{1}{\varphi}}^2 \le C_3\left(\tau^2 + h^2\right)^2. \tag{3.58}$$

Noticing (3.39), we know

$$\|\delta_x e^n\|^2 = \left\|\frac{\xi^n - R_2^n}{\varphi}\right\|^2 \le \frac{1}{C_1^2}\left(2\|\xi^n\|^2 + 2\|R_2^n\|^2\right).$$

According to Lemma 3.2, (3.58) and (2.18), we obtain

$$\|\delta_x e^n\|^2 \le C_4\left(\tau^2 + h^2\right)^2, \tag{3.59}$$

where $C_4 = \frac{2}{C_1^2}(C_2 \cdot C_3 + LC_R^2)$.

We now estimate $\|e^n\|$ by the following analysis.

Multiplying (3.37) and (3.38) by $he_{j+\frac{1}{2}}^n$ and $h\xi_{j+\frac{1}{2}}^{n-1+\sigma}$, respectively, and summing up for j from 0 to $M-1$, respectively, we obtain

$$\left\langle \Delta_{t_{n-1+\sigma}}^\alpha e^n, e^n\right\rangle = \left\langle \delta_x\xi^{n-1+\sigma}, e^n\right\rangle + \left\langle R_1^n, e^n\right\rangle, \tag{3.60}$$

$$\left\langle \xi^n, \xi^{n-1+\sigma}\right\rangle_{\frac{1}{\varphi}} = \left\langle \delta_x e^n, \xi^{n-1+\sigma}\right\rangle + \left\langle R_2^n, \xi^{n-1+\sigma}\right\rangle_{\frac{1}{\varphi}}. \tag{3.61}$$

Noticing that $\xi_0^{n-1+\sigma} = \xi_M^{n-1+\sigma} = 0$, we have

$$\left\langle \delta_x\xi^{n-1+\sigma}, e^n\right\rangle + \left\langle \delta_x e^n, \xi^{n-1+\sigma}\right\rangle = e_M^n\xi_M^{n-1+\sigma} - e_0^n\xi_0^{n-1+\sigma} = 0. \tag{3.62}$$

Adding (3.60) and (3.61), then applying (3.62), we obtain

$$\left\langle \Delta_{t_{n-1+\sigma}}^\alpha e^n, e^n\right\rangle + \left\langle \xi^n, \xi^{n-1+\sigma}\right\rangle_{\frac{1}{\varphi}} = \left\langle R_1^n, e^n\right\rangle + \left\langle R_2^n, \xi^{n-1+\sigma}\right\rangle_{\frac{1}{\varphi}}. \tag{3.63}$$

Transposing $\langle \xi^n, \xi^{n-1+\sigma}\rangle_{\frac{1}{\varphi}}$ into the right-hand side of the identity above, then using the Cauchy-Schwarz inequality, we get

$$\left\langle \Delta_{t_{n-1+\sigma}}^\alpha e^n, e^n\right\rangle \le \left\langle R_1^n, e^n\right\rangle + \frac{1}{2}\|\xi^{n-1+\sigma}\|_{\frac{1}{\varphi}}^2 + \frac{1}{2}\|R_2^n\|_{\frac{1}{\varphi}}^2$$

$$+ \frac{1}{2}\|\xi^n\|_{\frac{1}{\varphi}}^2 + \frac{1}{2}\|\xi^{n-1+\sigma}\|_{\frac{1}{\varphi}}^2. \tag{3.64}$$

From (3.28) and (3.58) we know

$$\left\|\xi^{n-1+\sigma}\right\|_{\frac{1}{\varphi}}^2 \le C_3\left(\tau^2+h^2\right)^2. \tag{3.65}$$

Substituting (3.58), (3.65) and (2.18) into (3.64), we obtain

$$\left\langle \Delta_{t_{n-1+\sigma}}^{\alpha} e^n, e^n\right\rangle \le \left\langle R_1^n, e^n\right\rangle + \frac{1}{2C_1} L C_R^2\left(\tau^2+h^2\right)^2 + \frac{3}{2} C_3\left(\tau^2+h^2\right)^2, \tag{3.66}$$

i.e.,

$$C_0^{(n)}\left\|e^n\right\|^2 \le \sum_{k=1}^{n-1}\left(C_{n-k-1}^{(n)} - C_{n-k}^{(n)}\right)\left\langle e^k, e^n\right\rangle + C_{n-1}^{(n)}\left\langle e^0, e^n\right\rangle \tag{3.67}$$

$$+ \mu\left\langle R_1^n, e^n\right\rangle + \mu\left(\frac{1}{2C_1} L C_R^2\left(\tau^2+h^2\right)^2 + \frac{3}{2} C_3\left(\tau^2+h^2\right)^2\right). \tag{3.68}$$

Using the Cauchy-Schwarz inequality and (2.18) again, we obtain

$$\begin{aligned} C_0^{(n)}\left\|e^n\right\|^2 \le{}& \sum_{k=1}^{n-1}\left(C_{n-k-1}^{(n)} - C_{n-k}^{(n)}\right)\frac{\|e^k\|^2+\|e^n\|^2}{2} + \frac{C_{n-1}^{(n)}}{4}\left\|e^n\right\|^2 + C_{n-1}^{(n)}\left\|e^0\right\|^2 \\ &+ \frac{C_{n-1}^{(n)}}{4}\left\|e^n\right\|^2 + \frac{\mu^2}{C_{n-1}^{(n)}}\cdot L C_R^2\left(\tau^2+h^2\right)^2 \\ &+ \mu\left(\frac{1}{2C_1} L C_R^2\left(\tau^2+h^2\right)^2 + \frac{3}{2} C_3\left(\tau^2+h^2\right)^2\right). \end{aligned} \tag{3.69}$$

From (3.40) and (3.19), we have

$$\begin{aligned} C_0^{(n)}\left\|e^n\right\|^2 \le{}& \sum_{k=1}^{n-1}\left(C_{n-k-1}^{(n)} - C_{n-k}^{(n)}\right)\left\|e^k\right\|^2 \\ &+ 8C_{n-1}^{(n)}\left[T^\alpha\Gamma(1-\alpha)\right]^2\cdot L C_R^2\left(\tau^2+h^2\right)^2 \\ &+ 4C_{n-1}^{(n)} T^\alpha\Gamma(1-\alpha)\left[\frac{1}{2C_1} L C_R^2\left(\tau^2+h^2\right)^2 + \frac{3}{2} C_3\left(\tau^2+h^2\right)^2\right]. \end{aligned}$$

Let

$$C_5 = 8\left[T^\alpha\Gamma(1-\alpha)\right]^2\cdot L C_R^2 + \frac{2}{C_1} T^\alpha\Gamma(1-\alpha) L C_R^2 + 6C_3 T^\alpha\Gamma(1-\alpha),$$

and apply the mathematic induction method again, then we can prove that

$$\left\|e^n\right\|^2 \le C_5\left(\tau^2+h^2\right)^2. \tag{3.70}$$

Now, according to Lemma 3.3, (3.59) and (3.70), the proof is completed ultimately. □

4 Numerical examples

In this section, we carry out numerical experiments to testify the efficiency and convergence orders of our new developed box-type scheme (2.25)-(2.28) for problem (2.1)-(2.3).

Table 1 The numerical convergence orders in temporal direction with $h = \frac{1}{3{,}000}$

τ	$\alpha = 0.2$		$\alpha = 0.5$		$\alpha = 0.8$	
	$E_\infty(h,\tau)$	Order(τ)	$E_\infty(h,\tau)$	Order(τ)	$E_\infty(h,\tau)$	Order(τ)
1/4	1.0303e–001	*	1.9759e–001	*	2.0353e–001	*
1/8	2.7328e–002	1.9146	5.2296e–002	1.9177	5.2924e–002	1.9432
1/16	7.0561e–003	1.9534	1.3532e–002	1.9503	1.3588e–002	1.9616
1/32	1.7959e–003	1.9742	3.4573e–003	1.9687	3.4673e–003	1.9704

All our tests were done in MATLAB. The maximum norm errors between the exact and the numerical solutions are denoted by

$$E_\infty(h,\tau) = \max_{1\le n\le N} \left\| u^n - U^n \right\|_\infty.$$

Furthermore, the temporal and spatial convergence orders are defined respectively by

$$\text{Order}(\tau) = \log_2\left(\frac{E_\infty(h,2\tau)}{E_\infty(h,\tau)}\right), \qquad \text{Order}(h) = \log_2\left(\frac{E_\infty(2h,\tau)}{E_\infty(h,\tau)}\right),$$

where τ and h are sufficiently small.

Firstly, we consider the following problem with zero initial value.

Example 1 Let $L = T = 1$, and take $\varphi(x) = e^x$. We consider the following problem:

$$\begin{aligned} &{}_0^C\mathcal{D}_t^\alpha u(x,t) = \frac{\partial}{\partial x}\left(e^x \frac{\partial u}{\partial x}\right) + e^x \frac{\Gamma(4+\alpha)}{6} t^3 - 2e^{2x} t^{3+\alpha}, \\ &\quad 0 < x < 1, 0 < t \le 1, \end{aligned} \tag{4.1}$$

$$u(x,0) = 0, \quad 0 < x < 1, \tag{4.2}$$

$$u_x(0,t) = t^{3+\alpha}, \qquad u_x(L,t) = et^{3+\alpha}, \quad 0 \le t \le 1. \tag{4.3}$$

The exact solution is $u(x,t) = e^x t^{3+\alpha}$.

We solve the problem with the proposed box-type scheme (2.25)-(2.28). Firstly, the numerical accuracy of this scheme in temporal direction is tested by taking a sufficiently small spatial step $h = 1/3{,}000$ and taking $\alpha = 0.2, 0.5, 0.8$, respectively. We present the computational errors and temporal convergence orders in the maximum norm in Table 1. We can see that our scheme generates the temporal convergence order of nearly $O(\tau^2)$. Secondly, the numerical accuracy of the scheme in spacial direction is verified by the example. We fix a sufficiently small temporal step size $\tau = 1/10{,}000$ and take different values of α again. Table 2 shows the errors and the spatial convergence orders for different spatial mesh sizes. The results are also in good agreement with our theoretical analysis.

In Figures 1 and 2, we plot the error ($|u(x_i,t_n) - u_i^n|$) surface figures with different mesh sizes by taking $\alpha = 0.2$, 0.8, respectively. We find that the maximum error becomes relatively smaller as the mesh size becomes smaller in these figures, which provides the validation of our results once again.

Secondly, we consider an example with nonzero initial value.

Table 2 The numerical convergence orders in spatial direction with $\tau = \frac{1}{10{,}000}$

h	$\alpha = 0.2$		$\alpha = 0.5$		$\alpha = 0.8$	
	$E_\infty(h,\tau)$	Order(h)	$E_\infty(h,\tau)$	Order(h)	$E_\infty(h,\tau)$	Order(h)
1/8	1.6761e–002	*	1.1880e–002	*	8.3067e–003	*
1/16	4.1978e–003	1.9974	2.9752e–003	1.9975	2.0799e–003	1.9978
1/32	1.0499e–003	1.9994	7.4416e–004	1.9993	5.2021e–004	1.9993
1/64	2.6253e–004	1.9997	1.8609e–004	1.9996	1.3010e–004	1.9995

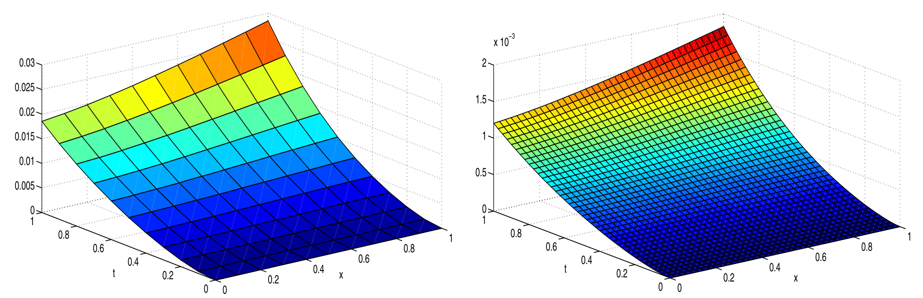

Figure 1 The error surface figures with $h = \tau = \frac{1}{10}$ (left) and $h = \tau = \frac{1}{40}$ (right) when $\alpha = 0.2$.

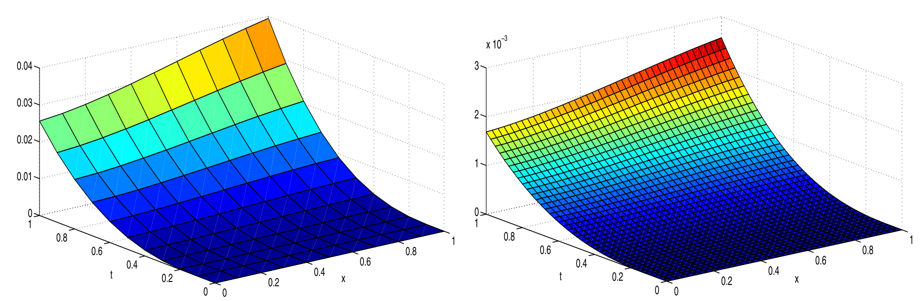

Figure 2 The error surface figures with $h = \tau = \frac{1}{10}$ (left) and $h = \tau = \frac{1}{40}$ (right) when $\alpha = 0.8$.

Example 2 Let $L = T = 1$, and take $\varphi(x) = x^2 + 1$. We consider the following problem:

$$\begin{aligned}
{}_0^C\mathcal{D}_t^\alpha u(x,t) &= \frac{\partial}{\partial x}\left(\left(x^2+1\right)\frac{\partial u}{\partial x}\right) + \cos(\pi x)\frac{\Gamma(4+\alpha)}{6}t^3 + \pi\left(t^{3+\alpha}+1\right) \\
&\quad \cdot \left[2x\sin(\pi x) + \pi\cos(\pi x)\left(x^2+1\right)\right], \quad 0 \le x \le 1, 0 < t \le 1,
\end{aligned} \tag{4.4}$$

$$u(x,0) = \cos(\pi x), \quad 0 \le x \le 1, \tag{4.5}$$

$$u_x(0,t) = 0, \qquad u_x(L,t) = 0, \quad 0 \le t \le 1. \tag{4.6}$$

The exact solution is $u(x,t) = \cos(\pi x)(t^{3+\alpha} + 1)$.

We solve the problem with the box-type scheme (2.25)-(2.28). Firstly, the numerical accuracy of this scheme in temporal direction is tested by taking a sufficiently small spatial step $h = 1/3{,}000$ and taking $\alpha = 0.1, 0.5, 0.9$, respectively. We list the computational errors and temporal convergence orders in the maximum norm in Table 3. We find that our scheme generates the temporal convergence order of nearly $O(\tau^2)$. Secondly, the numer-

Table 3 The numerical convergence orders in temporal direction with $h = \frac{1}{3,000}$

τ	$\alpha = 0.1$		$\alpha = 0.5$		$\alpha = 0.9$	
	$E_\infty(h,\tau)$	Order(τ)	$E_\infty(h,\tau)$	Order(τ)	$E_\infty(h,\tau)$	Order(τ)
1/4	8.2922e–003	*	4.4237e–002	*	7.7457e–002	*
1/8	2.1607e–003	1.9403	1.1423e–002	1.9533	1.9665e–002	1.9778
1/16	5.5117e–004	1.9709	2.8960e–003	1.9798	4.9383e–003	1.9935
1/32	1.3948e–004	1.9824	7.2848e–004	1.9911	1.2356e–003	1.9988

Table 4 The numerical convergence orders in spatial direction with $\tau = \frac{1}{10,000}$

h	$\alpha = 0.1$		$\alpha = 0.5$		$\alpha = 0.9$	
	$E_\infty(h,\tau)$	Order(h)	$E_\infty(h,\tau)$	Order(h)	$E_\infty(h,\tau)$	Order(h)
1/8	7.8990e–002	*	6.9592e–002	*	5.8868e–002	*
1/16	1.9618e–002	2.0095	1.7280e–002	2.0098	1.4612e–002	2.0103
1/32	4.8964e–003	2.0024	4.3128e–003	2.0024	3.6464e–003	2.0026
1/64	1.2236e–003	2.0006	1.0778e–003	2.0005	9.1120e–004	2.0006

ical accuracy of the scheme in spacial direction is verified by the example. We fix a sufficiently small temporal step size $\tau = 1/10,000$ and take different values of α again. Table 4 shows the errors and the spatial convergence orders for different spatial mesh sizes. The convergence orders of the numerical results are also in accordance with our theoretical analysis.

5 Conclusion

In this manuscript, we construct a box-type difference scheme with convergence order $O(\tau^2 + h^2)$ for the fractional sub-diffusion equation with spatially variable coefficient under Neumann boundary conditions. The scheme is established by introducing the auxiliary variable and applying the $L2-1_\sigma$ formula to approximate the time Caputo fractional derivative. With the help of the special properties of the $L2-1_\sigma$ formula and the mathematical induction method, we give the detailed deduction of unconditional stability and convergence for our scheme by the discrete energy method. Numerical examples are carried out to verify the theoretical analysis. It is meaningful to construct a $O(\tau^2 + h^4)$ accuracy difference scheme for this problem, which will be our work in the future.

Competing interests
The author declares that there is no conflict of interests regarding the publication of this paper.

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