Jin and Sun *Advances in Difference Equations* (2017) 2017:146
DOI 10.1186/s13662-017-1170-x

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH

Open Access

On a coupled system of fractional compartmental models for a biological system

Nana Jin and Shurong Sun*

*Correspondence:
sshrong@163.com
School of Mathematical Sciences,
University of Jinan, Jinan, Shandong
250022, P.R. China

Abstract

In this paper, we investigate the existence of solutions for a coupled system of fractional compartmental models as differential inclusions with coupled nonlocal and integral boundary conditions, whose multivalued terms depend on lower-order fractional derivatives. By means of nonlinear alternative of Leray-Schauder type, continuous and measurable selection theorems together with Leray-Schauder degree theory, some sufficient conditions for the existence of solutions are presented, which extend some known results. Several examples are given to demonstrate the application of our main results.

MSC: 26A33; 34A12; 34A60; 34B10; 34A34

Keywords: fractional differential inclusions; nonlocal boundary value problems; existence of solutions; coupled system; compartmental model

1 Introduction

Fractional calculus has emerged as an interesting field of investigation in the last few decades. Fractional differential equations and inclusions arise in the mathematical modeling of systems and processes occurring in many engineering and scientific disciplines such as physics, chemistry, viscoelasticity, electrochemistry, electromagnetism, aerodynamics, economics, polymer rheology, control theory, signal and image processing, biophysics, blood flow phenomena, etc. [1–4]. In consequence, the subject of fractional differential equations and inclusions is gaining much importance and attention. For details and examples, see [5–12] and the references therein.

On the other hand, coupled systems of fractional differential equations arise in various problems of applied nature, for example, HIV is a retrovirus that targets the $CD4^+$ T lymphocytes, which are the most abundant white blood cells of the immune system. Perelson [13, 14] developed a simple model for the primary infection with HIV. In this model, four categories of cells were defined: uninfected $CD4^+$ T cells, latently infected $CD4^+$ T cells, productively infected $CD4^+$ T cells and virus population. AAM Arafal et al. introduced fractional-order model of infection of $CD4^+$ T-cells. The coupled system is described by the following set of fractional ordinary differential equations of order $\alpha_1, \alpha_2, \alpha_3 > 0$:

$$\begin{cases} D^{\alpha_1}(T) = s - KVT - dT + bI, \\ D^{\alpha_2}(I) = KVT - (b+\delta)I, \\ D^{\alpha_3}(I) = N\delta I - cV, \end{cases}$$

where T, I and V denote the concentration of uninfected CD4$^+$ T cells, infected CD4$^+$ T cells, and free HIV virus particles in the blood, respectively. δ represents the death rate of infected T cells and includes the possibility of death by bursting of infected T cells, hence δd. The parameter b is the rate at which infected cells return to uninfected class, while c is the death rate of virus and N is the average number of viral particles produced by an infected cell. Besides, [15–19] have established the existence and uniqueness for solutions of some systems of nonlinear fractional differential equations.

Compartmental analysis initially developed from studies of the uptake and distribution of radioactive tracers, but today it plays a fundamental role in many parts of medicine, bioengineering and environmental science [20–22]. Compartmental models of pharmacokinetics were recently generalized using fractional calculus to extend the governing systems for the form of fractional-order differential equations with specified initial conditions [23, 24].

In 2011, Ivo Petráš and Richard L. Magin [25] considered the two-compartmental pharmacokinetic model for oral drug administration as follows:

$$\begin{cases} {}^cD^{\alpha_1}q_1(t) = -k_1q_1(t), \\ {}^cD^{\alpha_2}q_2(t) = k_2q_1(t) + k_3q_2(t), \end{cases} \tag{1.1}$$

with the initial condition

$$q_1(0) = d_1, \qquad q_2(0) = d_2, \tag{1.2}$$

where ${}^cD^{\alpha_i}$ $(i = 1,2)$ denotes the Caputo fractional derivative, $\alpha_1, \alpha_2 > 0$, $t \geq 0$, $q_i(t)$ $(i = 1,2)$ denotes the amount of drug in a specific compartment, k_i $(i = 1,2)$ is the fractional rate of transfer to compartment. In their results, they assumed $q_1'(0) = q_2'(0) = 0$ if $\alpha_1, \alpha_2 \in (0,2]$. They pointed out a very effective numerical method for the solution of system (1.1)-(1.2), and the numerical solution is also created as a Matlab function.

Let $q_1(t) \in F(t, q_1(t), q_2(t), {}^cD^\gamma q_2(t))$ and $q_2(t) \in G(t, q_1(t), {}^cD^\delta q_1(t), q_2(t))$. We get the converted coupled system of nonlinear fractional differential inclusions:

$$\begin{cases} {}^cD^\alpha x(t) \in F(t, x(t), y(t), {}^cD^\gamma y(t)), \quad 1 < \alpha \leq 2, 0 < \gamma < 1, \\ {}^cD^\beta y(t) \in G(t, x(t), {}^cD^\delta x(t), y(t)), \quad 1 < \beta \leq 2, 0 < \delta < 1, \end{cases} \tag{1.3}$$

supplemented with coupled nonlocal and integral boundary conditions of the form

$$\begin{cases} x(0) = h(y), \qquad \int_0^T y(s)\,ds = \mu_1 x(\eta), \quad \eta \in (0,T), \\ y(0) = \phi(x), \qquad \int_0^T x(s)\,ds = \mu_2 y(\xi), \quad \xi \in (0,T), \end{cases} \tag{1.4}$$

where $t \in J := [0, T]$, ${}^cD^i$ denotes the Caputo fractional derivatives of order i, $i = \alpha, \beta, \gamma, \delta$ respectively, $F, G : J \times \mathbb{R}^3 \to \mathcal{P}(\mathbb{R})$, h, ϕ are given continuous functionals and μ_1, μ_2 are real constants.

To the best of our knowledge, there are few people to study the coupled system of fractional differential inclusions for compartmental models. For the part of theoretical results, our basic tools are the theory of fractional calculus, the methods and results for differential inclusions, and several fixed point theorems for multifunctions due to [26, 27].

Next, we compare our theoretical results with the other literature in details as follows:

(i) Our approach is adapted to the case when the right-hand sides have convex values as well as nonconvex for system (1.3)-(1.4), which was not considered in [13, 14, 18, 19].

(ii) The present work is an extension of a recent paper of the authors [19], which was considered for a single-valued case. This adds to the uncertainty about the unknown function.

(iii) The fractional compartment form of (1.1)-(1.2) is a particular case of the coupled system (1.3)-(1.4) if we take $F = \{-k_1 x(t)\}$, $G = \{k_1 x(t) + k_2 y(t)\}$, $h(y) = \phi(x) = \mu_1 = \mu_2 = 0$.

(iv) Coupled system (1.3) is called a commensurate-order system if $\alpha = \beta$, otherwise it is an incommensurate-order system [25].

(v) The coupled system for a single-valued map [19] is a particular case of the corresponding multivalued map if we take $F = \{f\}$, $G = \{g\}$ and f, g are continuous functions.

(vi) We adopt the ideas of selection theorems to reduce the condition that the right-hand side has convex values.

(vii) The mentioned methods are also useful for further investigations concerning the existence of solutions of a coupled system of fractional differential inclusions and other types.

The structure of this paper is as follows. In Section 2, we introduce some notations, definitions of fractional calculus and multivalued maps, together with some basic lemmas to prove our main results. In Section 3, we will consider the sufficient conditions for the existence results. Finally, in Section 4, several examples are given to illustrate our main results.

2 Preliminaries

This section is devoted to presenting some notation and preliminary lemmas that will be used in the proofs of the main results [28, 29].

Let $(X, \|\cdot\|)$ be a normed space, and let Y be a subset of X. We denote

(i) $\mathcal{P}(X) = \{Y \subseteq X : Y \neq \emptyset\}$;

(ii) $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$;

(iii) $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$;

(iv) $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$;

(v) $\mathcal{P}_{cv}(X) = \{Y \in \mathcal{P}(X) : Y \text{ convex}\}$;

(vi) $\mathcal{P}_{cp,cv}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact, convex}\}$.

Consider the Pompeiu-Hausdorff metric $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ given by

$$H_d(A, B) = \max\Big\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\Big\},$$

where $d(A,b) = \inf_{a\in A} d(a;b)$, $d(a,B) = \inf_{b\in B} d(a;b)$. Then $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized (complete) metric space (see [28, 30]).

A multivalued map $F : X \to \mathcal{P}(X)$

(i) is convex (closed) valued if $F(x)$ is convex (closed) for all $x \in X$;

(ii) is bounded on bounded sets if $F(B) = \bigcup_{x\in B} F(x)$ is bounded in X for all $B \in \mathcal{P}_b(X)$ (i.e., $\sup_{x\in B}\{\sup\{|y| : y \in F(x)\}\} < \infty$);

(iii) is called upper semicontinuous *(u.s.c)* on X if, for each $x_0 \in X$, the set $F(x_0)$ is a nonempty closed subset of X and if, for each open set N of X containing $F(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $F(\mathcal{N}_0) \subseteq N$;

(iv) is called lower semicontinuous if the set $\{x \in X : F(x) \cap O \neq \emptyset\}$ is open for each open set O in X;

(v) is said to be completely continuous if $F(B)$ is relatively compact for every $B \in \mathcal{P}_b(X)$;

(vi) is said to be measurable if, for every $y \in \mathbb{R}$, the function

$$t \mapsto d\bigl(y, F(t)\bigr) = \inf\bigl\{|y - z| : z \in G(t)\bigr\}$$

is measurable;

(vii) has a fixed point if there is $x \in X$ such that $x \in F(x)$. The fixed point set of the multivalued operator F will be denoted by *Fix F*.

Definition 2.1 ([31]) The fractional integral of order $q > 0$ of a Lebesgue integrable function $f(\cdot) : (0,\infty) \to \mathbb{R}$ is defined by

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{f(s)}{(t-s)^{1-q}}\, ds,$$

provided that the right-hand side is pointwise defined on $(0,\infty)$ and $\Gamma(\cdot)$ is the (Euler) gamma function defined by $\Gamma(q) = \int_0^\infty t^{q-1}e^{-t}\, dt$.

Definition 2.2 ([31]) The Caputo fractional derivative of order $q > 0$ of a function $f(\cdot) : [0,\infty) \to \mathbb{R}$ is defined by

$$^cD^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} f^{(n)}(s)\, ds,$$

where $n = [q] + 1$. It is assumed implicitly that $f(\cdot)$ is n times differentiable whose nth derivative is absolutely continuous.

Lemma 2.1 ([32]) *Let $\alpha > 0$. If we assume $h \in AC^n(0,1)$, then the differential equation*

$$^cD^\alpha h(t) = 0$$

has a unique solution

$$h(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0,1,2,\ldots,n-1$, n is the smallest integer greater than or equal to α.

In view of Lemma 2.1, *it follows that*

$$I^{\alpha\,c}D^{\alpha}h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1},$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, *n is the smallest integer greater than or equal to* α.

Lemma 2.2 ([19]) *Let* $\omega, z \in L[0,T]$ *and* $x, y \in AC^2[0,T]$. *Then the unique solution of the problem*

$$\begin{cases} {}^cD^{\alpha}x(t) = \omega(t), & t \in (0,T), 1 < \alpha \le 2, \\ {}^cD^{\beta}y(t) = z(t), & t \in (0,T), 1 < \beta \le 2, \\ x(0) = h(y), & \int_0^T y(s)\,ds = \mu_1 x(\eta), \\ y(0) = \phi(x), & \int_0^T x(s)\,ds = \mu_2 y(\xi), \quad \eta, \xi \in (0,T), \end{cases} \tag{2.1}$$

is

$$\begin{aligned} x(t) = {} & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\omega(s)\,ds + (\sigma_1 t + 1)h(y) + t\sigma_2\phi(x) \\ & + \frac{t}{\Delta}\biggl[\mu_2\xi\biggl(\mu_1\int_0^{\eta}\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}\omega(s)\,ds - \int_0^T \frac{(T-s)^{\beta}}{\Gamma(\beta+1)}z(s)\,ds\biggr) \\ & + \frac{T^2}{2}\biggl(\mu_2\int_0^{\xi}\frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}z(s)\,ds - \int_0^T \frac{(T-s)^{\alpha}}{\Gamma(\alpha+1)}\omega(s)\,ds\biggr)\biggr] \end{aligned}$$

and

$$\begin{aligned} y(t) = {} & \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}z(s)\,ds + (\sigma_3 t + 1)\phi(x) + t\sigma_4 h(y) \\ & + \frac{t}{\Delta}\biggl[\mu_1\eta\biggl(\mu_2\int_0^{\xi}\frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}z(s)\,ds - \int_0^T \frac{(T-s)^{\alpha}}{\Gamma(\alpha+1)}\omega(s)\,ds\biggr) \\ & + \frac{T^2}{2}\biggl(\mu_1\int_0^{\eta}\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}\omega(s)\,ds - \int_0^T \frac{(T-s)^{\beta}}{\Gamma(\beta+1)}z(s)\,ds\biggr)\biggr], \end{aligned}$$

where

$$\begin{aligned} \Delta &= \frac{T^4 - 4\mu_1\mu_2\eta\xi}{4}, \\ \sigma_1 &= \frac{2\mu_1\mu_2\xi - T^3}{2\Delta}, \\ \sigma_2 &= \frac{T\mu_2(T-2\xi)}{2\Delta}, \\ \sigma_3 &= \frac{2\mu_1\mu_2\eta - T^3}{2\Delta}, \\ \sigma_4 &= \frac{T\mu_1(T-2\eta)}{2\Delta}. \end{aligned}$$

Definition 2.3 A function $(x,y) \in AC^2(J,\mathbb{R}) \times AC^2(J,\mathbb{R})$ is a solution of the coupled system (1.3) if it satisfies the coupled nonlocal and integral boundary conditions (1.4) and there exist functions $f,g \in L^1(J,\mathbb{R})$ such that $f(t) \in F(t,x(t),y(t),{}^cD^\gamma y(t))$, $g(t) \in G(t,x(t),{}^cD^\delta x(t),y(t))$ a.e. on $t \in J$ and

$$\begin{aligned} x(t) = {} & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\,ds + (\sigma_1 t + 1)h(y) + t\sigma_2 \phi(x) \\ & + \frac{t}{\Delta}\biggl[\mu_2\xi\biggl(\mu_1 \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\,ds - \int_0^T \frac{(T-s)^\beta}{\Gamma(\beta+1)} g(s)\,ds\biggr) \\ & + \frac{T^2}{2}\biggl(\mu_2 \int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} g(s)\,ds - \int_0^T \frac{(T-s)^\alpha}{\Gamma(\alpha+1)} f(s)\,ds\biggr)\biggr] \end{aligned}$$

and

$$\begin{aligned} y(t) = {} & \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g(s)\,ds + (\sigma_3 t + 1)\phi(x) + t\sigma_4 h(y) \\ & + \frac{t}{\Delta}\biggl[\mu_1\eta\biggl(\mu_2 \int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} g(s)\,ds - \int_0^T \frac{(T-s)^\alpha}{\Gamma(\alpha+1)} f(s)\,ds\biggr) \\ & + \frac{T^2}{2}\biggl(\mu_1 \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\,ds - \int_0^T \frac{(T-s)^\beta}{\Gamma(\beta+1)} g(s)\,ds\biggr)\biggr], \end{aligned}$$

where Δ and σ_i ($i = 1,2,3,4$) are defined in Lemma 2.2.

Definition 2.4 A multivalued map $F : X \to \mathcal{P}_{cl}(X)$ is called

(1) γ-Lipschitz if there exists $\gamma > 0$ such that

$$H_d\bigl(F(x),F(y)\bigr) \le \gamma d(x,y) \quad \text{for each } x,y \in X;$$

(2) a contraction if it is γ-Lipschitz with $\gamma < 1$.

Definition 2.5 A multivalued map $F : J \times \mathbb{R}^3 \to P(\mathbb{R})$ is said to be L^1-Carathéodory if

(1) $t \mapsto F(t,x,y,z)$ is measurable for each $x,y,z \in \mathbb{R}$;

(2) $(x,y,z) \mapsto F(t,x,y,z)$ is upper semicontinuous for almost $t \in J$;

(3) for each $l > 0$, there exists $\varphi_l \in L^1(J,\mathbb{R}^+)$ such that

$$\bigl\|F(t,x,y,z)\bigr\| = \sup\bigl\{|v| : v \in F(t,x,y,z)\bigr\} \le \varphi_l(t)$$

for all $\|x\| \le l$, $\|y\| \le l$, $\|z\| \le l$ and for a.e. $t \in J$.

Lemma 2.3 ([33]) *Let X be a Banach space. Let $F : J \times X^3 \to \mathcal{P}_{cp,cv}(X)$ be an L^1-Carathéodory multivalued map and T be a linear continuous mapping from $L^1(J,X)$ to $C(J,X)$. Then the operator*

$$T \circ S_F : C(J,X) \to \mathcal{P}_{cp,cv}\bigl(C(J,X)\bigr), \qquad y \mapsto (T \circ S_F)(y) = T(S_F,y)$$

is a closed graph operator in $C(J,X) \times C(J,X)$.

To prove our main results, we will use the following fixed point theorems.

Lemma 2.4 (Nonlinear alternative of Leray-Schauder type [26]) *Let X be a Banach space, C be a closed convex subset of X, and $\mathcal{U}$ be an open subset of C with $0 \in \mathcal{U}$. Suppose that $F : \bar{\mathcal{U}} \to P_{cp,cv}(C)$ is an upper semicontinuous compact map. Then either* (1) *F has a fixed point in $\bar{\mathcal{U}}$, or* (2) *there is $x \in \partial\mathcal{U}$ and $\lambda \in (0,1)$ such that $x \in \lambda F(x)$.*

Lemma 2.5 (Covitz and Nadler [27]) *Let (X,d) be a complete metric space. If $F : X \to P_{cl}(X)$ is a contraction, then F has a fixed point, i.e., a point $z \in X$ such that $z \in F(z)$.*

3 Main results

In this section, we will give some existence results for the coupled system (1.3)-(1.4).

Let $C(J,\mathbb{R})$ denote the space of all continuous functions defined on J. Let $X = \{x : x \in C(J,\mathbb{R}) \text{ and } {}^cD^\delta x \in C(J,\mathbb{R})\}$ be a Banach space endowed with the norm $\|x\|_X = \|x\| + \|{}^cD^\delta x\| = \max_{t\in J}|x(t)| + \max_{t\in J}|{}^cD^\delta x(t)|$, where $0 < \delta < 1$ [34], and let $Y = \{y : y \in C(J,\mathbb{R}) \text{ and } {}^cD^\gamma y \in C(J,\mathbb{R})\}$ be a Banach space equipped with the norm $\|y\|_Y = \|y\| + \|{}^cD^\gamma y\| = \max_{t\in J}|y(t)| + \max_{t\in J}|{}^cD^\gamma y(t)|$, where $0 < \gamma < 1$. Obviously the product space $(X \times Y, \|\cdot\|_{X\times Y})$ is a Banach space with the norm $\|(x,y)\|_{X\times Y} = \|x\|_X + \|y\|_Y$ for $(x,y) \in X \times Y$.

For each $(x,y) \in X \times Y$, define the sets of selections of F, G by

$$S_{F,(x,y)} = \{f \in L^1(J,\mathbb{R}) : f(t) \in F(t,x(t),y(t),{}^cD^\gamma y(t)) \text{ for a.e. } t \in J\}$$

and

$$S_{G,(x,y)} = \{g \in L^1(J,\mathbb{R}) : g(t) \in G(t,x(t),{}^cD^\delta x(t),y(t)) \text{ for a.e. } t \in J\}.$$

In view of Lemma 2.2, we define operators $N_1 : X \times Y \to \mathcal{P}(X \times Y)$ and $N_2 : X \times Y \to \mathcal{P}(X \times Y)$ as

$$\begin{aligned} N_1(x,y) = \{&h_1 \in X \times Y : \text{ there exist } f \in S_{F,(x,y)} \text{ and } g \in S_{G,(x,y)} \\ &\text{such that } h_1(x,y)(t) = A_1(t,x,y), \forall t \in J\} \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} N_2(x,y) = \{&h_2 \in X \times Y : \text{ there exist } f \in S_{F,(x,y)} \text{ and } g \in S_{G,(x,y)} \\ &\text{such that } h_2(x,y)(t) = A_2(t,x,y), \forall t \in J\}, \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} A_1(t,x,y) = {}&\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\,ds + (\sigma_1 t + 1)h(y) + t\sigma_2\phi(x) \\ &+ \frac{t}{\Delta}\left[\mu_2\xi\left(\mu_1\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\,ds - \int_0^T \frac{(T-s)^\beta}{\Gamma(\beta+1)} g(s)\,ds\right)\right. \\ &\left.+ \frac{T^2}{2}\left(\mu_2\int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} g(s)\,ds - \int_0^T \frac{(T-s)^\alpha}{\Gamma(\alpha+1)} f(s)\,ds\right)\right] \end{aligned}$$

and

$$\begin{aligned} A_2(t,x,y) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g(s)\,ds + (\sigma_3 t+1)\phi(x) + t\sigma_4 h(y) \\ &\quad + \frac{t}{\Delta}\left[\mu_1\eta\left(\mu_2\int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} g(s)\,ds - \int_0^T \frac{(T-s)^\alpha}{\Gamma(\alpha+1)} f(s)\,ds\right)\right. \\ &\quad \left. + \frac{T^2}{2}\left(\mu_1\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\,ds - \int_0^T \frac{(T-s)^\beta}{\Gamma(\beta+1)} g(s)\,ds\right)\right]. \end{aligned}$$

Then we define an operator $N: X\times Y \to \mathcal{P}(X\times Y)\times \mathcal{P}(X\times Y)$ by

$$N(x,y) = \bigl(N_1(x,y), N_2(x,y)\bigr), \tag{3.3}$$

where N_1, N_2 are defined as (3.1) and (3.2).

It is clear that if $(x,y)\in X\times Y$ is a fixed point of the operator N, then (x,y) is a solution of the coupled system (1.3)-(1.4).

For computational convenience, we introduce the notations:

$$\begin{aligned} P_1 &= \left(1 + \frac{T|\mu_1\mu_2|\xi}{|\Delta|} + \frac{T^4}{2|\Delta|(\alpha+1)}\right)\cdot \frac{T^\alpha}{\Gamma(\alpha+1)}, \\ P_2 &= \left(\frac{\xi}{|\Delta|(\beta+1)} + \frac{T}{2|\Delta|}\right)\cdot \frac{|\mu_2|T^{\beta+2}}{\Gamma(\beta+1)}, \\ P_3 &= \left(1 + \frac{T|\mu_1\mu_2|\eta}{|\Delta|} + \frac{T^4}{2|\Delta|(\beta+1)}\right)\cdot \frac{T^\beta}{\Gamma(\beta+1)}, \\ P_4 &= \left(\frac{\eta}{|\Delta|(\alpha+1)} + \frac{T}{2|\Delta|}\right)\cdot \frac{|\mu_1|T^{\alpha+2}}{\Gamma(\alpha+1)}, \\ \overline{P}_1 &= \frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{|\mu_1\mu_2|\xi T^\alpha}{|\Delta|\Gamma(\alpha+1)} + \frac{T^{\alpha+3}}{2|\Delta|\Gamma(\alpha+2)}, \\ \overline{P}_2 &= \left(\frac{\xi}{|\Delta|(\beta+1)} + \frac{T}{2|\Delta|}\right)\cdot \frac{|\mu_2|T^{\beta+1}}{\Gamma(\beta+1)}, \\ \overline{P}_3 &= \frac{T^{\beta-1}}{\Gamma(\beta)} + \frac{|\mu_1\mu_2|\eta T^\beta}{|\Delta|\Gamma(\beta+1)} + \frac{T^{\beta+3}}{2|\Delta|\Gamma(\beta+2)}, \\ \overline{P}_4 &= \left(\frac{\eta}{|\Delta|(\alpha+1)} + \frac{T}{2|\Delta|}\right)\cdot \frac{|\mu_1|T^{\alpha+1}}{\Gamma(\alpha+1)}, \\ Q_1 &= L_1\bigl(|\sigma_1|T+1\bigr) + L_2|\sigma_2|T, \\ \overline{Q}_1 &= L_1|\sigma_1| + L_2|\sigma_2|, \\ Q_2 &= L_2\bigl(|\sigma_3|T+1\bigr) + L_1|\sigma_4|T, \\ \overline{Q}_2 &= L_2|\sigma_3| + L_1|\sigma_4|. \end{aligned}$$

3.1 The Carathéodory case

First we consider the case when F, G are convex valued.

We give the following conditions:

(H1) $F, G: J\times \mathbb{R}^3 \to \mathcal{P}_{cp,cv}(\mathbb{R})$ are L^1-Carathéodory multivalued maps;

(H2) There exist $m_1, m_2 \in C(J, \mathbb{R}^+)$ and $\varphi_1, \varphi_2, \psi_1, \psi_2, \rho_1, \rho_2 : [0, \infty) \to (0, \infty)$ continuous, nondecreasing such that

$$\begin{aligned}\|F(t,x,y,z)\| &= \sup\{|f| : f \in F(t,x,y,z)\}\\ &\le m_1(t)\big(\varphi_1(|x|) + \psi_1(|y|) + \rho_1(|z|)\big)\end{aligned}$$

and

$$\begin{aligned}\|G(t,x,y,z)\| &= \sup\{|g| : g \in G(t,x,y,z)\}\\ &\le m_2(t)\big(\varphi_2(|x|) + \psi_2(|y|) + \rho_2(|z|)\big)\end{aligned}$$

for $x, y, z \in \mathbb{R}$ and a.e. $t \in J$;

(H3) h, ϕ are continuous functionals with $h(0) = \phi(0) = 0$, and there exist constants $L_1 > 0$, $L_2 > 0$ such that

$$\begin{aligned}&\big|h(x_1) - h(x_2)\big| \le L_1\|x_1 - x_2\|,\\ &\big|\phi(x_1) - \phi(x_2)\big| \le L_2\|x_1 - x_2\|, \quad \forall x_1, x_2 \in C(J, \mathbb{R}).\end{aligned}$$

Theorem 3.1 *Assume that* (H1)-(H3) *are satisfied and there exists* $K > 0$ *such that*

$$K > \sum_{i=1}^{2} \Lambda_i \|m_i\| \big(\varphi_i(K) + \psi_i(K) + \rho_i(K)\big)(1 - \Lambda_3)^{-1}. \tag{3.4}$$

If, in addition, $\Lambda_3 < 1$, *where*

$$\begin{aligned}\Lambda_1 &= P_1 + P_4 + \frac{T^{1-\delta}}{\Gamma(2-\delta)}\overline{P}_1 + \frac{T^{1-\gamma}}{\Gamma(2-\gamma)}\overline{P}_4,\\ \Lambda_2 &= P_2 + P_3 + \frac{T^{1-\delta}}{\Gamma(2-\delta)}\overline{P}_2 + \frac{T^{1-\gamma}}{\Gamma(2-\gamma)}\overline{P}_3,\\ \Lambda_3 &= Q_1 + Q_2 + \frac{T^{1-\delta}}{\Gamma(2-\delta)}\overline{Q}_1 + \frac{T^{1-\gamma}}{\Gamma(2-\gamma)}\overline{Q}_2,\end{aligned}$$

then the coupled system (1.3)-(1.4) *has at least one solution on* J.

Proof Consider the operators $N_1 : X \times Y \to \mathcal{P}(X \times Y)$ and $N_2 : X \times Y \to \mathcal{P}(X \times Y)$ defined by (3.1) and (3.2). From (H1), we have that for each $(x, y) \in X \times Y$, the sets $S_{F,(x,y)}$ and $S_{G,(x,y)}$ are nonempty [33]. For $(x, y) \in X \times Y$, let $f \in S_{F,(x,y)}$, $g \in S_{G,(x,y)}$ and

$$\begin{aligned}&h_1(x,y)(t)\\ &\quad= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\,ds + (\sigma_1 t + 1)h(y) + t\sigma_2\phi(x)\\ &\qquad+ \frac{t}{\Delta}\bigg[\mu_2\xi\bigg(\mu_1\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\,ds - \int_0^T \frac{(T-s)^\beta}{\Gamma(\beta+1)} g(s)\,ds\bigg)\\ &\qquad+ \frac{T^2}{2}\bigg(\mu_2\int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} g(s)\,ds - \int_0^T \frac{(T-s)^\alpha}{\Gamma(\alpha+1)} f(s)\,ds\bigg)\bigg], \quad \text{for } t \in J,\end{aligned}$$

and

$$\begin{aligned}
& h_2(x,y)(t) \\
&= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g(s)\,ds + (\sigma_3 t+1)\phi(x) + t\sigma_4 h(y) \\
&\quad + \frac{t}{\Delta}\biggl[\mu_1\eta\biggl(\mu_2\int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} g(s)\,ds - \int_0^T \frac{(T-s)^\alpha}{\Gamma(\alpha+1)} f(s)\,ds\biggr) \\
&\quad + \frac{T^2}{2}\biggl(\mu_1\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\,ds - \int_0^T \frac{(T-s)^\beta}{\Gamma(\beta+1)} g(s)\,ds\biggr)\biggr], \quad \text{for } t\in J,
\end{aligned}$$

that is, $h_1 \in N_1(x,y)$, $h_2 \in N_2(x,y)$, so $(h_1,h_2) \in N(x,y)$.

We will show that N satisfies the requirements of the nonlinear alternative of Leray-Schauder type. The proof will be given in five steps.

Step 1. $N(x,y)$ is convex valued.

Suppose $(h_i,\widetilde{h}_i) \in (N_1,N_2)$ $(i=1,2)$. Then there exist $f_i \in S_{F,(x,y)}$, $g_i \in S_{G,(x,y)}$ $(i=1,2)$ such that for any $t\in J$, $i=1,2$, we have

$$\begin{aligned}
h_i(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_i(s)\,ds + (\sigma_1 t+1)h(y) + t\sigma_2\phi(x) \\
&\quad + \frac{t}{\Delta}\biggl[\mu_2\xi\biggl(\mu_1\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} f_i(s)\,ds - \int_0^T \frac{(T-s)^\beta}{\Gamma(\beta+1)} g_i(s)\,ds\biggr) \\
&\quad + \frac{T^2}{2}\biggl(\mu_2\int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} g_i(s)\,ds - \int_0^T \frac{(T-s)^\alpha}{\Gamma(\alpha+1)} f_i(s)\,ds\biggr)\biggr]
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{h}_i(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g_i(s)\,ds + (\sigma_3 t+1)\phi(x) + t\sigma_4 h(y) \\
&\quad + \frac{t}{\Delta}\biggl[\mu_1\eta\biggl(\mu_2\int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} g_i(s)\,ds - \int_0^T \frac{(T-s)^\alpha}{\Gamma(\alpha+1)} f_i(s)\,ds\biggr) \\
&\quad + \frac{T^2}{2}\biggl(\mu_1\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} f_i(s)\,ds - \int_0^T \frac{(T-s)^\beta}{\Gamma(\beta+1)} g_i(s)\,ds\biggr)\biggr].
\end{aligned}$$

Let $0 \le \theta \le 1$. Then, for any $t\in J$, we have

$$\begin{aligned}
& \bigl(\theta h_1 + (1-\theta)h_2\bigr)(t) \\
&= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \bigl(\theta f_1(s) + (1-\theta)f_2(s)\bigr)\,ds + (\sigma_1 t+1)h(y) + t\sigma_2\phi(x) \\
&\quad + \frac{t}{\Delta}\biggl[\mu_2\xi\biggl(\mu_1\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} \bigl(\theta f_1(s) + (1-\theta)f_2(s)\bigr)\,ds \\
&\quad - \int_0^T \frac{(T-s)^\beta}{\Gamma(\beta+1)} \bigl(\theta g_1(s) + (1-\theta)g_2(s)\bigr)\,ds\biggr) \\
&\quad + \frac{T^2}{2}\biggl(\mu_2\int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} \bigl(\theta g_1(s) + (1-\theta)g_2(s)\bigr)\,ds \\
&\quad - \int_0^T \frac{(T-s)^\alpha}{\Gamma(\alpha+1)} \bigl(\theta f_1(s) + (1-\theta)f_2(s)\bigr)\,ds\biggr)\biggr]
\end{aligned}$$

and

$$
\begin{aligned}
&\big(\theta\widetilde{h}_1 + (1-\theta)\widetilde{h}_2\big)(t) \\
&\quad = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\big(\theta g_1(s) + (1-\theta)g_2(s)\big)\,ds + (\sigma_3 t + 1)\phi(x) + t\sigma_4 h(y) \\
&\qquad + \frac{t}{\Delta}\bigg[\mu_1\eta\bigg(\mu_2\int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}\big(\theta g_1(s) + (1-\theta)g_2(s)\big)\,ds \\
&\qquad - \int_0^T \frac{(T-s)^\alpha}{\Gamma(\alpha+1)}\big(\theta f_1(s) + (1-\theta)f_2(s)\big)\,ds\bigg) \\
&\qquad + \frac{T^2}{2}\bigg(\mu_1\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}\big(\theta f_1(s) + (1-\theta)f_2(s)\big)\,ds \\
&\qquad - \int_0^T \frac{(T-s)^\beta}{\Gamma(\beta+1)}\big(\theta g_1(s) + (1-\theta)g_2(s)\big)\,ds\bigg)\bigg].
\end{aligned}
$$

Since F and G are convex valued, we deduce that $S_{F,(x,y)}$ and $S_{G,(x,y)}$ are convex. Obviously, $\theta h_1 + (1-\theta)h_2 \in N_1$, $\theta\widetilde{h}_1 + (1-\theta)\widetilde{h}_2 \in N_2$. Therefore, $\theta(h_1,\widetilde{h}_1) + (1-\theta)(h_2,\widetilde{h}_2) \in N$.

Step 2. N maps bounded sets into bounded sets in $X \times Y$.

Let $r > 0$, $B_r = \{(x,y) \in X \times Y : \|(x,y)\|_{X\times Y} \le r\}$ be a bounded subset of $X \times Y$, $(h_1,h_2) \in N(x,y)$ and $(x,y) \in B_r$. Then there exist $f \in S_{F,(x,y)}$ and $g \in S_{G,(x,y)}$ such that for any $t \in J$,

$$
\begin{aligned}
h_1(x,y)(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)\,ds + (\sigma_1 t + 1)h(y) + t\sigma_2\phi(x) \\
&\quad + \frac{t}{\Delta}\bigg[\mu_2\xi\bigg(\mu_1\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)\,ds - \int_0^T \frac{(T-s)^\beta}{\Gamma(\beta+1)}g(s)\,ds\bigg) \\
&\quad + \frac{T^2}{2}\bigg(\mu_2\int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}g(s)\,ds - \int_0^T \frac{(T-s)^\alpha}{\Gamma(\alpha+1)}f(s)\,ds\bigg)\bigg]
\end{aligned}
$$

and

$$
\begin{aligned}
h_2(x,y)(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}g(s)\,ds + (\sigma_3 t + 1)\phi(x) + t\sigma_4 h(y) \\
&\quad + \frac{t}{\Delta}\bigg[\mu_1\eta\bigg(\mu_2\int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}g(s)\,ds - \int_0^T \frac{(T-s)^\alpha}{\Gamma(\alpha+1)}f(s)\,ds\bigg) \\
&\quad + \frac{T^2}{2}\bigg(\mu_1\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)\,ds - \int_0^T \frac{(T-s)^\beta}{\Gamma(\beta+1)}g(s)\,ds\bigg)\bigg].
\end{aligned}
$$

Based on assumptions (H3), we find the following estimates:

$$
\begin{aligned}
&\big|h(y)\big| \le \big|h(y) - h(0)\big| + \big|h(0)\big| \le L_1\|y\|_Y \le L_1 r, \quad \forall y \in \mathbb{R}, \\
&\big|\phi(x)\big| \le \big|\phi(x) - \phi(0)\big| + \big|\phi(0)\big| \le L_2\|x\|_X \le L_2 r, \quad \forall x \in \mathbb{R}.
\end{aligned}
$$

Using these estimates, we get

$$
\begin{aligned}
&\big|h_1(x,y)(t)\big| \\
&\quad \le \frac{T^\alpha}{\Gamma(\alpha+1)} \cdot \|m_1\|\big(\varphi_1(r) + \psi_1(r) + \rho_1(r)\big) + L_1 r \cdot |\sigma_1 t + 1| + L_2 r \cdot |\sigma_2 t|
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{T|\mu_1\mu_2|\xi}{|\Delta|}\cdot\frac{T^{\alpha}}{\Gamma(\alpha+1)}\cdot\|m_1\|\bigl(\varphi_1(r)+\psi_1(r)+\rho_1(r)\bigr)+\frac{|\mu_2|\xi T}{|\Delta|}\cdot\frac{T^{\beta+1}}{\Gamma(\beta+2)}\\
&\times\|m_2\|\bigl(\varphi_2(r)+\psi_2(r)+\rho_2(r)\bigr)\\
&+\frac{|\mu_2|T^3}{2|\Delta|}\cdot\frac{T^{\beta}}{\Gamma(\beta+1)}\cdot\|m_2\|\bigl(\varphi_2(r)+\psi_2(r)+\rho_2(r)\bigr)\\
&+\frac{T^3}{2|\Delta|}\cdot\frac{T^{\alpha+1}}{\Gamma(\alpha+2)}\cdot\|m_1\|\bigl(\varphi_1(r)+\psi_1(r)+\rho_1(r)\bigr)\\
\le{}& P_1\cdot\|m_1\|\bigl(\varphi_1(r)+\psi_1(r)+\rho_1(r)\bigr)+P_2\cdot\|m_2\|\bigl(\varphi_2(r)+\psi_2(r)+\rho_2(r)\bigr)+Q_1r
\end{aligned}
$$

and

$$
\begin{aligned}
&\bigl|h_1(x,y)'(t)\bigr|\\
&=\biggl|\int_0^t\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s)\,ds+\sigma_1h(y)+\sigma_2\phi(x)+\frac{1}{\Delta}\biggl[\mu_2\xi\biggl(\mu_1\int_0^{\eta}\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)\,ds\\
&\quad-\int_0^T\frac{(T-s)^{\beta}}{\Gamma(\beta+1)}g(s)\,ds\biggr)+\frac{T^2}{2}\biggl(\mu_2\int_0^{\xi}\frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}g(s)\,ds-\int_0^T\frac{(T-s)^{\alpha}}{\Gamma(\alpha+1)}f(s)\,ds\biggr)\biggr]\biggr|\\
&\le\frac{T^{\alpha-1}}{\Gamma(\alpha)}\cdot\|m_1\|\bigl(\varphi_1(r)+\psi_1(r)+\rho_1(r)\bigr)+L_1|\sigma_1|r+L_2|\sigma_2|r\\
&\quad+\frac{|\mu_1\mu_2|\xi}{|\Delta|}\cdot\frac{T^{\alpha}}{\Gamma(\alpha+1)}\cdot\|m_1\|\bigl(\varphi_1(r)+\psi_1(r)+\rho_1(r)\bigr)+\frac{|\mu_2|\xi}{|\Delta|}\cdot\frac{T^{\beta+1}}{\Gamma(\beta+2)}\\
&\quad\times\|m_2\|\bigl(\varphi_2(r)+\psi_2(r)+\rho_2(r)\bigr)+\frac{|\mu_2|T^2}{2|\Delta|}\cdot\frac{T^{\beta}}{\Gamma(\beta+1)}\cdot\|m_2\|\bigl(\varphi_2(r)+\psi_2(r)+\rho_2(r)\bigr)\\
&\quad+\frac{T^2}{2|\Delta|}\cdot\frac{T^{\alpha+1}}{\Gamma(\alpha+2)}\cdot\|m_1\|\bigl(\varphi_1(r)+\psi_1(r)+\rho_1(r)\bigr)\\
&=\overline{P}_1\cdot\|m_1\|\bigl(\varphi_1(r)+\psi_1(r)+\rho_1(r)\bigr)+\overline{P}_2\cdot\|m_2\|\bigl(\varphi_2(r)+\psi_2(r)+\rho_2(r)\bigr)+\overline{Q}_1r,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\bigl|{}^cD^{\delta}h_1(x,y)(t)\bigr|&\le\int_0^t\frac{(t-s)^{-\delta}}{\Gamma(1-\delta)}\bigl|h_1(x,y)'(s)\bigr|\,ds\\
&\le\frac{T^{1-\delta}}{\Gamma(2-\delta)}\cdot\bigl[\overline{P}_1\cdot\|m_1\|\bigl(\varphi_1(r)+\psi_1(r)+\rho_1(r)\bigr)\\
&\quad+\overline{P}_2\cdot\|m_2\|\bigl(\varphi_2(r)+\psi_2(r)+\rho_2(r)\bigr)+\overline{Q}_1r\bigr].
\end{aligned}
$$

Thus

$$
\begin{aligned}
\bigl\|h_1(x,y)\bigr\|_X&=\bigl\|h_1(x,y)\bigr\|+\bigl\|{}^cD^{\delta}h_1(x,y)\bigr\|\\
&\le\biggl(P_1+\frac{T^{1-\delta}}{\Gamma(2-\delta)}\overline{P}_1\biggr)\cdot\|m_1\|\bigl(\varphi_1(r)+\psi_1(r)+\rho_1(r)\bigr)+\biggl(P_2+\frac{T^{1-\delta}}{\Gamma(2-\delta)}\overline{P}_2\biggr)\\
&\quad\times\|m_2\|\bigl(\varphi_2(r)+\psi_2(r)+\rho_2(r)\bigr)+\biggl(Q_1+\frac{T^{1-\delta}}{\Gamma(2-\delta)}\overline{Q}_1\biggr)r.
\end{aligned}
$$

In a similar manner, for any $t \in J$, we obtain

$$\begin{aligned}
&\bigl|h_2(x,y)(t)\bigr| \\
&\quad\leq \frac{T^\beta}{\Gamma(\beta+1)} \cdot \|m_2\|\bigl(\varphi_2(r)+\psi_2(r)+\rho_2(r)\bigr) + L_2 r \cdot |\sigma_3 T + 1| + L_1 r \cdot |\sigma_2| T \\
&\qquad + \frac{T|\mu_1\mu_2|\eta}{|\Delta|} \cdot \frac{T^\beta}{\Gamma(\beta+1)} \cdot \|m_2\|\bigl(\varphi_2(r)+\psi_2(r)+\rho_2(r)\bigr) + \frac{|\mu_1|\eta T}{|\Delta|} \cdot \frac{T^{\alpha+1}}{\Gamma(\alpha+2)} \\
&\qquad \times \|m_1\|\bigl(\varphi_1(r)+\psi_1(r)+\rho_1(r)\bigr) + \frac{|\mu_1|T^3}{2|\Delta|} \cdot \frac{T^\alpha}{\Gamma(\alpha+1)} \cdot \|m_1\|\bigl(\varphi_1(r)+\psi_1(r)+\rho_1(r)\bigr) \\
&\qquad + \frac{T^3}{2|\Delta|} \cdot \frac{T^{\beta+1}}{\Gamma(\beta+2)} \cdot \|m_2\|\bigl(\varphi_2(r)+\psi_2(r)+\rho_2(r)\bigr) \\
&\quad\leq P_3 \cdot \|m_2\|\bigl(\varphi_2(r)+\psi_2(r)+\rho_2(r)\bigr) + P_4 \cdot \|m_1\|\bigl(\varphi_1(r)+\psi_1(r)+\rho_1(r)\bigr) + Q_2 r, \\
&\bigl|h_2(x,y)'(t)\bigr| \\
&\quad= \biggl|\int_0^t \frac{(t-s)^{\beta-2}}{\Gamma(\beta-1)} g(s)\,ds + \sigma_3\phi(x) + \sigma_4 h(y) + \frac{1}{\Delta}\biggl[\mu_1\eta\biggl(\mu_2\int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} g(s)\,ds \\
&\qquad - \int_0^T \frac{(T-s)^\alpha}{\Gamma(\alpha+1)} f(s)\,ds\biggr) + \frac{T^2}{2}\biggl(\mu_1\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\,ds - \int_0^T \frac{(T-s)^\beta}{\Gamma(\beta+1)} g(s)\,ds\biggr)\biggr]\biggr| \\
&\quad\leq \frac{T^{\beta-1}}{\Gamma(\beta)} \cdot \|m_2\|\bigl(\varphi_2(r)+\psi_2(r)+\rho_2(r)\bigr) + L_2|\sigma_3| r + L_1|\sigma_4| r \\
&\qquad + \frac{|\mu_1\mu_2|\eta}{|\Delta|} \cdot \frac{T^\beta}{\Gamma(\beta+1)} \cdot \|m_2\|\bigl(\varphi_2(r)+\psi_2(r)+\rho_2(r)\bigr) + \frac{|\mu_1|\eta}{|\Delta|} \cdot \frac{T^{\alpha+1}}{\Gamma(\alpha+2)} \\
&\qquad \times \|m_1\|\bigl(\varphi_1(r)+\psi_1(r)+\rho_1(r)\bigr) + \frac{|\mu_1|T^2}{2|\Delta|} \cdot \frac{T^\alpha}{\Gamma(\alpha+1)} \cdot \|m_1\|\bigl(\varphi_1(r)+\psi_1(r)+\rho_1(r)\bigr) \\
&\qquad + \frac{T^2}{2|\Delta|} \cdot \frac{T^{\beta+1}}{\Gamma(\beta+2)} \cdot \|m_2\|\bigl(\varphi_2(r)+\psi_2(r)+\rho_2(r)\bigr) \\
&\quad= \overline{P}_3 \cdot \|m_2\|\bigl(\varphi_2(r)+\psi_2(r)+\rho_2(r)\bigr) + \overline{P}_4 \cdot \|m_1\|\bigl(\varphi_1(r)+\psi_1(r)+\rho_1(r)\bigr) + \overline{Q}_2 r
\end{aligned}$$

and

$$\begin{aligned}
\bigl|{}^cD^\gamma h_2(x,y)(t)\bigr| &\leq \int_0^t \frac{(t-s)^{-\gamma}}{\Gamma(1-\gamma)} \bigl|h_2(x,y)'(s)\bigr|\,ds \\
&\leq \frac{T^{1-\gamma}}{\Gamma(2-\gamma)} \cdot \bigl[\overline{P}_3 \cdot \|m_2\|\bigl(\varphi_2(r)+\psi_2(r)+\rho_2(r)\bigr) \\
&\qquad + \overline{P}_4 \cdot \|m_1\|\bigl(\varphi_1(r)+\psi_1(r)+\rho_1(r)\bigr) + \overline{Q}_2 r\bigr].
\end{aligned}$$

In consequence, we get

$$\begin{aligned}
\bigl\|h_2(x,y)\bigr\|_Y &= \bigl\|h_2(x,y)\bigr\| + \bigl\|{}^cD^\gamma h_2(x,y)\bigr\| \\
&\leq \biggl(P_3 + \frac{T^{1-\gamma}}{\Gamma(2-\gamma)}\overline{P}_3\biggr) \cdot \|m_2\|\bigl(\varphi_2(r)+\psi_2(r)+\rho_2(r)\bigr) + \biggl(P_4 + \frac{T^{1-\gamma}}{\Gamma(2-\gamma)}\overline{P}_4\biggr) \\
&\quad \times \|m_1\|\bigl(\varphi_1(r)+\psi_1(r)+\rho_1(r)\bigr) + \biggl(Q_2 + \frac{T^{1-\gamma}}{\Gamma(2-\gamma)}\overline{Q}_2\biggr) r.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\left\|(h_1,h_2)\right\|_{X\times Y} &= \left\|h_1(x,y)\right\|_X + \left\|h_2(x,y)\right\|_Y \\ &\leq \Lambda_1\cdot\|m_1\|\big(\varphi_1(r)+\psi_1(r)+\rho_1(r)\big) \\ &\quad + \Lambda_2\cdot\|m_2\|\big(\varphi_2(r)+\psi_2(r)+\rho_2(r)\big)+\Lambda_3 r \\ &= l \quad \text{(a constant).}\end{aligned}$$

Step 3. N maps bounded sets into equicontinuous sets in $X\times Y$.

Let B_r be a bounded set of $X\times Y$ as in Step 2. Let $0\leq t_1<t_2\leq T$ and $(x,y)\in B_r$. For each $(h_1,h_2)\in N(x,y)$, then there exist $f\in S_{F,(x,y)}$ and $g\in S_{G,(x,y)}$ such that

$$\begin{aligned}h_1(x,y)(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)\,ds+(\sigma_1 t+1)h(y)+t\sigma_2\phi(x) \\ &\quad + \frac{t}{\Delta}\left[\mu_2\xi\left(\mu_1\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)\,ds-\int_0^T\frac{(T-s)^\beta}{\Gamma(\beta+1)}g(s)\,ds\right)\right. \\ &\quad \left.+\frac{T^2}{2}\left(\mu_2\int_0^\xi\frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}g(s)\,ds-\int_0^T\frac{(T-s)^\alpha}{\Gamma(\alpha+1)}f(s)\,ds\right)\right]\end{aligned}$$

and

$$\begin{aligned}h_2(x,y)(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}g(s)\,ds+(\sigma_3 t+1)\phi(x)+t\sigma_4 h(y) \\ &\quad + \frac{t}{\Delta}\left[\mu_1\eta\left(\mu_2\int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}g(s)\,ds-\int_0^T\frac{(T-s)^\alpha}{\Gamma(\alpha+1)}f(s)\,ds\right)\right. \\ &\quad \left.+\frac{T^2}{2}\left(\mu_1\int_0^\eta\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)\,ds-\int_0^T\frac{(T-s)^\beta}{\Gamma(\beta+1)}g(s)\,ds\right)\right].\end{aligned}$$

Since

$$\begin{aligned}&\left|h_1(x,y)(t_2)-h_1(x,y)(t_1)\right| \\ &\quad\leq \int_0^{t_1}\frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\left|f(s)\right|ds+\left|\sigma_1 h(y)\right|(t_2-t_1)+\left|\sigma_2\phi(x)\right|(t_2-t_1) \\ &\qquad+\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\left|f(s)\right|ds+\frac{t_2-t_1}{|\Delta|}\left(|\mu_1\mu_2|\xi\int_0^\eta\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}\left|f(s)\right|ds\right. \\ &\qquad+|\mu_2|\xi\int_0^T\frac{(T-s)^\beta}{\Gamma(\beta+1)}\left|g(s)\right|ds \\ &\qquad\left.+\frac{|\mu_2|T^2}{2}\int_0^\xi\frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}\left|g(s)\right|ds+\frac{T^2}{2}\int_0^T\frac{(T-s)^\alpha}{\Gamma(\alpha+1)}\left|f(s)\right|ds\right) \\ &\quad\leq \int_0^{t_1}\frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\,ds\cdot\|m_1\|\big(\varphi_1(r)+\psi_1(r)+\rho_1(r)\big) \\ &\qquad+\big(|\sigma_1|L_1 r+|\sigma_2|L_2 r\big)(t_2-t_1) \\ &\qquad+\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\,ds\cdot\|m_1\|\big(\varphi_1(r)+\psi_1(r)+\rho_1(r)\big)+\frac{t_2-t_1}{|\Delta|}\left[\frac{|\mu_1\mu_2|\xi T^\alpha}{\Gamma(\alpha+1)}\right.\end{aligned}$$

$$
\begin{aligned}
&\times \|m_1\|\big(\varphi_1(r)+\psi_1(r)+\rho_1(r)\big)+\frac{|\mu_2|\xi T^{\beta+1}}{\Gamma(\beta+2)}\cdot\|m_2\|\big(\varphi_2(r)+\psi_2(r)+\rho_2(r)\big)\\
&+\frac{|\mu_2|T^{\beta+2}}{2\Gamma(\beta+1)}\cdot\|m_2\|\big(\varphi_2(r)+\psi_2(r)+\rho_2(r)\big)\\
&+\frac{T^{\alpha+3}}{2\Gamma(\alpha+2)}\cdot\|m_1\|\big(\varphi_1(r)+\psi_1(r)+\rho_1(r)\big)\bigg]\\
\le{}&\frac{t_2^\alpha-t_1^\alpha}{\alpha\Gamma(\alpha)}\cdot\|m_1\|\big(\varphi_1(r)+\psi_1(r)+\rho_1(r)\big)+\big(|\sigma_1|L_1r+|\sigma_2|L_2r\big)(t_2-t_1)\\
&+\frac{(t_2-t_1)^\alpha}{\alpha\Gamma(\alpha)}\cdot\|m_1\|\big(\varphi_1(r)+\psi_1(r)+\rho_1(r)\big)+\frac{t_2-t_1}{|\Delta|}\bigg[\frac{|\mu_1\mu_2|\xi T^\alpha}{\Gamma(\alpha+1)}\\
&\times \|m_1\|\big(\varphi_1(r)+\psi_1(r)+\rho_1(r)\big)+\frac{|\mu_2|\xi T^{\beta+1}}{\Gamma(\beta+2)}\cdot\|m_2\|\big(\varphi_2(r)+\psi_2(r)+\rho_2(r)\big)\\
&+\frac{|\mu_2|T^{\beta+2}}{2\Gamma(\beta+1)}\cdot\|m_2\|\big(\varphi_2(r)+\psi_2(r)+\rho_2(r)\big)\\
&+\frac{T^{\alpha+3}}{2\Gamma(\alpha+2)}\cdot\|m_1\|\big(\varphi_1(r)+\psi_1(r)+\rho_1(r)\big)\bigg],
\end{aligned}
$$

$$
\begin{aligned}
&\big|h_1(x,y)'(t_2)-h_1(x,y)'(t_1)\big|\\
&\quad\le\bigg(\int_0^{t_1}\frac{(t_2-s)^{\alpha-2}-(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\,ds+\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-2}}{\Gamma(\alpha-1)}\,ds\bigg)\\
&\qquad\times\|m_1\|\big(\varphi_1(r)+\psi_1(r)+\rho_1(r)\big)\\
&\quad\le\frac{t_2^{\alpha-1}-t_1^{\alpha-1}}{(\alpha-1)\Gamma(\alpha-1)}\cdot\|m_1\|\big(\varphi_1(r)+\psi_1(r)+\rho_1(r)\big)
\end{aligned}
$$

and

$$
\begin{aligned}
&\big|{}^cD^\delta h_1(x,y)(t_2)-{}^cD^\delta h_1(x,y)(t_1)\big|\\
&\quad=\bigg|\int_0^{t_2}\frac{(t_2-s)^{-\delta}}{\Gamma(1-\delta)}h_1(x,y)'(s)\,ds-\int_0^{t_1}\frac{(t_1-s)^{-\delta}}{\Gamma(1-\delta)}h_1(x,y)'(s)\,ds\bigg|\\
&\quad\le\int_0^{t_1}\frac{(t_2-s)^{-\delta}-(t_1-s)^{-\delta}}{\Gamma(1-\delta)}\big|h_1(x,y)'(s)\big|\,ds\\
&\qquad+\int_{t_1}^{t_2}\frac{(t_2-s)^{-\delta}}{\Gamma(1-\delta)}\big|h_1(x,y)'(s)\big|\,ds\\
&\quad\le\bigg(\int_0^{t_1}\frac{(t_2-s)^{-\delta}-(t_1-s)^{-\delta}}{\Gamma(1-\delta)}\,ds+\int_{t_1}^{t_2}\frac{(t_2-s)^{-\delta}}{\Gamma(1-\delta)}\,ds\bigg)\\
&\qquad\times\big(\overline{P}_1\cdot\|m_1\|\big(\varphi_1(r)+\psi_1(r)+\rho_1(r)\big)\\
&\qquad+\overline{P}_2\cdot\|m_2\|\big(\varphi_2(r)+\psi_2(r)+\rho_2(r)\big)+\overline{Q}_1r\big)\\
&\quad\le\frac{t_2^{1-\delta}-t_1^{1-\delta}}{(1-\delta)\Gamma(1-\delta)}\big(\overline{P}_1\cdot\|m_1\|\big(\varphi_1(r)+\psi_1(r)+\rho_1(r)\big)\\
&\qquad+\overline{P}_2\cdot\|m_2\|\big(\varphi_2(r)+\psi_2(r)+\rho_2(r)\big)+\overline{Q}_1r\big).
\end{aligned}
$$

Hence $|h_1(x,y)(t_2)-h_1(x,y)(t_1)|\to 0$ as $t_2\to t_1$.

Analogously, one can obtain

$$\begin{aligned}
&\bigl|h_2(x,y)(t_2)-h_2(x,y)(t_1)\bigr| \\
&\quad\le \frac{t_2^{\beta}-t_1^{\beta}}{\beta\Gamma(\beta)}\cdot\|m_2\|\bigl(\varphi_2(r)+\psi_2(r)+\rho_2(r)\bigr)+\bigl(|\sigma_3|L_2r+|\sigma_4|L_1r\bigr)(t_2-t_1) \\
&\qquad+\frac{(t_2-t_1)^{\beta}}{\beta\Gamma(\beta)}\cdot\|m_2\|\bigl(\varphi_2(r)+\psi_2(r)+\rho_2(r)\bigr) \\
&\qquad+\frac{t_2-t_1}{|\Delta|}\biggl[\frac{|\mu_1\mu_2|\eta T^{\beta}}{\Gamma(\beta+1)}\cdot\|m_2\|\bigl(\varphi_2(r)+\psi_2(r)+\rho_2(r)\bigr) \\
&\qquad+\frac{|\mu_1|\eta T^{\alpha+1}}{\Gamma(\alpha+2)}\cdot\|m_1\|\bigl(\varphi_1(r)+\psi_1(r)+\rho_1(r)\bigr) \\
&\qquad+\frac{|\mu_1|T^{\alpha+2}}{2\Gamma(\alpha+1)}\cdot\|m_1\|\bigl(\varphi_1(r)+\psi_1(r)+\rho_1(r)\bigr) \\
&\qquad+\frac{T^{\beta+3}}{2\Gamma(\beta+2)}\cdot\|m_2\|\bigl(\varphi_2(r)+\psi_2(r)+\rho_2(r)\bigr)\biggr], \\
&\bigl|h_2(x,y)'(t_2)-h_2(x,y)'(t_1)\bigr| \\
&\quad\le \frac{t_2^{\beta-1}-t_1^{\beta-1}}{(\beta-1)\Gamma(\beta-1)}\cdot\|m_2\|\bigl(\varphi_2(r)+\psi_2(r)+\rho_2(r)\bigr)
\end{aligned}$$

and

$$\begin{aligned}
&\bigl|{}^cD^{\gamma}h_2(x,y)(t_2)-{}^cD^{\gamma}h_2(x,y)(t_1)\bigr| \\
&\quad\le \frac{t_2^{1-\gamma}-t_1^{1-\gamma}}{(1-\gamma)\Gamma(1-\gamma)}\bigl(\overline{P}_3\cdot\|m_2\|\bigl(\varphi_2(r)+\psi_2(r)+\rho_2(r)\bigr) \\
&\qquad+\overline{P}_4\cdot\|m_1\|\bigl(\varphi_1(r)+\psi_1(r)+\rho_1(r)\bigr)+\overline{Q}_2r\bigr).
\end{aligned}$$

Hence $|h_2(x,y)(t_2)-h_2(x,y)(t_1)|\to 0$ as $t_2\to t_1$. Therefore, the operator $N(x,y)$ is equicontinuous.

From the foregoing arguments, we infer that the operator $N(x,y)$ is completely continuous by the Arzelá-Ascoli theorem.

Step 4. N has a closed graph.

Letting $(x_n,y_n)\to(x_*,y_*)$, $(h_n,\overline{h}_n)\in N(x_n,y_n)$ and $(h_n,\overline{h}_n)\to(h_*,\overline{h}_*)$, we need to show $(h_*,\overline{h}_*)\in N(x_*,y_*)$. Now $(h_n,\overline{h}_n)\in N(x_n,y_n)$ implies that there exist $f_n\in S_{F,(x_n,y_n)}$ and $g_n\in S_{G,(x_n,y_n)}$ such that for all $t\in J$,

$$\begin{aligned}
h_n(x_n,y_n)(t)={}&\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f_n(s)\,ds+(\sigma_1t+1)h(y)+t\sigma_2\phi(x) \\
&+\frac{t}{\Delta}\biggl[\mu_2\xi\biggl(\mu_1\int_0^{\eta}\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}f_n(s)\,ds-\int_0^T\frac{(T-s)^{\beta}}{\Gamma(\beta+1)}g_n(s)\,ds\biggr) \\
&+\frac{T^2}{2}\biggl(\mu_2\int_0^{\xi}\frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}g_n(s)\,ds-\int_0^T\frac{(T-s)^{\alpha}}{\Gamma(\alpha+1)}f_n(s)\,ds\biggr)\biggr]
\end{aligned}$$

and

$$\begin{aligned}\overline{h}_n(x_n,y_n)(t) = {}& \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g_n(s)\,ds + (\sigma_3 t+1)\phi(x) + t\sigma_4 h(y) \\ & + \frac{t}{\Delta}\biggl[\mu_1\eta\biggl(\mu_2\int_0^{\xi}\frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}g_n(s)\,ds - \int_0^T\frac{(T-s)^{\alpha}}{\Gamma(\alpha+1)}f_n(s)\,ds\biggr) \\ & + \frac{T^2}{2}\biggl(\mu_1\int_0^{\eta}\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}f_n(s)\,ds - \int_0^T\frac{(T-s)^{\beta}}{\Gamma(\beta+1)}g_n(s)\,ds\biggr)\biggr].\end{aligned}$$

Let us consider the continuous linear operators $\Phi_1, \Phi_2 : L^1(J, X \times Y) \to C(J, X \times Y)$ given by

$$\begin{aligned}\Phi_1(x,y)(t) = {}& \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\,ds + (\sigma_1 t+1)h(y) + t\sigma_2 \phi(x) \\ & + \frac{t}{\Delta}\biggl[\mu_2\xi\biggl(\mu_1\int_0^{\eta}\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)\,ds - \int_0^T\frac{(T-s)^{\beta}}{\Gamma(\beta+1)}g(s)\,ds\biggr) \\ & + \frac{T^2}{2}\biggl(\mu_2\int_0^{\xi}\frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}g(s)\,ds - \int_0^T\frac{(T-s)^{\alpha}}{\Gamma(\alpha+1)}f(s)\,ds\biggr)\biggr]\end{aligned}$$

and

$$\begin{aligned}\Phi_2(x,y)(t) = {}& \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g(s)\,ds + (\sigma_3 t+1)\phi(x) + t\sigma_4 h(y) \\ & + \frac{t}{\Delta}\biggl[\mu_1\eta\biggl(\mu_2\int_0^{\xi}\frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}g(s)\,ds - \int_0^T\frac{(T-s)^{\alpha}}{\Gamma(\alpha+1)}f(s)\,ds\biggr) \\ & + \frac{T^2}{2}\biggl(\mu_1\int_0^{\eta}\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)\,ds - \int_0^T\frac{(T-s)^{\beta}}{\Gamma(\beta+1)}g(s)\,ds\biggr)\biggr].\end{aligned}$$

From Lemma 2.3, we know that $(\Phi_1, \Phi_2) \circ (S_F, S_G)$ is a closed graph operator. Moreover, we get $(h_n, \overline{h}_n) \in (\Phi_1, \Phi_2) \circ (S_{F,(x_n,y_n)}, S_{G,(x_n,y_n)})$ for all n. Since $(x_n, y_n) \to (x_*, y_*)$, $(h_n, \overline{h}_n) \to (h_*, \overline{h}_*)$ it follows the existence of $f_* \in S_{F,(x_*,y_*)}$ and $g_* \in S_{G,(x_*,y_*)}$ such that

$$\begin{aligned}h_*(x_*,y_*)(t) = {}& \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_*(s)\,ds + (\sigma_1 t+1)h(y) + t\sigma_2 \phi(x) \\ & + \frac{t}{\Delta}\biggl[\mu_2\xi\biggl(\mu_1\int_0^{\eta}\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}f_*(s)\,ds - \int_0^T\frac{(T-s)^{\beta}}{\Gamma(\beta+1)}g_*(s)\,ds\biggr) \\ & + \frac{T^2}{2}\biggl(\mu_2\int_0^{\xi}\frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}g_*(s)\,ds - \int_0^T\frac{(T-s)^{\alpha}}{\Gamma(\alpha+1)}f_*(s)\,ds\biggr)\biggr]\end{aligned}$$

and

$$\begin{aligned}\overline{h}_*(x_*,y_*)(t) = {}& \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g_*(s)\,ds + (\sigma_3 t+1)\phi(x) + t\sigma_4 h(y) \\ & + \frac{t}{\Delta}\biggl[\mu_1\eta\biggl(\mu_2\int_0^{\xi}\frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}g_*(s)\,ds - \int_0^T\frac{(T-s)^{\alpha}}{\Gamma(\alpha+1)}f_*(s)\,ds\biggr) \\ & + \frac{T^2}{2}\biggl(\mu_1\int_0^{\eta}\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}f_*(s)\,ds - \int_0^T\frac{(T-s)^{\beta}}{\Gamma(\beta+1)}g_*(s)\,ds\biggr)\biggr],\end{aligned}$$

that is, $(h_*, \overline{h}_*) \in N(x_*, y_*)$.

Step 5. A priori bounds on solutions.

Let $(x,y) \in \lambda N(x,y)$ for some $\lambda \in (0,1)$. Then there exist $f \in S_{F,(x,y)}$ and $g \in S_{G,(x,y)}$ such that for all $t \in J$,

$$\begin{aligned} x(t) &= \lambda \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\,ds + (\sigma_1 t + 1)h(y) + t\sigma_2 \phi(x) \\ &\quad + \frac{\lambda t}{\Delta}\biggl[\mu_2 \xi \biggl(\mu_1 \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\,ds - \int_0^T \frac{(T-s)^\beta}{\Gamma(\beta+1)} g(s)\,ds\biggr) \\ &\quad + \frac{T^2}{2}\biggl(\mu_2 \int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} g(s)\,ds - \int_0^T \frac{(T-s)^\alpha}{\Gamma(\alpha+1)} f(s)\,ds\biggr)\biggr] \end{aligned}$$

and

$$\begin{aligned} y(t) &= \lambda \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g(s)\,ds + (\sigma_3 t + 1)\phi(x) + t\sigma_4 h(y) \\ &\quad + \frac{\lambda t}{\Delta}\biggl[\mu_1 \eta \biggl(\mu_2 \int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} g(s)\,ds - \int_0^T \frac{(T-s)^\alpha}{\Gamma(\alpha+1)} f(s)\,ds\biggr) \\ &\quad + \frac{T^2}{2}\biggl(\mu_1 \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\,ds - \int_0^T \frac{(T-s)^\beta}{\Gamma(\beta+1)} g(s)\,ds\biggr)\biggr]. \end{aligned}$$

With the same arguments as in Step 2 of our proof, for each $(x,y) \in X \times Y$, we obtain

$$\begin{aligned} \|x\|_X &\le \biggl(P_1 + \frac{T^{1-\delta}}{\Gamma(2-\delta)}\overline{P}_1\biggr) \cdot \|m_1\| \bigl(\varphi_1\bigl(\|(x,y)\|_{X\times Y}\bigr) \\ &\quad + \psi_1\bigl(\|(x,y)\|_{X\times Y}\bigr) + \rho_1\bigl(\|(x,y)\|_{X\times Y}\bigr)\bigr) \\ &\quad + \biggl(P_2 + \frac{T^{1-\delta}}{\Gamma(2-\delta)}\overline{P}_2\biggr) \cdot \|m_2\| \bigl(\varphi_2\bigl(\|(x,y)\|_{X\times Y}\bigr) \\ &\quad + \psi_2\bigl(\|(x,y)\|_{X\times Y}\bigr) + \rho_2\bigl(\|(x,y)\|_{X\times Y}\bigr)\bigr) \\ &\quad + \biggl(Q_1 + \frac{T^{1-\delta}}{\Gamma(2-\delta)}\overline{Q}_1\biggr)\|(x,y)\|_{X\times Y} \end{aligned}$$

and

$$\begin{aligned} \|y\|_Y &\le \biggl(P_3 + \frac{T^{1-\gamma}}{\Gamma(2-\gamma)}\overline{P}_3\biggr) \cdot \|m_2\| \bigl(\varphi_2\bigl(\|(x,y)\|_{X\times Y}\bigr) \\ &\quad + \psi_2\bigl(\|(x,y)\|_{X\times Y}\bigr) + \rho_2\bigl(\|(x,y)\|_{X\times Y}\bigr)\bigr) \\ &\quad + \biggl(P_4 + \frac{T^{1-\gamma}}{\Gamma(2-\gamma)}\overline{P}_4\biggr) \cdot \|m_1\| \bigl(\varphi_1\bigl(\|(x,y)\|_{X\times Y}\bigr) \\ &\quad + \psi_1\bigl(\|(x,y)\|_{X\times Y}\bigr) + \rho_1\bigl(\|(x,y)\|_{X\times Y}\bigr)\bigr) + \biggl(Q_2 + \frac{T^{1-\gamma}}{\Gamma(2-\gamma)}\overline{Q}_2\biggr)\|(x,y)\|_{X\times Y}. \end{aligned}$$

Thus

$$\begin{aligned} (1-\Lambda_3)\|(x,y)\|_{X\times Y} &\le \sum_{i=1}^{2} \Lambda_i \cdot \|m_i\| \bigl(\varphi_i\bigl(\|(x,y)\|_{X\times Y}\bigr) \\ &\quad + \psi_i\bigl(\|(x,y)\|_{X\times Y}\bigr) + \rho_i\bigl(\|(x,y)\|_{X\times Y}\bigr)\bigr). \end{aligned}$$

Now we set

$$\mathcal{U} = \{(x,y) \in X \times Y : \|(x,y)\|_{X\times Y} < K\}.$$

Clearly, $\mathcal{U}$ is an open subset of $X \times Y$ and $(0,0) \in \mathcal{U}$. As a consequence of Steps 1-4, together with the Arzelá-Ascoli theorem, we can conclude that $N : \overline{\mathcal{U}} \to \mathcal{P}_{cp,cv}(X) \times \mathcal{P}_{cp,cv}(Y)$ is upper semicontinuous and completely continuous. From the choice of $\mathcal{U}$, there is no $(x,y) \in \partial\mathcal{U}$ such that $(x,y) \in \lambda N(x,y)$ for some $\lambda \in (0,1)$. Therefore, by Theorem 2.4, we deduce that N has a fixed point $(x,y) \in \overline{\mathcal{U}}$, which is a solution of the coupled system (1.3)-(1.4). This completes the proof. □

3.2 The lower semicontinuous case

Now we study the case when F, G are not necessarily convex valued.

In this result, we need to give the following conditions:

(H4) $F, G : J \times \mathbb{R}^3 \to \mathcal{P}_{cp}(\mathbb{R})$ are multivalued maps such that

(1) $(t,x,y,z) \to F(t,x,y,z)$ and $(t,x,y,z) \to G(t,x,y,z)$ are $\Sigma \otimes \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B}$ measurable;

(2) $(x,y,z) \to F(t,x,y,z)$ and $(x,y,z) \to G(t,x,y,z)$ are lower semicontinuous for a.e. $t \in J$.

Theorem 3.2 *Assume that* (H2)-(H4) *and relation* (3.4) *hold. Then the coupled system* (1.3)-(1.4) *has at least one solution on J.*

Proof From (H2), (H4) and [35], Lemma 4.1, maps

$$\mathcal{F}_1 : X \to \mathcal{P}(L^1(J,\mathbb{R})), \qquad x \to \mathcal{F}_1(x,y) = S_{F,(x,y)},$$
$$\mathcal{F}_2 : Y \to \mathcal{P}(L^1(J,\mathbb{R})), \qquad y \to \mathcal{F}_2(x,y) = S_{G,(x,y)},$$

are lower semicontinuous and have nonempty closed and decomposable values. Then from the selection theorem due to Bressan and Colombo [36], there exist continuous functions $f : X \to L^1(J,\mathbb{R})$ and $g : Y \to L^1(J,\mathbb{R})$ such that $f \in \mathcal{F}_1(x,y)$ and $g \in \mathcal{F}_2(x,y)$ for all $x \in X$, $y \in Y$. That is to say, we have $f(t,x(t),y(t),{}^cD^\gamma y(t)) \in F(t,x(t),y(t),{}^cD^\gamma y(t))$ and $g(t,x(t),{}^cD^\delta x(t),y(t)) \in G(t,x(t),{}^cD^\delta x(t),y(t))$ for a.e. $t \in J$. Now consider the problem

$$\begin{cases} {}^cD^\alpha x(t) = f(t,x(t),y(t),{}^cD^\gamma y(t)), & t \in J, \\ {}^cD^\beta y(t) = g(t,x(t),{}^cD^\delta x(t),y(t)), & t \in J, \end{cases} \tag{3.5}$$

with the boundary conditions (1.4). Note that if $(x,y) \in X \times Y$ is a solution of the coupled system (3.5), then (x,y) is a solution to the coupled system (1.3)-(1.4).

A solution of the boundary value problem (3.5), (1.4) is then reformulated as a fixed point problem for the operator $\overline{N} : X \times Y \to X \times Y$ defined by

$$\overline{N}(x,y)(t) = (\overline{N}_1(x,y)(t), \overline{N}_2(x,y)(t)),$$

where

$$\begin{aligned}\overline{N}_1(x,y)(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s,x(s),y(s),{}^cD^\gamma y(s)\bigr)\,ds + (\sigma_1 t+1)h(y) + t\sigma_2\phi(x)\\ &\quad + \frac{t}{\Delta}\biggl[\mu_2\xi\biggl(\mu_1\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s,x(s),y(s),{}^cD^\gamma y(s)\bigr)\,ds\\ &\quad - \int_0^T \frac{(T-s)^\beta}{\Gamma(\beta+1)} g\bigl(s,x(s),{}^cD^\delta x(s),y(s)\bigr)\,ds\biggr) + \frac{T^2}{2}\biggl(\mu_2\int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}\\ &\quad \times g\bigl(s,x(s),{}^cD^\delta x(s),y(s)\bigr)\,ds - \int_0^T \frac{(T-s)^\alpha}{\Gamma(\alpha+1)} f\bigl(s,x(s),y(s),{}^cD^\gamma y(s)\bigr)\,ds\biggr)\biggr]\end{aligned}$$

and

$$\begin{aligned}\overline{N}_2(x,y)(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g\bigl(s,x(s),{}^cD^\delta x(s),y(s)\bigr)\,ds + (\sigma_3 t+1)\phi(x) + t\sigma_4 h(y)\\ &\quad + \frac{t}{\Delta}\biggl[\mu_1\eta\biggl(\mu_2\int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} g\bigl(s,x(s),{}^cD^\delta x(s),y(s)\bigr)\,ds\\ &\quad - \int_0^T \frac{(T-s)^\alpha}{\Gamma(\alpha+1)} f\bigl(s,x(s),y(s),{}^cD^\gamma y(s)\bigr)\,ds\biggr) + \frac{T^2}{2}\biggl(\mu_1\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}\\ &\quad \times f\bigl(s,x(s),y(s),{}^cD^\gamma y(s)\bigr)\,ds - \int_0^T \frac{(T-s)^\beta}{\Gamma(\beta+1)} g\bigl(s,x(s),{}^cD^\delta x(s),y(s)\bigr)\,ds\biggr)\biggr].\end{aligned}$$

It can easily be shown that $\overline{N}$ is continuous and completely continuous and satisfies all the conditions of the Leray-Schauder nonlinear alternative for single-valued maps [37]. The remaining part of the proof is similar to that of Theorem 3.1, so we omit it. This completes the proof. □

3.3 The Lipschitz case

In this section, we need to give the following conditions:

(H5) $F,G : J\times\mathbb{R}^3 \to \mathcal{P}_{cp}(\mathbb{R})$ are multivalued maps such that

(1) F and G are integrable bounded, and maps $t \to F(t,x,y,z)$ and $t \to G(t,x,y,z)$ are measurable for all $x,y,z\in\mathbb{R}$;

(2) There exist $m_3,m_4 \in C(J,\mathbb{R}^+)$ such that for a.e. $t\in J$ and all $x_1,x_2,y_1,y_2,z_1,z_2\in\mathbb{R}$,

$$H_d\bigl(F(t,x_1,y_1,z_1),F(t,x_2,y_2,z_2)\bigr) \le m_3(t)\bigl(|x_1-x_2|+|y_1-y_2|+|z_1-z_2|\bigr)$$

and

$$H_d\bigl(G(t,x_1,y_1,z_1),G(t,x_2,y_2,z_2)\bigr) \le m_4(t)\bigl(|x_1-x_2|+|y_1-y_2|+|z_1-z_2|\bigr).$$

Theorem 3.3 *Assume that* (H3) *and* (H5) *hold. If, in addition,*

$$\Lambda_4 + \frac{T^{1-\delta}}{\Gamma(2-\delta)}\Lambda_5 + \Lambda_6 + \frac{T^{1-\gamma}}{\Gamma(2-\gamma)}\Lambda_7 < 1, \tag{3.6}$$

where

$$\begin{aligned}\Lambda_4 =& \frac{T^\alpha \|m_3\|}{\Gamma(\alpha+1)} + \frac{|\mu_1\mu_2|T^{\alpha+1}\xi\|m_3\|}{|\Delta|\Gamma(\alpha+1)} + \frac{|\mu_2|T^{\beta+2}\xi\|m_4\|}{|\Delta|\Gamma(\beta+2)} + \frac{|\mu_2|T^{\beta+3}\|m_4\|}{2|\Delta|\Gamma(\beta+1)} \\ &+ \frac{T^{\alpha+4}\|m_3\|}{2|\Delta|\Gamma(\alpha+2)} + \max\{L_1|\sigma_1+1|, L_2|\sigma_2|\},\end{aligned}$$

$$\begin{aligned}\Lambda_5 =& \frac{T^{\alpha-1} \|m_3\|}{\Gamma(\alpha)} + \frac{|\mu_1\mu_2|T^{\alpha}\xi\|m_3\|}{|\Delta|\Gamma(\alpha+1)} + \frac{|\mu_2|T^{\beta+1}\xi\|m_4\|}{|\Delta|\Gamma(\beta+2)} + \frac{|\mu_2|T^{\beta+2}\|m_4\|}{2|\Delta|\Gamma(\beta+1)} \\ &+ \frac{T^{\alpha+3}\|m_3\|}{2|\Delta|\Gamma(\alpha+2)} + \max\{L_1|\sigma_1|, L_2|\sigma_2|\},\end{aligned}$$

$$\begin{aligned}\Lambda_6 =& \frac{T^\beta \|m_4\|}{\Gamma(\beta+1)} + \frac{|\mu_1\mu_2|T^{\beta+1}\eta\|m_4\|}{|\Delta|\Gamma(\beta+1)} + \frac{|\mu_1|T^{\alpha+2}\eta\|m_3\|}{|\Delta|\Gamma(\alpha+2)} + \frac{|\mu_1|T^{\alpha+3}\|m_3\|}{2|\Delta|\Gamma(\alpha+1)} \\ &+ \frac{T^{\beta+4}\|m_4\|}{2|\Delta|\Gamma(\beta+2)} + \max\{L_2|\sigma_3+1|, L_1|\sigma_4|\},\end{aligned}$$

$$\begin{aligned}\Lambda_7 =& \frac{T^{\beta-1} \|m_4\|}{\Gamma(\beta)} + \frac{|\mu_1\mu_2|T^{\beta}\eta\|m_4\|}{|\Delta|\Gamma(\beta+1)} + \frac{|\mu_1|T^{\alpha+1}\eta\|m_3\|}{|\Delta|\Gamma(\alpha+2)} + \frac{|\mu_1|T^{\alpha+2}\|m_3\|}{2|\Delta|\Gamma(\alpha+1)} \\ &+ \frac{T^{\beta+3}\|m_4\|}{2|\Delta|\Gamma(\beta+2)} + \max\{L_2|\sigma_3|, L_1|\sigma_4|\},\end{aligned}$$

then the coupled system (1.3)-(1.4) *has at least one solution on* J.

Proof From (H5), we have that the multivalued maps $t \to F(t, x(t), y(t), {}^cD^\gamma y(t))$ and $t \to G(t, x(t), {}^cD^\delta x(t), y(t))$ are measurable [29], Proposition 2.7.9, and closed valued for each $(x, y) \in X \times Y$. Hence they have measurable selection [29], Theorem 2.2.1, and the sets $S_{F,(x,y)}$ and $S_{G,(x,y)}$ are nonempty. Let N be defined in (3.3). We will show that, under this situation, N satisfies the requirements of Lemma 2.5.

Step 1. For each $(x, y) \in X \times Y$, $N(x, y) \in \mathcal{P}_{cl}(X) \times \mathcal{P}_{cl}(Y)$. Let $(h_n, \overline{h}_n) \in N(x_n, y_n)$ $(n \geq 1)$ such that $(h_n, \overline{h}_n) \to (h, \overline{h})$ in $X \times Y$. Then $(h, \overline{h}) \in X \times Y$ and there exist $f_n \in S_{F,(x_n,y_n)}$ and $g_n \in S_{G,(x_n,y_n)}$ $(n \geq 1)$ such that for all $t \in J$,

$$\begin{aligned}h_n(x_n, y_n)(t) =& \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_n(s)\,ds + (\sigma_1 t + 1)h(y) + t\sigma_2\phi(x) \\ &+ \frac{t}{\Delta}\left[\mu_2\xi\left(\mu_1\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} f_n(s)\,ds - \int_0^T \frac{(T-s)^\beta}{\Gamma(\beta+1)} g_n(s)\,ds\right)\right. \\ &\left. + \frac{T^2}{2}\left(\mu_2\int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} g_n(s)\,ds - \int_0^T \frac{(T-s)^\alpha}{\Gamma(\alpha+1)} f_n(s)\,ds\right)\right]\end{aligned}$$

and

$$\begin{aligned}\overline{h}_n(x_n, y_n)(t) =& \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g_n(s)\,ds + (\sigma_3 t + 1)\phi(x) + t\sigma_4 h(y) \\ &+ \frac{t}{\Delta}\left[\mu_1\eta\left(\mu_2\int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} g_n(s)\,ds - \int_0^T \frac{(T-s)^\alpha}{\Gamma(\alpha+1)} f_n(s)\,ds\right)\right. \\ &\left. + \frac{T^2}{2}\left(\mu_1\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} f_n(s)\,ds - \int_0^T \frac{(T-s)^\beta}{\Gamma(\beta+1)} g_n(s)\,ds\right)\right].\end{aligned}$$

By (H5), the sequences f_n and g_n are integrable bounded. Since F and G have compact values, we may pass to subsequences if necessary to get that f_n and g_n converge to f and g in $L^1(J,\mathbb{R})$. Thus $f \in S_{F,(x,y)}$, $g \in S_{G,(x,y)}$ and for each $t \in J$,

$$\begin{aligned} h_n(x_n,y_n)(t) \to h(x,y)(t) \\ &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\,ds + (\sigma_1 t+1)h(y) + t\sigma_2\phi(x) \\ &\quad + \frac{t}{\Delta}\biggl[\mu_2\xi\biggl(\mu_1\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\,ds - \int_0^T \frac{(T-s)^\beta}{\Gamma(\beta+1)} g(s)\,ds\biggr) \\ &\quad + \frac{T^2}{2}\biggl(\mu_2\int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} g(s)\,ds - \int_0^T \frac{(T-s)^\alpha}{\Gamma(\alpha+1)} f(s)\,ds\biggr)\biggr] \end{aligned}$$

and

$$\begin{aligned} \overline{h}_n(x_n,y_n)(t) \to \overline{h}(x,y)(t) \\ &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g(s)\,ds + (\sigma_3 t+1)\phi(x) + t\sigma_4 h(y) \\ &\quad + \frac{t}{\Delta}\biggl[\mu_1\eta\biggl(\mu_2\int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} g(s)\,ds - \int_0^T \frac{(T-s)^\alpha}{\Gamma(\alpha+1)} f(s)\,ds\biggr) \\ &\quad + \frac{T^2}{2}\biggl(\mu_1\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\,ds - \int_0^T \frac{(T-s)^\beta}{\Gamma(\beta+1)} g(s)\,ds\biggr)\biggr]. \end{aligned}$$

This means that $(h,\overline{h}) \in N$ and N is closed.

Step 2. There exists $\tau < 1$ such that

$$H_d\bigl(N(x,y),N(\overline{x},\overline{y})\bigr) \le \tau\bigl(\|x-\overline{x}\|_X + \|y-\overline{y}\|_Y\bigr), \quad \forall x,\overline{x} \in X, y,\overline{y} \in Y.$$

Let $(x,y),(\overline{x},\overline{y}) \in X \times Y$ and $(h_1,h_2) \in N(x,y)$. Then there exist $f_1 \in S_{F,(x,y)}$ and $g_1 \in S_{G,(x,y)}$ such that for all $t \in J$,

$$\begin{aligned} h_1(x,y)(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s)\,ds + (\sigma_1 t+1)h(y) + t\sigma_2\phi(x) \\ &\quad + \frac{t}{\Delta}\biggl[\mu_2\xi\biggl(\mu_1\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s)\,ds - \int_0^T \frac{(T-s)^\beta}{\Gamma(\beta+1)} g_1(s)\,ds\biggr) \\ &\quad + \frac{T^2}{2}\biggl(\mu_2\int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} g_1(s)\,ds - \int_0^T \frac{(T-s)^\alpha}{\Gamma(\alpha+1)} f_1(s)\,ds\biggr)\biggr] \end{aligned}$$

and

$$\begin{aligned} h_2(x,y)(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g_1(s)\,ds + (\sigma_3 t+1)\phi(x) + t\sigma_4 h(y) \\ &\quad + \frac{t}{\Delta}\biggl[\mu_1\eta\biggl(\mu_2\int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} g_1(s)\,ds - \int_0^T \frac{(T-s)^\alpha}{\Gamma(\alpha+1)} f_1(s)\,ds\biggr) \\ &\quad + \frac{T^2}{2}\biggl(\mu_1\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s)\,ds - \int_0^T \frac{(T-s)^\beta}{\Gamma(\beta+1)} g_1(s)\,ds\biggr)\biggr]. \end{aligned}$$

From (H5)(2), we deduce

$$\begin{aligned}&H_d\big(F\big(t,x(t),y(t),{}^cD^\gamma y(t)\big),F\big(t,\overline{x}(t),\overline{y}(t),{}^cD^\gamma\overline{y}(t)\big)\big)\\&\quad\le m_3(t)\big(\big|x(t)-\overline{x}(t)\big|+\big|y(t)-\overline{y}(t)\big|+\big|{}^cD^\gamma y(t)-{}^cD^\gamma\overline{y}(t)\big|\big)\end{aligned}$$

and

$$\begin{aligned}&H_d\big(G\big(t,x(t),{}^cD^\delta x(t),y(t)\big),G\big(t,\overline{x}(t),{}^cD^\delta\overline{x}(t),\overline{y}(t)\big)\big)\\&\quad\le m_4(t)\big(\big|x(t)-\overline{x}(t)\big|+\big|{}^cD^\delta x(t)-{}^cD^\delta\overline{x}(t)\big|+\big|y(t)-\overline{y}(t)\big|\big).\end{aligned}$$

Hence, for a.e. $t\in J$, there exist $f\in F(t,\overline{x}(t),\overline{y}(t),{}^cD^\gamma\overline{y}(t))$ and $g\in G(t,\overline{x}(t),{}^cD^\delta\overline{x}(t),\overline{y}(t))$ such that

$$\big|f_1(t)-f\big|\le m_3(t)\big(\big|x(t)-\overline{x}(t)\big|+\big|y(t)-\overline{y}(t)\big|+\big|{}^cD^\gamma y(t)-{}^cD^\gamma\overline{y}(t)\big|\big)\tag{3.7}$$

and

$$\big|g_1(t)-g\big|\le m_4(t)\big(\big|x(t)-\overline{x}(t)\big|+\big|{}^cD^\delta x(t)-{}^cD^\delta\overline{x}(t)\big|+\big|y(t)-\overline{y}(t)\big|\big).\tag{3.8}$$

Consider the multivalued maps $V_1,V_2:J\to\mathcal{P}(\mathbb{R})$ given by

$$V_1(t)=\left\{\begin{array}{l}f\in\mathbb{R}:\\|f_1(t)-f|\le m_3(t)(|x(t)-\overline{x}(t)|+|y(t)-\overline{y}(t)|+|{}^cD^\gamma y(t)-{}^cD^\gamma\overline{y}(t)|)\end{array}\right\}$$

and

$$V_2(t)=\left\{\begin{array}{l}g\in\mathbb{R}:\\|g_1(t)-g|\le m_4(t)(|x(t)-\overline{x}(t)|+|{}^cD^\delta x(t)-{}^cD^\delta\overline{x}(t)|+|y(t)-\overline{y}(t)|)\end{array}\right\}.$$

Define

$$\begin{aligned}&\kappa_1(t)=m_3(t)\big(\big|x(t)-\overline{x}(t)\big|+\big|y(t)-\overline{y}(t)\big|+\big|{}^cD^\gamma y(t)-{}^cD^\gamma\overline{y}(t)\big|\big),\\&\kappa_2(t)=m_4(t)\big(\big|x(t)-\overline{x}(t)\big|+\big|{}^cD^\delta x(t)-{}^cD^\delta\overline{x}(t)\big|+\big|y(t)-\overline{y}(t)\big|\big).\end{aligned}$$

Since $f_1(t)$, $g_1(t)$, $\kappa_1(t)$, $\kappa_2(t)$ are measurable, [38], Theorem III.41, implies that V_1 and V_2 are measurable. It follows from (H5) that the maps $t\to F(t,x(t),y(t),{}^cD^\gamma y(t))$ and $t\to G(t,x(t),{}^cD^\delta x(t),y(t))$ are measurable. Hence, by (3.7)-(3.8) and [29], Proposition 2.1.43, the multivalued maps $t\to V_1(t)\cap F(t,\overline{x}(t),\overline{y}(t),{}^cD^\gamma\overline{y}(t))$ and $t\to V_2(t)\cap G(t,\overline{x}(t),{}^cD^\delta\overline{x}(t),\overline{y}(t))$ are measurable and nonempty closed valued. Therefore, we can find $f_2(t)\in F(t,\overline{x}(t),\overline{y}(t),{}^cD^\gamma\overline{y}(t))$ and $g_2(t)\in G(t,\overline{x}(t),{}^cD^\delta\overline{x}(t),\overline{y}(t))$ such that for a.e. $t\in J$,

$$\big|f_1(t)-f_2(t)\big|\le m_3(t)\big(\big|x(t)-\overline{x}(t)\big|+\big|y(t)-\overline{y}(t)\big|+\big|{}^cD^\gamma y(t)-{}^cD^\gamma\overline{y}(t)\big|\big)$$

and

$$\big|g_1(t)-g_2(t)\big|\le m_4(t)\big(\big|x(t)-\overline{x}(t)\big|+\big|{}^cD^\delta x(t)-{}^cD^\delta\overline{x}(t)\big|+\big|y(t)-\overline{y}(t)\big|\big).$$

Let

$$\begin{aligned}
\overline{h}_1(\overline{x},\overline{y})(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_2(s)\,ds + (\sigma_1 t+1)h(\overline{y}) + t\sigma_2\phi(\overline{x}) \\
&\quad + \frac{t}{\Delta}\biggl[\mu_2\xi\biggl(\mu_1\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} f_2(s)\,ds - \int_0^T \frac{(T-s)^\beta}{\Gamma(\beta+1)} g_2(s)\,ds\biggr) \\
&\quad + \frac{T^2}{2}\biggl(\mu_2\int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} g_2(s)\,ds - \int_0^T \frac{(T-s)^\alpha}{\Gamma(\alpha+1)} f_2(s)\,ds\biggr)\biggr]
\end{aligned}$$

and

$$\begin{aligned}
\overline{h}_2(\overline{x},\overline{y})(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g_2(s)\,ds + (\sigma_3 t+1)\phi(\overline{x}) + t\sigma_4 h(\overline{y}) \\
&\quad + \frac{t}{\Delta}\biggl[\mu_1\eta\biggl(\mu_2\int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} g_2(s)\,ds - \int_0^T \frac{(T-s)^\alpha}{\Gamma(\alpha+1)} f_2(s)\,ds\biggr) \\
&\quad + \frac{T^2}{2}\biggl(\mu_1\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} f_2(s)\,ds - \int_0^T \frac{(T-s)^\beta}{\Gamma(\beta+1)} g_2(s)\,ds\biggr)\biggr],
\end{aligned}$$

that is, $(\overline{h}_1,\overline{h}_2) \in N(\overline{x},\overline{y})$. Since

$$\begin{aligned}
&\bigl|h_1(x,y)(t) - \overline{h}_1(\overline{x},\overline{y})(t)\bigr| \\
&\quad\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\bigl|f_1(s)-f_2(s)\bigr|\,ds \\
&\qquad + \frac{t}{\Delta}\biggl[\mu_2\xi\biggl(\mu_1\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}\bigl|f_1(s)-f_2(s)\bigr|\,ds - \int_0^T \frac{(T-s)^\beta}{\Gamma(\beta+1)}\bigl|g_1(s)-g_2(s)\bigr|\,ds\biggr) \\
&\qquad + \frac{T^2}{2}\biggl(\mu_2\int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}\bigl|g_1(s)-g_2(s)\bigr|\,ds - \int_0^T \frac{(T-s)^\alpha}{\Gamma(\alpha+1)}\bigl|f_1(s)-f_2(s)\bigr|\,ds\biggr)\biggr] \\
&\qquad + |t\sigma_1+1|\cdot\bigl|h(y)-h(\overline{y})\bigr| + t|\sigma_2|\cdot\bigl|\phi(x)-\phi(\overline{x})\bigr| \\
&\quad\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} m_3(s)\bigl(\bigl|x(s)-\overline{x}(s)\bigr| + \bigl|y(s)-\overline{y}(s)\bigr| + \bigl|{}^cD^\gamma y(s){}^cD^\gamma\overline{y}(s)\bigr|\bigr)\,ds \\
&\qquad + \frac{t}{\Delta}\biggl[\mu_2\xi\biggl(\mu_1\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} m_3(s)\bigl(\bigl|x(s)-\overline{x}(s)\bigr| \\
&\qquad + \bigl|y(s)-\overline{y}(s)\bigr| + \bigl|{}^cD^\gamma y(s){}^cD^\gamma\overline{y}(s)\bigr|\bigr)\,ds \\
&\qquad - \int_0^T \frac{(T-s)^\beta}{\Gamma(\beta+1)} m_4(s)\bigl(\bigl|x(s)-\overline{x}(s)\bigr| + \bigl|{}^cD^\delta x(s){}^cD^\delta\overline{x}(s)\bigr| + \bigl|y(s)-\overline{y}(s)\bigr|\bigr)\,ds\biggr) \\
&\qquad + \frac{T^2}{2}\biggl(\mu_2\int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} m_4(s)\bigl(\bigl|x(s)-\overline{x}(s)\bigr| + \bigl|{}^cD^\delta x(s){}^cD^\delta\overline{x}(s)\bigr| + \bigl|y(s)-\overline{y}(s)\bigr|\bigr)\,ds \\
&\qquad - \int_0^T \frac{(T-s)^\alpha}{\Gamma(\alpha+1)} m_3(s)\bigl(\bigl|x(s)-\overline{x}(s)\bigr| + \bigl|y(s)-\overline{y}(s)\bigr| + \bigl|{}^cD^\gamma y(s){}^cD^\gamma\overline{y}(s)\bigr|\bigr)\,ds\biggr)\biggr] \\
&\qquad + L_1|\sigma_1+1|\cdot\|y-\overline{y}\|_Y + L_2|\sigma_2|\cdot\|x-\overline{x}\|_X \\
&\quad\le \frac{T^\alpha}{\Gamma(\alpha+1)}\cdot\|m_3\|\bigl(\|x-\overline{x}\|_X + \|y-\overline{y}\|_Y\bigr)
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{|\mu_1\mu_2|T^{\alpha+1}\xi}{|\Delta|\Gamma(\alpha+1)} \cdot \|m_3\|\big(\|x-\overline{x}\|_X + \|y-\overline{y}\|_Y\big) \\
&+ \frac{|\mu_2|T^{\beta+2}\xi}{|\Delta|\Gamma(\beta+2)} \cdot \|m_4\|\big(\|x-\overline{x}\|_X + \|y-\overline{y}\|_Y\big) \\
&+ \frac{|\mu_2|T^{\beta+3}}{2|\Delta|\Gamma(\beta+1)} \cdot \|m_4\|\big(\|x-\overline{x}\|_X + \|y-\overline{y}\|_Y\big) \\
&+ \frac{T^{\alpha+4}}{2|\Delta|\Gamma(\alpha+2)} \cdot \|m_3\|\big(\|x-\overline{x}\|_X + \|y-\overline{y}\|_Y\big) \\
&+ \max\big\{L_1|\sigma_1+1|, L_2|\sigma_2|\big\}\big(\|x-\overline{x}\|_X + \|y-\overline{y}\|_Y\big) \\
= {} & \Lambda_4 \cdot \big(\|x-\overline{x}\|_X + \|y-\overline{y}\|_Y\big),
\end{aligned}
$$

$$
\begin{aligned}
&\big|h_1(x,y)'(t) - \overline{h}_1(\overline{x},\overline{y})'(t)\big| \\
&\leq \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\big|f_1(s)-f_2(s)\big|\,ds + \frac{|\mu_1\mu_2|\xi}{|\Delta|}\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}\big|f_1(s)-f_2(s)\big|\,ds \\
&\quad + \frac{|\mu_2|\xi}{|\Delta|}\int_0^T \frac{(T-s)^{\beta}}{\Gamma(\beta+1)}\big|g_1(s)-g_2(s)\big|\,ds + \frac{|\mu_2|T^2}{2|\Delta|}\int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}\big|g_1(s)-g_2(s)\big|\,ds \\
&\quad + \frac{T^2}{2|\Delta|}\int_0^T \frac{(T-s)^{\alpha}}{\Gamma(\alpha+1)}\big|f_1(s)-f_2(s)\big|\,ds + |\sigma_1| \cdot \big|h(y)-h(\overline{y})\big| + |\sigma_2| \cdot \big|\phi(x)-\phi(\overline{x})\big| \\
&\leq \Lambda_5 \cdot \big(\|x-\overline{x}\|_X + \|y-\overline{y}\|_Y\big)
\end{aligned}
$$

and

$$
\begin{aligned}
&\big|{}^cD^\delta h_1(x,y)(t) - {}^cD^\delta \overline{h}_1(\overline{x},\overline{y})(t)\big| \\
&\leq \int \frac{(t-s)^{-\delta}}{\Gamma(1-\delta)}\big|h_1(x,y)'(t) - \overline{h}_1(\overline{x},\overline{y})'(t)\big|\,ds \\
&\leq \frac{T^{1-\delta}}{\Gamma(2-\delta)}\Lambda_5 \cdot \big(\|x-\overline{x}\|_X + \|y-\overline{y}\|_Y\big),
\end{aligned}
$$

we obtain

$$
\big\|h_1(x,y) - \overline{h}_1(\overline{x},\overline{y})\big\|_X \leq \left(\Lambda_4 + \frac{T^{1-\delta}}{\Gamma(2-\delta)}\Lambda_5\right) \cdot \big(\|x-\overline{x}\|_X + \|y-\overline{y}\|_Y\big).
$$

In a similar manner, we can have

$$
\begin{aligned}
&\big|h_2(x,y)(t) - \overline{h}_2(\overline{x},\overline{y})(t)\big| \leq \Lambda_6 \cdot \big(\|x-\overline{x}\|_X + \|y-\overline{y}\|_Y\big), \\
&\big|h_2(x,y)'(t) - \overline{h}_2(\overline{x},\overline{y})'(t)\big| \leq \Lambda_7 \cdot \big(\|x-\overline{x}\|_X + \|y-\overline{y}\|_Y\big)
\end{aligned}
$$

and

$$
\big|{}^cD^\gamma h_2(x,y)(t) - {}^cD^\gamma \overline{h}_2(\overline{x},\overline{y})(t)\big| \leq \frac{T^{1-\gamma}}{\Gamma(2-\gamma)}\Lambda_7 \cdot \big(\|x-\overline{x}\|_X + \|y-\overline{y}\|_Y\big).
$$

We deduce that

$$
\big\|h_2(x,y) - \overline{h}_2(\overline{x},\overline{y})\big\|_Y \leq \left(\Lambda_6 + \frac{T^{1-\gamma}}{\Gamma(2-\gamma)}\Lambda_7\right) \cdot \big(\|x-\overline{x}\|_X + \|y-\overline{y}\|_Y\big).
$$

Thus,

$$\begin{aligned}&\left\|(h_1-\overline{h}_1,h_2-\overline{h}_2)\right\|_{X\times Y}\\&\quad\le\left(\Lambda_4+\frac{T^{1-\delta}}{\Gamma(2-\delta)}\Lambda_5+\Lambda_6+\frac{T^{1-\gamma}}{\Gamma(2-\gamma)}\Lambda_7\right)\left(\|x-\overline{x}\|_X+\|y-\overline{y}\|_Y\right).\end{aligned}$$

Denote

$$\tau=\Lambda_4+\frac{T^{1-\delta}}{\Gamma(2-\delta)}\Lambda_5+\Lambda_6+\frac{T^{1-\gamma}}{\Gamma(2-\gamma)}\Lambda_7.$$

By using an analogous relation obtained by interchanging the roles of (x,y) and $(\overline{x},\overline{y})$, we get

$$H_d\big(N(x,y),N(\overline{x},\overline{y})\big)\le\tau\left(\|x-\overline{x}\|_X+\|y-\overline{y}\|_Y\right).$$

Therefore, from (3.6), Lemma 2.5 implies that N has a fixed point, which is a solution of the coupled system (1.3)-(1.4). This completes the proof. □

Remark 3.1 If $f(t,x(t),y(t),{}^cD^\gamma y(t))$ and $g(t,x(t),{}^cD^\delta x(t),y(t))$ are continuous functions, $F(t,x(t),y(t),{}^cD^\gamma y(t))$ only contains the function $f(t,x(t),y(t),{}^cD^\gamma y(t))$ and $G(t,x(t),{}^cD^\delta x(t),y(t))$ only contains the function $g(t,x(t),{}^cD^\delta x(t),y(t))$. Then the existence results Theorem 3.1 and Theorem 3.2 are just the ones, respectively, in paper [19].

Remark 3.2 The new existence results for a class of second-order coupled system differential inclusions with coupled nonlocal and integral boundary conditions follow as a special case by taking $\alpha=\beta=2$ in the results of this paper.

4 Examples for fractional compartmental models

In this section, we will give some examples to illustrate our main results.

Example 4.1 Consider the following fractional differential inclusions:

$$\begin{cases}{}^cD^{\frac{5}{4}}x(t)\in F(t,x(t),y(t),{}^cD^\gamma y(t)), & t\in[0,1],\\ {}^cD^{\frac{3}{2}}y(t)\in G(t,x(t),{}^cD^\delta x(t),y(t)),\end{cases}\tag{4.1}$$

with boundary conditions of the form

$$\begin{cases}x(0)=\frac{1}{180}y(t), & \int_0^T y(s)\,ds=\frac{1}{2}x(\frac{1}{9}),\\ y(0)=\frac{1}{144}x(t), & \int_0^T x(s)\,ds=\frac{1}{2}y(\frac{1}{9}),\end{cases}\tag{4.2}$$

where $\alpha=\frac{5}{4}$, $\beta=\frac{3}{2}$, $\gamma=\frac{1}{4}$, $\delta=\frac{1}{2}$, $T=1$, $\mu_1=\mu_2=\frac{1}{2}$, $\eta=\xi=\frac{1}{9}$, $L_1=\frac{1}{180}$, $L_2=\frac{1}{144}$, and $F,G:[0,1]\times\mathbb{R}^3\to\mathbb{R}$ are multivalued maps given by

$$\begin{aligned}F\big(t,x,y,{}^cD^\gamma y\big)&=\left\{f\in\mathbb{R}:0\le f\le\frac{|x|}{1+|x|}+\sin y+\tan^{-1}\big({}^cD^{\frac{1}{4}}y\big)+\frac{e^{-t}}{1+t^2}\right\},\\ G\big(t,x,{}^cD^\delta x,y\big)&=\left\{g\in\mathbb{R}:0\le g\le\sin x+\frac{{}^cD^{\frac{1}{2}}x}{1+{}^cD^{\frac{1}{2}}x}+\tan^{-1}y+\cos t+1\right\}.\end{aligned}$$

It is clear that F, G are L^1-Carathéodory and have convex values satisfying

$$\begin{aligned}&\left\|F(t,x,y,z)\right\| = \sup\left\{|f| : f \in F(t,x,y,z)\right\} \le 4, \quad \text{for each } (t,x,y,z) \in J \times \mathbb{R}^3,\\&\left\|G(t,x,y,z)\right\| = \sup\left\{|g| : f \in G(t,x,y,z)\right\} \le 5, \quad \text{for each } (t,x,y,z) \in J \times \mathbb{R}^3,\end{aligned}$$

with $m_1(t) = m_2(t) = \varphi_1(|x|) = \varphi_2(|x|) = \rho_1(|z|) \equiv 1$, $\psi_1(|y|) = \psi_2(|y|) = \rho_2(|z|) \equiv 2$.

Using the given data, we find that $P_1 \approx 1.7762$, $P_2 \approx 1.8657$, $P_3 \approx 1.4459$, $P_4 \approx 0.9819$, $\overline{P}_1 \approx 1.9968$, $\overline{P}_2 \approx 1.8657$, $\overline{P}_3 \approx 1.8220$, $\overline{P}_4 \approx 0.9819$, $Q_1 \approx 0.0216$, $Q_2 \approx 0.0246$, $\overline{Q}_1 \approx 0.0161$, $\overline{Q}_2 \approx 0.0177$, $\Lambda_1 \approx 6.0862$, $\Lambda_2 \approx 7.3976$, $\Lambda_3 \approx 0.0836 < 1$.

Furthermore, let K be any number satisfying

$$K > \frac{6.0862 \times 4 + 7.3976 \times 5}{1 - 0.0836} > 66.9280.$$

Clearly, all the conditions of Theorem 3.1 are satisfied. So there exists at least one solution of problem (4.1)-(4.2) on $[0,1]$.

Example 4.2 Consider the following coupled system of fractional compartmental models:

$$\begin{cases} {}^cD^{\frac{3}{2}}x(t) \in F(t,x(t),y(t),{}^cD^{\gamma}y(t)), \quad t \in [0,1],\\ {}^cD^{\frac{3}{2}}y(t) \in G(t,x(t),{}^cD^{\delta}x(t),y(t)),\end{cases} \tag{4.3}$$

with boundary conditions of the form

$$\begin{cases} x(0) = 0, \qquad \int_0^T y(s)\,ds = 0,\\ y(0) = 0, \qquad \int_0^T x(s)\,ds = 0,\end{cases} \tag{4.4}$$

where $\alpha = \beta = \frac{3}{2}$, $\gamma = \delta = \frac{1}{2}$, $T = 1$, $\mu_1 = \mu_2 = 0$, $\eta = \xi = \frac{1}{9}$, $L_1 = \frac{1}{180}$, $L_2 = \frac{1}{144}$ and $F, G : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ are multivalued maps given by

$$F(t,x,y,z) = \left\{-k_1x(t)\right\}, \qquad G(t,x,y,z) = \left\{k_2x(t) + k_3y(t)\right\}, \quad k_1,k_2,k_3 \in \mathbb{R}^+.$$

It is clear that F, G satisfy (H4) and

$$\begin{aligned}&\left\|F(t,x,y,z)\right\| \le k_1\|x\|, \quad \text{for each } (t,x,y,z) \in J \times \mathbb{R}^3,\\&\left\|G(t,x,y,z)\right\| \le k_2\|x\| + k_3\|x\| \le \max\{k_2,k_3\}\left(\|x\| + \|y\|\right),\\&\quad \text{for each } (t,x,y,z) \in J \times \mathbb{R}^3,\end{aligned}$$

with $m_1(t) \equiv k_1$, $m_2(t) \equiv \max\{k_2,k_3\}$, $\varphi_1(|x|) = \varphi_2(|x|) \equiv \|x\|$, $\psi_2(|y|) \equiv \|y\|$, $\psi_1(|y|) = \rho_1(|z|) = \rho_2(|z|) \equiv 0$. Because $x(t)$, $y(t)$ denote the amount of a drug in a specific compartment in [25], $\|x\|$ and $\|y\|$ are constants. Letting $\varphi_1(|x|) = \varphi_2(|x|) \equiv c_1$, $\psi_2(|y|) \equiv c_2$ (c_1, c_2 are constants). Using the given data, we find that $P_1 \approx 1.3544$, $P_2 \approx 0$, $P_3 \approx 1.3544$, $P_4 \approx 0$, $\overline{P}_1 \approx 1.7306$, $\overline{P}_2 \approx 0$, $\overline{P}_3 \approx 1.7306$, $\overline{P}_4 \approx 0$, $Q_1 \approx 0.0167$, $Q_2 \approx 0.0208$, $\overline{Q}_1 \approx 0.0111$, $\overline{Q}_2 \approx 0.0139$, $\Lambda_1 \approx 3.3077$, $\Lambda_2 \approx 3.3077$, $\Lambda_3 \approx 0.0657$.

Furthermore, let K be any number satisfying

$$K > \frac{3.3077k_1c_1 + 3.3077\max\{k_2,k_3\} \cdot (c_1 + c_2)}{1 - 0.0657}.$$

Clearly, all the conditions of Theorem 3.2 are satisfied. So there exists at least one solution of problem (4.3)-(4.4) on $[0,1]$.

At the same time, we can see that the numerical solutions and simulations for problem (4.3)-(4.4) (single-valued) are obtained in [25]. That is, the conclusions we have gained are correct.

Example 4.3 Consider problem (4.1)-(4.2), where $F, G : [0,1] \times \mathbb{R}^3 \to \mathbb{R}^+$ are multivalued maps given by

$$F(t,x,y,z) = \left[0, \frac{|\sin x + \sin y + \sin z|}{4(2+t)^4}\right],$$
$$G(t,x,y,z) = \left[0, \frac{|\cos x + \cos y + \cos z|}{4(2+t)^6}\right].$$

Now

$$\sup\bigl\{|f| : f \in F(t,x,y,z)\bigr\} \le \frac{1}{(2+t)^4} \le 116, \quad \text{for each } (t,x,y,z) \in [0,1] \times \mathbb{R}^3,$$
$$\sup\bigl\{|g| : g \in G(t,x,y,z)\bigr\} \le \frac{1}{(2+t)^6} \le 164, \quad \text{for each } (t,x,y,z) \in [0,1] \times \mathbb{R}^3,$$

and

$$d_H\bigl(F(t,x_1,y_1,z_1), F(t,x_2,y_2,z_2)\bigr) \le \frac{1}{(2+t)^4}\bigl(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|\bigr),$$
$$d_H\bigl(G(t,x_1,y_1,z_1), G(t,x_2,y_2,z_2)\bigr) \le \frac{1}{(2+t)^6}\bigl(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|\bigr).$$

Here $m_3(t) = \frac{1}{(2+t)^4}$, $m_4(t) = \frac{1}{(2+t)^6}$ with $\|m_3\| = \frac{1}{16}$ and $\|m_4\| = \frac{1}{64}$.

Using the given data, we find that $\Lambda_4 \approx 0.0853$, $\Lambda_5 \approx 0.1403$, $\Lambda_6 \approx 0.1190$, $\Lambda_7 \approx 0.1390$ and

$$\Lambda_4 + \frac{T^{1-\delta}}{\Gamma(2-\delta)}\Lambda_5 + \Lambda_6 + \frac{T^{1-\gamma}}{\Gamma(2-\gamma)}\Lambda_7 \approx 0.5138 < 1.$$

The compactness of F, G together with the above calculations leads to the existence of solution of problem (4.1)-(4.2) by Theorem 3.3.

5 Conclusion

In this article, we present existence conditions of solutions, which are the prerequisites for solving the numerical solutions. Before solving the numerical solution, we can know whether there is a solution. This reduces a lot of unnecessary calculations to a certain extent. It is very significant for fractional compartmental model for a biological system.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

Acknowledgements
The authors sincerely thank the reviewers for their valuable suggestions and useful comments that have led to the present improved version of the original manuscript.
This research is supported by the Natural Science Foundation of China (11571202), by Shandong Provincial Natural Science Foundation (ZR2016AM17), and by Graduate Innovation Foundation of University of Jinan (YCXS15005).

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 23 November 2016 Accepted: 5 April 2017 Published online: 25 May 2017

References
1. Podlubny, I: Fractional Differential Equations. Academic Press, San Diego (1999)
2. Kilbas, AA, Srivastava, HM, Trujillo, JJ: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam (2006)
3. Miller, KS, Ross, B: An Introduction to the Fractional Calculus and Fractional Differential Equations. Wiley, New York (1993)
4. Sabatier, J, Agrawal, O, Machado, J: Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering. Springer, Dordrecht (2007)
5. Jin, N, Sun, S, Zhang, C, Han, Z: The existence of solutions for boundary value problems of fractional differential inclusion with a parameter. J. Appl. Math. Comput., 1-16 (2015)
6. Wang, J, Ibrahim, AG, Fečkan, M: Nonlocal Cauchy problems for semilinear differential inclusions with fractional order in Banach spaces. Commun. Nonlinear Sci. Numer. Simul. **27**, 281-293 (2015)
7. Ahmad, B, Ntouyas, SK, Tariboon, J: A study of mixed Hadamard and Riemann-Liouville fractional integro-differential inclusions via endpoint theory. Appl. Math. Lett. **52**, 9-14 (2016)
8. Bayour, B, Torres, DFM: Existence of solution to a local fractional nonlinear differential equation. J. Comput. Appl. Math. **312**, 127-133 (2016)
9. Benchaabane, A, Sakthivel, R: Sobolev-type fractional stochastic differential equations with non-Lipschitz coefficients. J. Comput. Appl. Math. **312**, 65-73 (2016)
10. Balasubramaniam, P, Tamilalagan, P: Approximate controllability of a class of fractional neutral stochastic integro-differential inclusions with infinite delay by using Mainardi's function. Appl. Math. Comput. **256**, 232-246 (2015)
11. Liu, X, Liu, X, Liu, Y: Solvability for fractional differential inclusions with fractional nonseparated boundary conditions. J. Comput. Anal. Appl. **20**(4), 734-749 (2016)
12. Liu, X, Liu, Z: Existence Results for Fractional Differential Inclusions with Multivalued Term Depending on Lower-Order Derivative. Abstract and Applied Analysis **2012**, Article ID 423796 (2012)
13. Perelson, A: Modeling the interaction of the immune system with HIV. In: Castillo-Chavez, C (ed.) Mathematical and Statistical Approaches to AIDS Epidemiology. Lecture Notes in Biomathematics, vol. 83, p. 350. Springer, Berlin (1989)
14. Perelson, A, Kirschner, D, Boer, R: Dynamics of HIV infection of $CD4^+$ T cells. Math. Biosci. **114**, 81-125 (1993)
15. Liu, S, Wang, G, Zhang, L: Existence results for a coupled system of nonlinear neutral fractional differential equations. Appl. Math. Lett. **26**, 1120-1124 (2013)
16. Jiang, J, Liu, L, Wu, Y: Positive solutions to singular fractional differential system with coupled boundary conditions. Commun. Nonlinear Sci. Numer. Simul. **18**, 3061-3074 (2013)
17. Zhao, Y, Chen, H, Qin, B: Multiple solutions for a coupled system of nonlinear fractional differential equations via variational methods. Appl. Math. Comput. **257**, 417-427 (2015)
18. Henderson, J, Luca, R: Positive solutions for a system of fractional differential equations with coupled integral boundary conditions. Appl. Math. Comput. **249**, 182-197 (2014)
19. Ahmad, B, Ntouyas, SK, Alsaedi, A: On a coupled system of fractional differential equations with coupled nonlocal and integral boundary conditions. Chaos Solitons Fractals **83**, 234-241 (2016)
20. Gunn, RN, Gunn, SR, Cunningham, VJ: Positron emission tomography compartmental models. J. Cereb. Blood Flow Metab. **21**(6), 635-652 (2001)
21. Brauer, F: Compartmental models in epidemiology. In: Mathematical Epidemiology, pp. 19-79 (2008)
22. Graham, BP, Ooyen, AV: Compartmental models of growing neurites. Neurocomputing **38-40**(6), 31-36 (2011)
23. Dokoumetzidis, A, Macheras, P: Fractional kinetics in drug absorption and disposition processes. J. Pharmacokinet. Pharmacodyn. **36**, 165-178 (2009)
24. Popović, JK, Atanacković, MT, Pilipović, AS, Rapaić, MR, Pilipović, S, Atanacković, TM: A new approach to the compartmental analysis in pharmacokinetics: fractional time evolution of diclofenac. J. Pharmacokinet. Pharmacodyn. **37**, 119-134 (2010)
25. Petráš, I, Magin, RL: Simulation of drug uptake in a two compartmental fractional model for a biological system. Commun. Nonlinear Sci. Numer. Simul. **16**(12), 4588-4595 (2011)
26. Granas, A, Dugundji, J: Fixed Point Theory. Springer Monographs in Mathematics. Springer, New York (2003)
27. Covitz, H, Nadler, SB Jr.: Multi-valued contraction mappings in generalized metric spaces. Isr. J. Math. **8**, 5-11 (1970)
28. Deimling, K: Multivalued Differential Equations. de Gruyter, Berlin (1992)
29. Hu, S, Papageorgiou, NS: Handbook of Multivalued Analysis, Theory I. Kluwer Academic, Dordrecht (1997)
30. Kisielewicz, M: Differential Inclusions and Optimal Control. Mathematics and Its Applications (East European Series), vol. 44. Kluwer Academic, Dordrecht (1991)

31. Samko, SG, Kilbas, AA, Marichev, OI: Fractional Integrals and Derivatives Theory and Applications. Gordon and Breach, Yverdon (1993)
32. Kilbas, AA, Srivastava, HM, Trujillo, JJ: Theory and Applications of Fractional Differential. Equation North-Holland Mathematics Studies. Elsevier, Amsterdam (2006)
33. Lasota, A, Opial, Z: An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations. Bull. Acad. Pol. Sci., Sér. Sci. Math. Astron. Phys. **13**, 781-786 (1965)
34. Su, X: Boundary value problem for a coupled system of nonlinear fractional differential equations. Appl. Math. Lett. **22**(1), 64-69 (2009)
35. Tolstonogov, AA: A theorem of Bogolyubov with constraints generated by a second-order evolutionary control system. Izv. Math. **67**(5), 177-206 (2003)
36. Bressan, A, Colombo, G: Extensions and selections of maps with decomposable values. Stud. Math. **90**(1), 69-86 (1988)
37. Granas, A, Dugundji, J: Fixed Point Theory. Springer Monographs in Mathematics. Springer, New York (2003)
38. Castaing, C, Valadier, M: Convex Analysis and Measurable Multifunctions. Lecture Notes in Mathematics, vol. 580. Springer, Berlin (1977)