Alhothuali et al. *Advances in Difference Equations* (2017) 2017:104
DOI 10.1186/s13662-017-1152-z

Advances in Difference Equations
a SpringerOpen Journal

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Nonlinear sequential fractional differential equations with nonlocal boundary conditions involving lower-order fractional derivatives

MS Alhothuali, Ahmed Alsaedi, Bashir Ahmad* and Mohammed H Aqlan

*Correspondence:
bashirahmad_qau@yahoo.com
Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia

Abstract

This paper studies a new class of boundary value problems of sequential fractional differential equations of order $q \in (2, 3]$ supplemented with nonlocal non-separated boundary conditions involving lower-order fractional derivatives. Existence and uniqueness results for the given problem are obtained by applying standard fixed-point theorems and are illustrated with the aid of examples. Some interesting observations are presented.

1 Introduction

Fractional calculus has been extensively studied and developed during the last few decades. It has been mainly due to an overwhelming interest shown by the modelers and researchers in the subject. One can find the applications of fractional-order derivatives and integrals in diverse disciplines such as applied mathematics, physics, control theory, mechanical structures, thermodynamics, etc. [1–5]. In contrast to the classical Laplacian, which is a local operator, the fractional Laplacian is a paradigm of the vast family of nonlocal linear operators and has immediate consequences in the formulation of basic equations like the diffusion equation. In particular, there has been a significant progress on fractional-order initial/boundary value problems, and the literature on the topic is now much enriched, covering theoretical development as well as applications of this important topic. The advancement in the study of fractional-order boundary value problems includes different kinds of boundary conditions such as two-point, multi-point, nonlocal, periodic/anti-periodic, and integral conditions. For details and examples, we refer the reader to a series of papers [6–16]. For some works on sequential fractional differential equations, for example, see [17, 18] and the references cited therein. In a recent work [19], the authors obtained some existence results for sequential fractional differential equations with anti-periodic type boundary conditions.

In this paper, we plan to develop the existence theory for nonlinear sequential fractional differential equations equipped with nonlocal non-separated fractional boundary conditions. Precisely, we consider the following problem:

$$\begin{cases} ({}^cD^q + k^cD^{q-1})u(t) = f(t,u(t)), \quad 2 < q \leq 3, 0 < t < T, \\ \alpha_1 u(\eta) + \beta_1 u(T) = a, \\ \alpha_2 {}^cD^{q-1}u(\eta) + \beta_2 {}^cD^{q-1}u(T) = b, \\ \alpha_3 {}^cD^{q-2}u(\eta) + \beta_3 {}^cD^{q-2}u(T) = c, \end{cases} \tag{1.1}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order $q, \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, a, b, c \in \mathbb{R}$, $k \in \mathbb{R}^+$, $0 < \eta < T$, and f is a given continuous function.

To the best of our knowledge, problem (1.1) considered in this paper is new and covers a variety of special cases for appropriate values of the parameters involved in the boundary conditions (see the 'Concluding remarks' section). The rest of the paper is organized as follows. In Section 2, we recall some preliminary concepts of fractional calculus and prove an auxiliary lemma associated with the linear variant of the given problem. Section 3 contains our main results which are supported with illustrative examples. Some interesting observations are presented in the last section.

2 Preliminary work

First of all, we recall some basic definitions [20, 21].

Definition 2.1 The fractional integral of order r with the lower limit zero for a function $f : [0,\infty) \to R$ is defined as

$$I^r f(t) = \frac{1}{\Gamma(r)} \int_0^t \frac{f(s)}{(t-s)^{1-r}}\,ds, \quad t > 0, r > 0,$$

provided the right-hand side is point-wise defined on $[0,\infty)$, where $\Gamma(\cdot)$ is the gamma function, which is defined by $\Gamma(r) = \int_0^\infty t^{r-1}e^{-t}\,dt$.

Definition 2.2 The Riemann-Liouville fractional derivative of order $r > 0$, $n-1 < r < n$, $n \in N$, is defined as

$$D^r_{0+} f(t) = \frac{1}{\Gamma(n-r)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-r-1} f(s)\,ds,$$

where the function $f : [0,\infty) \to R$ has an absolutely continuous derivative up to order $(n-1)$.

Definition 2.3 The Caputo derivative of order r for a function $f : [0,\infty) \to R$ can be written as

$${}^cD^r f(t) = D^r_{0+}\left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0)\right), \quad t > 0, n-1 < r < n.$$

Remark 2.4 If $f(t) \in C^n[0,\infty)$, then

$${}^cD^r f(t) = \frac{1}{\Gamma(n-r)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{r+1-n}}\,ds = I^{n-r} f^{(n)}(t), \quad t > 0, n-1 < q < n.$$

To define a solution for problem (1.1), we need the following lemma for its linear variant.

Lemma 2.5 *Let $h \in C([0,T],\mathbb{R})$. Then the following linear problem*

$$\begin{cases} ({}^cD^q + k{}^cD^{q-1})u(t) = h(t), \quad 2 < q \leq 3, 0 < t < T, \\ \alpha_1 u(\eta) + \beta_1 u(T) = a, \\ \alpha_2 {}^cD^{q-1}u(\eta) + \beta_2 {}^cD^{q-1}u(T) = b, \\ \alpha_3 {}^cD^{q-2}u(\eta) + \beta_3 {}^cD^{q-2}u(T) = c, \end{cases} \tag{2.1}$$

is equivalent to the fractional integral equation

$$\begin{aligned} u(t) = {}& \nu_1(t) + \int_0^t e^{-k(t-s)} \left(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} h(x)\,dx \right) ds \\ & - \frac{\alpha_1}{\rho} \int_0^\eta e^{-k(\eta-s)} \left(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} h(x)\,dx \right) ds \\ & - \frac{\beta_1}{\rho} \int_0^T e^{-k(T-s)} \left(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} h(x)\,dx \right) ds \\ & + \nu_2(t) \int_0^\eta (\eta-s)^{2-q} \left(\int_0^s e^{-k(s-x)} \left(\int_0^x \frac{(x-m)^{q-2}}{\Gamma(q-1)} h(m)\,dm \right) dx \right) ds \\ & + \nu_3(t) \int_0^T (T-s)^{2-q} \left(\int_0^s e^{-k(s-x)} \left(\int_0^x \frac{(x-m)^{q-2}}{\Gamma(q-1)} h(m)\,dm \right) dx \right) ds \\ & + \nu_4(t) \int_0^\eta (\eta-s)^{2-q} \left(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} h(x)\,dx \right) ds \\ & + \nu_5(t) \int_0^T (T-s)^{2-q} \left(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} h(x)\,dx \right) ds \\ & + \nu_6(t) \int_0^\eta (\eta-s)^{2-q} \left(\int_0^s \frac{(s-x)^{q-3}}{\Gamma(q-2)} h(x)\,dx \right) ds \\ & + \nu_7(t) \int_0^T (T-s)^{2-q} \left(\int_0^s \frac{(s-x)^{q-3}}{\Gamma(q-2)} h(x)\,dx \right) ds, \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} \nu_1(t) &= \frac{a}{\rho} + \frac{\Gamma(3-q)b}{k^2 N\rho\sigma} \left(N\left(\rho e^{-kt} - \alpha\right) - k(z-\rho t)\vartheta \right) + \frac{\Gamma(3-q)c}{N\rho}(\rho t - z), \\ \nu_2(t) &= \frac{\alpha_2}{\rho\sigma}\left(\alpha - \rho e^{-kt}\right) + \frac{k(t\rho - z)(\sigma\alpha_3 - \alpha_2\vartheta)}{\rho\sigma N}, \\ \nu_3(t) &= \frac{\beta_2}{\rho\sigma}\left(\alpha - \rho e^{-kt}\right) + \frac{k(\rho t - z)(\sigma\beta_3 - \beta_2\vartheta)}{\rho\sigma N}, \\ \nu_4(t) &= \frac{\alpha_2}{\rho k\sigma}\left(\rho e^{-kt} - \alpha\right) + \frac{(z - t\rho)(\sigma\alpha_3 - \alpha_2\vartheta)}{\rho\sigma N}, \\ \nu_5(t) &= \frac{\beta_2}{\rho k\sigma}\left(\rho e^{-kt} - \alpha\right) + \frac{(z - t\rho)(\sigma\beta_3 - \beta_2\vartheta)}{\rho\sigma N}, \\ \nu_6(t) &= \frac{\alpha_2}{\rho N k^2 \sigma}\left[N\left(\alpha - \rho e^{-kt}\right) + k\vartheta(z - \rho t) \right], \end{aligned} \tag{2.3}$$

$$\nu_7(t) = \frac{\beta_2}{\rho N k^2 \sigma}\bigl[N\bigl(\alpha - \rho e^{-kt}\bigr) + k\vartheta(z - \rho t)\bigr],$$

$$\rho = \alpha_1 + \beta_1, \qquad z = \eta\alpha_1 + T\beta_1, \qquad \sigma = \alpha_2\gamma + \beta_2\beta,$$

$$\alpha = \alpha_1 e^{-k\eta} + \beta_1 e^{-kT}, \qquad \vartheta = \alpha_3\gamma + \beta_3\beta$$

$$N = \frac{\alpha_3\eta^{3-q} + \beta_3 T^{3-q}}{3-q}, \qquad \gamma = \int_0^\eta (\eta - s)^{2-q} e^{-ks}\,ds,$$

$$\beta = \int_0^T (T - s)^{2-q} e^{-ks}\,ds.$$

Proof Writing the linear sequential fractional differential equation in (2.1) as ${}^cD^{q-1}(D + k)u(t) = h(t)$ and then applying the Riemann-Liouville integral operator I^{q-1} on both sides, followed by integration from 0 to t, we get

$$u(t) = A_0 e^{-kt} + A_1 + A_2 t + \int_0^t e^{-k(t-s)} I^{q-1}h(s)\,ds, \tag{2.4}$$

where A_0, A_1, and A_2 are arbitrary constants and

$$I^{q-1}h(t) = \int_0^t \frac{(t-x)^{q-2}}{\Gamma(q-1)} h(x)\,dx.$$

From (2.4), we have

$$\begin{aligned} {}^cD^{q-1}u(t) &= \frac{1}{\Gamma(3-q)} \int_0^t (t-s)^{2-q}\biggl(k^2 A_0 e^{-kt} + k^2 \int_0^s e^{-k(s-x)} I^{q-1}h(x)\,dx \\ &\quad - kI^{q-1}h(s) + I^{q-2}h(s)\biggr)\,ds, \qquad (2.5) \\ {}^cD^{q-2}u(t) &= \frac{1}{\Gamma(3-q)} \int_0^t (t-s)^{2-q}\biggl(-kA_0 e^{-ks} + A_2 - k\int_0^s e^{-k(s-x)} I^{q-1}h(x)\,dx \\ &\quad + I^{q-1}h(s)\biggr)\,ds. \qquad (2.6) \end{aligned}$$

Now, using the boundary conditions given by (2.1) in (2.4), (2.5), and (2.6) together with notations (2.3), we get

$$\alpha A_0 + \rho A_1 + zA_2 + \alpha_1 \int_0^\eta e^{-k(\eta-s)} I^{q-1}h(s)\,ds + \beta_1 \int_0^T e^{-k(T-s)} I^{q-1}h(s)\,ds = a, \tag{2.7}$$

$$\begin{aligned} &\frac{k^2 A_0}{\Gamma(3-q)}(\alpha_2\gamma + \beta_2\beta) \\ &\quad + \frac{\alpha_2}{\Gamma(3-q)} \int_0^\eta (\eta-s)^{2-q}\biggl(k^2\int_0^s e^{-k(s-x)} I^{q-1}h(x)\,dx - kI^{q-1}h(s) + I^{q-2}h(s)\biggr)\,ds \\ &\quad + \frac{\beta_2}{\Gamma(3-q)} \int_0^T (T-s)^{2-q}\biggl(k^2\int_0^s e^{-k(s-x)} I^{q-1}h(x)\,dx \\ &\quad - kI^{q-1}h(s) + I^{q-2}h(s)\biggr)\,ds = b, \qquad (2.8) \end{aligned}$$

$$\begin{aligned}
&\frac{-kA_0}{\Gamma(3-q)}(\alpha_3\gamma+\beta_3\beta)+\frac{A_2N}{\Gamma(3-q)} \\
&\quad+\frac{\alpha_3}{\Gamma(3-q)}\int_0^\eta(\eta-s)^{2-q}\left(-k\int_0^s e^{-k(s-x)}I^{q-1}h(x)\,dx+I^{q-1}h(s)\right)ds \\
&\quad+\frac{\beta_3}{\Gamma(3-q)}\int_0^T(T-s)^{2-q}\left(-k\int_0^s e^{-k(s-x)}I^{q-1}h(x)\,dx+I^{q-1}h(s)\right)ds=c. \qquad (2.9)
\end{aligned}$$

Solving system (2.7), (2.8), and (2.9) for A_0, A_1, and A_2, we find that

$$\begin{aligned}
A_0={}&\frac{\Gamma(3-q)b}{k^2\sigma} \\
&-\frac{\alpha_2}{k^2\sigma}\int_0^\eta(\eta-s)^{2-q}\left(k^2\int_0^s e^{-k(s-x)}I^{q-1}h(x)\,dx-kI^{q-1}h(s)+I^{q-2}h(s)\right)ds \\
&-\frac{\beta_2}{k^2\sigma}\int_0^T(T-s)^{2-q}\left(k^2\int_0^s e^{-k(s-x)}I^{q-1}h(x)\,dx-kI^{q-1}h(s)+I^{q-2}h(s)\right)ds, \\
A_1={}&\frac{a}{\rho}-\frac{\Gamma(3-q)}{k^2\rho\sigma N}(N\alpha+zk\vartheta)b-\frac{z\Gamma(3-q)}{N\rho}c \\
&+\frac{1}{N\rho\sigma}\big(N\alpha\alpha_2-zk(\sigma\alpha_3-\alpha_2\vartheta)\big)\int_0^\eta(\eta-s)^{2-q}\left(\int_0^s e^{-k(s-x)}I^{q-1}h(x)\,dx\right)ds \\
&+\frac{1}{N\rho\sigma}\big(N\alpha\beta_2-zk(\sigma\beta_3-\beta_2\vartheta)\big)\int_0^T(T-s)^{2-q}\left(\int_0^s e^{-k(s-x)}I^{q-1}h(x)\,dx\right)ds \\
&-\frac{1}{kN\rho\sigma}\big(N\alpha\alpha_2+kz(\alpha_2\vartheta-\sigma\alpha_3)\big)\int_0^\eta(\eta-s)^{2-q}I^{q-1}h(s)\,ds \\
&-\frac{1}{kN\rho\sigma}\big(N\alpha\beta_2+kz(\beta_2\vartheta-\sigma\beta_3)\big)\int_0^\eta(\eta-s)^{2-q}I^{q-1}h(s)\,ds \\
&+\frac{\alpha_2}{k^2N\rho\sigma}(N\alpha+zk\vartheta)\int_0^\eta(\eta-s)^{2-q}I^{q-2}h(s)\,ds \\
&+\frac{\beta_2}{k^2N\rho\sigma}(N\alpha+zk\vartheta)\int_0^T(T-s)^{2-q}I^{q-2}h(s)\,ds \\
&-\frac{\alpha_1}{\rho}\int_0^\eta e^{-k(\eta-s)}I^{q-1}h(s)\,ds-\frac{\beta_1}{\rho}\int_0^T e^{-k(T-s)}I^{q-1}h(s)\,ds, \\
A_2={}&\frac{\Gamma(3-q)}{Nk\sigma}(k\sigma c+b\vartheta)+\frac{k}{N\sigma}(\sigma\alpha_3-\alpha_2\vartheta)\int_0^\eta(\eta-s)^{2-q}\left(\int_0^s e^{-k(s-x)}I^{q-1}h(x)\,dx\right)ds \\
&+\frac{k}{N\sigma}(\sigma\beta_3-\beta_2\vartheta)\int_0^T(T-s)^{2-q}\left(\int_0^s e^{-k(s-x)}I^{q-1}h(x)\,dx\right)ds \\
&+\frac{1}{N\sigma}(\alpha_2\vartheta-\alpha_3\sigma)\int_0^\eta(\eta-s)^{2-q}I^{q-1}h(s)\,ds \\
&+\frac{1}{N\sigma}(\beta_2\vartheta-\beta_3\sigma)\int_0^T(T-s)^{2-q}I^{q-1}h(s)\,ds \\
&-\frac{\alpha_2\vartheta}{kN\sigma}\int_0^\eta(\eta-s)^{2-q}I^{q-2}h(s)\,ds-\frac{\beta_2\vartheta}{kN\sigma}\int_0^T(T-s)^{2-q}I^{q-2}h(s)\,ds.
\end{aligned}$$

Substituting the values of A_0, A_1, and A_2 in (2.4) and using notations (2.3), we get the solution (2.2). The converse follows by direct computation. This completes the proof. □

3 Main results

Let $\mathcal{P} = C([0,T],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,T] \to \mathbb{R}$ endowed with the norm defined by $\|u\| = \sup\{|u(t)|, t \in [0,T]\}$.

In view of Lemma 2.5, we transform problem (1.1) into an equivalent fixed-point problem as

$$u = \mathcal{H}u, \tag{3.1}$$

where $\mathcal{H} : \mathcal{P} \to \mathcal{P}$ is defined by

$$\begin{aligned}(\mathcal{H}u)(t) = {}& \nu_1(t) + \int_0^t e^{-k(t-s)}\left(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} f\bigl(x,u(x)\bigr)\,dx\right) ds \\
&- \frac{\alpha_1}{\rho}\int_0^\eta e^{-k(\eta-s)}\left(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} f\bigl(x,u(x)\bigr)\,dx\right) ds \\
&- \frac{\beta_1}{\rho}\int_0^T e^{-k(T-s)}\left(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} f\bigl(x,u(x)\bigr)\,dx\right) ds \\
&+ \nu_2(t)\int_0^\eta (\eta-s)^{2-q}\left(\int_0^s e^{-k(s-x)}\left(\int_0^x \frac{(x-m)^{q-2}}{\Gamma(q-1)} f\bigl(m,u(m)\bigr)\,dm\right) dx\right) ds \\
&+ \nu_3(t)\int_0^T (T-s)^{2-q}\left(\int_0^s e^{-k(s-x)}\left(\int_0^x \frac{(x-m)^{q-2}}{\Gamma(q-1)} f\bigl(m,u(m)\bigr)\,dm\right) dx\right) ds \\
&+ \nu_4(t)\int_0^\eta (\eta-s)^{2-q}\left(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} f\bigl(x,u(x)\bigr)\,dx\right) ds \\
&+ \nu_5(t)\int_0^T (T-s)^{2-q}\left(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} f\bigl(x,u(x)\bigr)\,dx\right) ds \\
&+ \nu_6(t)\int_0^\eta (\eta-s)^{2-q}\left(\int_0^s \frac{(s-x)^{q-3}}{\Gamma(q-2)} f\bigl(x,u(x)\bigr)\,dx\right) ds \\
&+ \nu_7(t)\int_0^T (T-s)^{2-q}\left(\int_0^s \frac{(s-x)^{q-3}}{\Gamma(q-2)} f\bigl(x,u(x)\bigr)\,dx\right) ds. \end{aligned} \tag{3.2}$$

Observe that problem (1.1) has solutions if operator equation (3.1) has fixed points.

For computational convenience, we set

$$\begin{aligned} Q = \sup_{t\in[0,T]} \biggl\{ & \frac{t^{q-1}|\rho|(1-e^{-kt}) + |\alpha_1|\eta^{q-1}(1-e^{-k\eta}) + |\beta_1|T^{q-1}(1-e^{-kT})}{|\rho|k\Gamma(q)} \\
&+ \frac{|\nu_2(t)|\eta^2(1-e^{-k\eta}) + |\nu_3(t)|T^2(1-e^{-kT}) + |\nu_4(t)|k\eta^2 + |\nu_5(t)|kT^2}{k(3-q)\Gamma(q)} \\
&+ \frac{|\nu_6(t)|\eta + |\nu_7(t)|T}{(3-q)\Gamma(q-1)} \biggr\}. \end{aligned} \tag{3.3}$$

In the following theorem, we establish the existence of a unique solution of problem (1.1) by means of Banach's fixed-point theorem.

Theorem 3.1 *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the Lipschitz condition*:

(A_1) *There exists a positive number ℓ such that $|f(t,u)-f(t,v)| \leq \ell|u-v|$, $\forall t \in [0,T]$, $u,v \in \mathbb{R}$.*

Then the boundary value problem (1.1) *has a unique solution on* $[0,T]$ *if* $\ell < 1/Q$, *where* Q *is given by* (3.3).

Proof Consider a set $B_r = \{u \in \mathcal{P} : \|u\| \leq r\}$, where $r \geq \frac{QM+\|\nu_1\|}{1-\ell Q}$, $\sup_{t\in[0,T]}|f(t,0)| = M$ and Q is given by (3.3). In the first step, we show that $\mathcal{H}B_r \subset B_r$, where the operator $\mathcal{H}$ is defined by (3.2). For any $u \in B_r$, $t \in [0,T]$, observe that

$$\begin{aligned}\bigl|f\bigl(t,u(t)\bigr)\bigr| &= \bigl|f\bigl(t,u(t)\bigr)-f(t,0)+f(t,0)\bigr| \leq \bigl|f\bigl(t,u(t)\bigr)-f(t,0)\bigr| + \bigl|f(t,0)\bigr| \\ &\leq \ell\|u\| + M \leq \ell r + M,\end{aligned}$$

which together with (3.2) yields

$$\begin{aligned}
\bigl\|(\mathcal{H}u)\bigr\| \leq{}& \sup_{t\in[0,T]}\biggl\{\bigl|\nu_1(t)\bigr| + \int_0^t e^{-k(t-s)}\biggl(\int_0^s \frac{(s-x)^{\alpha-2}}{\Gamma(\alpha-1)}\bigl|f\bigl(x,u(x)\bigr)\bigr|\,dx\biggr)\,ds \\
&+ \biggl|\frac{\alpha_1}{\rho}\biggr|\int_0^\eta e^{-k(\eta-s)}\biggl(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)}\bigl|f\bigl(x,u(x)\bigr)\bigr|\,dx\biggr)\,ds \\
&+ \biggl|\frac{\beta_1}{\rho}\biggr|\int_0^T e^{-k(T-s)}\biggl(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)}\bigl|f\bigl(x,u(x)\bigr)\bigr|\,dx\biggr)\,ds \\
&+ \bigl|\nu_2(t)\bigr|\int_0^\eta (\eta-s)^{2-q}\biggl(\int_0^s e^{-k(s-x)}\biggl(\int_0^x \frac{(x-m)^{q-2}}{\Gamma(q-1)}\bigl|f\bigl(m,u(m)\bigr)\bigr|\,dm\biggr)\,dx\biggr)\,ds \\
&+ \bigl|\nu_3(t)\bigr| \\
&\times \int_0^T (T-s)^{2-q}\biggl(\int_0^s e^{-k(s-x)}\biggl(\int_0^x \frac{(x-m)^{q-2}}{\Gamma(q-1)}\bigl|f\bigl(m,u(m)\bigr)\bigr|\,dm\biggr)\,dx\biggr)\,ds \\
&+ \bigl|\nu_4(t)\bigr|\int_0^\eta (\eta-s)^{2-q}\biggl(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)}\bigl|f\bigl(x,u(x)\bigr)\bigr|\,dx\biggr)\,ds \\
&+ \bigl|\nu_5(t)\bigr|\int_0^T (T-s)^{2-q}\biggl(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)}\bigl|f\bigl(x,u(x)\bigr)\bigr|\,dx\biggr)\,ds \\
&+ \bigl|\nu_6(t)\bigr|\int_0^\eta (\eta-s)^{2-q}\biggl(\int_0^s \frac{(s-x)^{q-3}}{\Gamma(q-2)}\bigl|f\bigl(x,u(x)\bigr)\bigr|\,dx\biggr)\,ds \\
&+ \bigl|\nu_7(t)\bigr|\int_0^T (T-s)^{2-q}\biggl(\int_0^s \frac{(s-x)^{q-3}}{\Gamma(q-2)}\bigl|f\bigl(x,u(x)\bigr)\bigr|\,dx\biggr)\,ds\biggr\} \\
\leq{}& (\ell r + M)\sup_{t\in[0,T]}\biggl\{|\nu_1| + \int_0^t e^{-k(t-s)}\biggl(\int_0^s \frac{(s-x)^{\alpha-2}}{\Gamma(\alpha-1)}\,dx\biggr)\,ds \\
&+ \biggl|\frac{\alpha_1}{\rho}\biggr|\int_0^\eta e^{-k(\eta-s)}\biggl(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)}\,dx\biggr)\,ds \\
&+ \biggl|\frac{\beta_1}{\rho}\biggr|\int_0^\eta e^{-k(\eta-s)}\biggl(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)}\,dx\biggr)\,ds \\
&+ \bigl|\nu_2(t)\bigr|\int_0^\eta (\eta-s)^{2-q}\biggl(\int_0^s e^{-k(s-x)}\biggl(\int_0^x \frac{(x-m)^{q-2}}{\Gamma(q-1)}\,dm\biggr)\,dx\biggr)\,ds \\
&+ \bigl|\nu_3(t)\bigr|\int_0^T (T-s)^{2-q}\biggl(\int_0^s e^{-k(s-x)}\biggl(\int_0^x \frac{(x-m)^{q-2}}{\Gamma(q-1)}\,dm\biggr)\,dx\biggr)\,ds
\end{aligned}$$

$$
\begin{aligned}
&+ \left|\nu_4(t)\right| \int_0^\eta (\eta - s)^{2-q} \left(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)}\, dx \right) ds \\
&+ \left|\nu_5(t)\right| \int_0^T (T - s)^{2-q} \left(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)}\, dx \right) ds \\
&+ \left|\nu_6(t)\right| \int_0^\eta (\eta - s)^{2-q} \left(\int_0^s \frac{(s-x)^{q-3}}{\Gamma(q-2)}\, dx \right) ds \\
&+ \left|\nu_7(t)\right| \int_0^T (T - s)^{2-q} \left(\int_0^s \frac{(s-x)^{q-3}}{\Gamma(q-2)}\, dx \right) ds \Bigg\} \\
\leq{}& (\ell r + M)Q + \|\nu_1\| \leq r.
\end{aligned}
$$

This shows that $\mathcal{H}B_r \subset B_r$. Next we show that the operator $\mathcal{H}$ is a contraction. Let $u, v \in \mathcal{P}$. Then

$$
\begin{aligned}
\|\mathcal{H}u - \mathcal{H}v\| \leq{}& \sup_{t\in[0,T]} \Bigg\{ \int_0^t e^{-k(t-s)} \left(\int_0^s \frac{(s-x)^{\alpha-2}}{\Gamma(\alpha-1)} \left| f\big(x,u(x)\big) - f\big(x,v(x)\big)\right| dx \right) ds \\
&+ \left|\frac{\alpha_1}{\rho}\right| \int_0^\eta e^{-k(\eta-s)} \left(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} \left| f\big(x,u(x)\big) - f\big(x,v(x)\big)\right| dx \right) ds \\
&+ \left|\frac{\beta_1}{\rho}\right| \int_0^T e^{-k(T-s)} \left(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} \left| f\big(x,u(x)\big) - f\big(x,v(x)\big)\right| dx \right) ds \\
&+ \left|\nu_2(t)\right| \int_0^\eta (\eta-s)^{2-q} \left(\int_0^s e^{-k(s-x)} \left(\int_0^x \frac{(x-m)^{q-2}}{\Gamma(q-1)} \left| f\big(m,u(m)\big) \right.\right.\right. \\
&\left.\left.\left. - f\big(m,v(m)\big)\right| dm \right) dx \right) ds \\
&+ \left|\nu_3(t)\right| \int_0^T (T-s)^{2-q} \left(\int_0^s e^{-k(s-x)} \left(\int_0^x \frac{(x-m)^{q-2}}{\Gamma(q-1)} \left| f\big(m,u(m)\big) \right.\right.\right. \\
&\left.\left.\left. - f\big(m,v(m)\big)\right| dm \right) dx \right) ds \\
&+ \left|\nu_4(t)\right| \int_0^\eta (\eta-s)^{2-q} \left(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} \left| f\big(x,u(x)\big) - f\big(x,v(x)\big)\right| dx \right) ds \\
&+ \left|\nu_5(t)\right| \int_0^T (T-s)^{2-q} \left(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} \left| f\big(x,u(x)\big) - f\big(x,v(x)\big)\right| dx \right) ds \\
&+ \left|\nu_6(t)\right| \int_0^\eta (\eta-s)^{2-q} \left(\int_0^s \frac{(s-x)^{q-3}}{\Gamma(q-2)} \left| f\big(x,u(x)\big) - f\big(x,v(x)\big)\right| dx \right) ds \\
&+ \left|\nu_7(t)\right| \int_0^T (T-s)^{2-q} \left(\int_0^s \frac{(s-x)^{q-3}}{\Gamma(q-2)} \left| f\big(x,u(x)\big) - f\big(x,v(x)\big)\right| dx \right) ds \Bigg\} \\
\leq{}& \ell\|u-v\| \sup_{t\in[0,T]} \Bigg\{ \int_0^t e^{-k(t-s)} \left(\int_0^s \frac{(s-x)^{\alpha-2}}{\Gamma(\alpha-1)}\, dx \right) ds \\
&+ \left|\frac{\alpha_1}{\rho}\right| \int_0^\eta e^{-k(\eta-s)} \left(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)}\, dx \right) ds \\
&+ \left|\frac{\beta_1}{\rho}\right| \int_0^T e^{-k(T-s)} \left(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)}\, dx \right) ds \\
&+ \left|\nu_2(t)\right| \int_0^\eta (\eta-s)^{2-q} \left(\int_0^s e^{-k(s-x)} \left(\int_0^x \frac{(x-m)^{q-2}}{\Gamma(q-1)}\, dm \right) dx \right) ds
\end{aligned}
$$

$$
\begin{aligned}
&+ \left|\nu_3(t)\right| \int_0^T (T-s)^{2-q}\left(\int_0^s e^{-k(s-x)}\left(\int_0^x \frac{(x-m)^{q-2}}{\Gamma(q-1)}\,dm\right)dx\right)ds \\
&+ \left|\nu_4(t)\right| \int_0^\eta (\eta-s)^{2-q}\left(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)}\,dx\right)ds \\
&+ \left|\nu_5(t)\right| \int_0^T (T-s)^{2-q}\left(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)}\,dx\right)ds \\
&+ \left|\nu_6(t)\right| \int_0^\eta (\eta-s)^{2-q}\left(\int_0^s \frac{(s-x)^{q-3}}{\Gamma(q-2)}\,dx\right)ds \\
&+ \left|\nu_7(t)\right| \int_0^T (T-s)^{2-q}\left(\int_0^s \frac{(s-x)^{q-3}}{\Gamma(q-2)}\,dx\right)ds\Bigg\} \\
\le{}& \ell Q\|u-v\|,
\end{aligned}
$$

where we have used (3.3). By the given assumption $\ell < 1/Q$, it follows that the operator $\mathcal{H}$ is a contraction. Thus, by Banach's contraction mapping principle, we deduce that the operator $\mathcal{H}$ has a fixed point, which equivalently implies that problem (1.1) has a unique solution on $[0,T]$. □

Next, we show the existence of solutions for problem (1.1) by applying Krasnoselskii's fixed-point theorem which is stated below.

Lemma 3.2 (Krasnoselskii's fixed-point theorem [22]) *Let $\mathcal{Y}$ be a closed bounded, convex, and nonempty subset of a Banach space $\mathcal{X}$. Let $\mathcal{G}_1, \mathcal{G}_2$ be the operators such that* (i) $\mathcal{G}_1 y_1 + \mathcal{G}_2 y_2 \in \mathcal{Y}$ *whenever* $y_1, y_2 \in \mathcal{Y}$; (ii) $\mathcal{G}_1$ *is compact and continuous; and* (iii) $\mathcal{G}_2$ *is a contraction mapping. Then there exists $\widehat{y} \in \mathcal{Y}$ such that $\widehat{y} = \mathcal{G}_1\widehat{y} + \mathcal{G}_2\widehat{y}$.*

Theorem 3.3 *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying condition* (A_1) *and that $|f(t,u)| \le g(t)$, $\forall (t,u) \in [0,T] \times \mathbb{R}$, where $g \in C([0,T], \mathbb{R}^+)$ with $\sup_{t\in[0,T]} |g(t)| = \|g\|$. In addition, it is assumed that $\ell Q_1 < 1$, where*

$$
\begin{aligned}
Q_1 = \sup_{t\in[0,T]} \Bigg\{ & \frac{|\alpha_1|\eta^{q-1}(1-e^{-k\eta}) + |\beta_1|T^{q-1}(1-e^{-kT})}{|\rho|k\Gamma(q)} \\
&+ \frac{|\nu_2(t)|\eta^2(1-e^{-k\eta}) + |\nu_3(t)|T^2(1-e^{-kT}) + |\nu_4(t)|k\eta^2 + |\nu_5(t)|kT^2}{k(3-q)\Gamma(q)} \\
&+ \frac{|\nu_6(t)|\eta + |\nu_7(t)|T}{(3-q)\Gamma(q-1)}\Bigg\}.
\end{aligned}
\tag{3.4}
$$

Then problem (1.1) *has at least one solution on* $[0,T]$.

Proof Let us consider the closed set $B_a = \{u \in \mathcal{P} : \|u\| \le a\}$, where $a \ge Q\|g\| + \|\nu_1\|$ and Q is given by (3.3). We define the operators $\mathcal{H}_1$ and $\mathcal{H}_2$ on B_a as

$$
(\mathcal{H}_1 u)(t) = \int_0^t e^{-k(t-s)}\left(\int_0^s \frac{(s-x)^{\alpha-2}}{\Gamma(\alpha-1)} f\bigl(x,u(x)\bigr)\,dx\right)ds,
$$

$$\begin{aligned}
(\mathcal{H}_2u)(t) = {}& \nu_1(t) - \frac{\alpha_1}{\rho}\int_0^{\eta} e^{-k(\eta-s)}\left(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} f\bigl(x,u(x)\bigr)\,dx\right)ds \\
& - \frac{\beta_1}{\rho}\int_0^{T} e^{-k(T-s)}\left(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} f\bigl(x,u(x)\bigr)\,dx\right)ds \\
& + \nu_2(t)\int_0^{\eta}(\eta-s)^{2-q}\left(\int_0^s e^{-k(s-x)}\left(\int_0^x \frac{(x-m)^{q-2}}{\Gamma(q-1)} f\bigl(m,u(m)\bigr)\,dm\right)dx\right)ds \\
& + \nu_3(t)\int_0^{T}(T-s)^{2-q}\left(\int_0^s e^{-k(s-x)}\left(\int_0^x \frac{(x-m)^{q-2}}{\Gamma(q-1)} f\bigl(m,u(m)\bigr)\,dm\right)dx\right)ds \\
& + \nu_4(t)\int_0^{\eta}(\eta-s)^{2-q}\left(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} f\bigl(x,u(x)\bigr)\,dx\right)ds \\
& + \nu_5(t)\int_0^{T}(T-s)^{2-q}\left(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} f\bigl(x,u(x)\bigr)\,dx\right)ds \\
& + \nu_6(t)\int_0^{\eta}(\eta-s)^{2-q}\left(\int_0^s \frac{(s-x)^{q-3}}{\Gamma(q-2)} f\bigl(x,u(x)\bigr)\,dx\right)ds \\
& + \nu_7(t)\int_0^{T}(T-s)^{2-q}\left(\int_0^s \frac{(s-x)^{q-3}}{\Gamma(q-2)} f\bigl(x,u(x)\bigr)\,dx\right)ds.
\end{aligned}$$

For $u, v \in B_a$, it is easy to verify that $\|\mathcal{H}_1u + \mathcal{H}_2v\| \le Q\|g\| + \|\nu_1\|$. Thus, $\mathcal{H}_1u + \mathcal{H}_2v \in B_a$. Using assumption ($A_1$) and (3.4), we can get $\|\mathcal{H}_2u - \mathcal{H}_2v\| \le \ell Q_1\|u - v\|$, which implies that $\mathcal{H}_2$ is a contraction in view of the given condition $\ell Q_1 < 1$.

Notice that the continuity of f implies that the operator $\mathcal{H}_1$ is continuous. Also, $\mathcal{H}_1$ is uniformly bounded on B_a as

$$\|\mathcal{H}_1u\| \le \frac{(1-e^{-kT})T^{q-1}\|g\|}{k\Gamma(q)}.$$

In the last step, it will be shown that the operator $\mathcal{H}_1$ is compact. Fixing $\sup_{(t,u)\in[0,T]\times B_a}|f(t,u)| = f_a$, for $t_1, t_2 \in [0,T]$, we have

$$\begin{aligned}
&\bigl\|(\mathcal{H}_1u)(t_2) - (\mathcal{H}_1u)(t_1)\bigr\| \\
&\quad \le f_a\left(\bigl|e^{-kt_2} - e^{-kt_1}\bigr|\int_0^{t_1} e^{ks}\left(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)}\,dx\right)ds \right.\\
&\qquad \left. + \int_{t_1}^{t_2} e^{-k(t_2-s)}\left(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)}\,dx\right)ds\right) \\
&\quad \to 0 \quad \text{as } t_2 - t_1 \to 0
\end{aligned}$$

independent of u. This implies that $\mathcal{H}_1$ is relatively compact on B_a. Hence, by the Arzelá-Ascoli theorem, the operator $\mathcal{H}_1$ is compact on B_a. Thus all the assumptions of Lemma 3.2 are satisfied. In consequence, it follows from the conclusion of Lemma 3.2 that problem (1.1) has at least one solution on $[0,T]$. □

Now we prove the existence of solutions for problem (1.1) via the Leray-Schauder alternative. Let us first recall the nonlinear alternative for single-valued maps [23].

Lemma 3.4 *Let C be a closed, convex subset of a Banach space E and U be an open subset of C such that $0 \in U$. Suppose that the operator $\mathcal{T} : \overline{U} \to C$ is a continuous and compact* *map, that is, $\mathcal{T}(\overline{U})$ is a relatively compact subset of C. Then either* (i) *$\mathcal{T}$ has a fixed point in $\overline{U}$, or* (ii) *there is $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0,1)$ such that $u = \lambda\mathcal{T}(u)$.*

Theorem 3.5 *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that*

(A_2) *there exist a function $p \in C([0,T],\mathbb{R}^+)$ and a nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $|f(t,u)| \le p(t)\psi(\|u\|)$, $\forall (t,u) \in [0,T] \times \mathbb{R}$;*

(A_3) *there exists a constant $\overline{M} > 0$ such that $\overline{M}/\overline{Q} > 1$, where*

$$\overline{Q} = \|\nu_1(t)\| + \|p\|\psi(\overline{M})Q. \tag{3.5}$$

Then problem (1.1) *has at least one solution on* $[0,T]$.

Proof We complete the proof in several steps. In the first step, it will be shown that the operator $\mathcal{H} : \mathcal{P} \to \mathcal{P}$ maps bounded sets into bounded sets in $C([0,T],\mathbb{R})$. For the positive number r, let $B_r = \{u \in C([0,T],\mathbb{R}) : \|u\| \le r\}$ be a bounded set in $C([0,T],\mathbb{R})$. Then, in view of assumption (A_2), one can get

$$\begin{aligned}
\big|(\mathcal{H}u)(t)\big| \le{}& \big|\nu_1(t)\big| + \int_0^t e^{-k(t-s)}\left(\int_0^s \frac{(s-x)^{\alpha-2}}{\Gamma(\alpha-1)}p(x)\psi\big(\|u\|\big)\,dx\right)ds \\
&+ \left|\frac{\alpha_1}{\rho}\right|\int_0^\eta e^{-k(\eta-s)}\left(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)}p(x)\psi\big(\|u\|\big)\,dx\right)ds \\
&+ \left|\frac{\beta_1}{\rho}\right|\int_0^T e^{-k(T-s)}\left(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)}p(x)\psi\big(\|u\|\big)\,dx\right)ds \\
&+ \big|\nu_2(t)\big| \\
&\times \int_0^\eta (\eta-s)^{2-q}\left(\int_0^s e^{-k(s-x)}\left(\int_0^x \frac{(x-m)^{q-2}}{\Gamma(q-1)}p(m)\psi\big(\|u\|\big)\,dm\right)dx\right)ds \\
&+ \big|\nu_3(t)\big| \\
&\times \int_0^T (T-s)^{2-q}\left(\int_0^s e^{-k(s-x)}\left(\int_0^x \frac{(x-m)^{q-2}}{\Gamma(q-1)}p(m)\psi\big(\|u\|\big)\,dm\right)dx\right)ds \\
&+ \big|\nu_4(t)\big|\int_0^\eta (\eta-s)^{2-q}\left(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)}p(x)\psi\big(\|u\|\big)\,dx\right)ds \\
&+ \big|\nu_5(t)\big|\int_0^T (T-s)^{2-q}\left(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)}p(x)\psi\big(\|u\|\big)\right)ds \\
&+ \big|\nu_6(t)\big|\int_0^\eta (\eta-s)^{2-q}\left(\int_0^s \frac{(s-x)^{q-3}}{\Gamma(q-2)}p(x)\psi\big(\|u\|\big)\,dx\right)ds \\
&+ \big|\nu_7(t)\big|\int_0^T (T-s)^{2-q}\left(\int_0^s \frac{(s-x)^{q-3}}{\Gamma(q-2)}p(x)\psi\big(\|u\|\big)\,dx\right)ds.
\end{aligned}$$

In consequence, taking the norm on $[0,T]$, we have that

$$\begin{aligned}
\big\|(\mathcal{H}u)\big\| \le{}& \|\nu_1\| + \|p\|\psi(r)\left\{\frac{t^{q-1}|\rho|(1-e^{-kt}) + |\alpha_1|\eta^{q-1}(1-e^{-k\eta}) + |\beta_1|T^{q-1}(1-e^{-kT})}{|\rho|k\Gamma(q)}\right. \\
&+ \frac{|\nu_2(t)|\eta^2(1-e^{-k\eta}) + |\nu_3(t)|T^2(1-e^{-kT}) + |\nu_4(t)|k\eta^2 + |\nu_5(t)|kT^2}{k(3-q)\Gamma(q)}
\end{aligned}$$

$$
\begin{aligned}
&\quad + \frac{|\nu_6(t)|\eta + |\nu_7(t)|T}{(3-q)\Gamma(q-1)} \Bigg\} \\
&= \|\nu_1\| + \|p\|\psi(r)Q,
\end{aligned}
$$

where Q is given by (3.3).

Next we show that $\mathcal{H}$ maps bounded sets into equicontinuous sets of $C([0,T],\mathbb{R})$. Let $t_1, t_2 \in [0,T]$ with $t_1 < t_2$ and $u \in B_r$, where B_r is a bounded set of $C([0,T],\mathbb{R})$. Then we obtain

$$
\begin{aligned}
&\bigl|(\mathcal{H}u)(t_2) - (\mathcal{H}u)(t_1)\bigr| \\
&\quad \le \bigl|\nu_1(t_2) - \nu_1(t_1)\bigr| \\
&\qquad + \left| \int_0^{t_2} \bigl(e^{-k(t_2-s)} - e^{-k(t_1-s)}\bigr) \left(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} f\bigl(x,u(x)\bigr)\,dx \right) ds \right| \\
&\qquad + \left| \int_{t_1}^{t_2} e^{-k(t_1-s)} \left(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} f\bigl(x,u(x)\bigr)\,dx \right) ds \right| \\
&\qquad + \bigl|\nu_2(t_2) - \nu_2(t_1)\bigr| \\
&\qquad \times \int_0^{\eta} (\eta - s)^{2-q} \left(\int_0^s e^{-k(s-x)} \left(\int_0^x \frac{(x-m)^{q-2}}{\Gamma(q-1)} \bigl|f\bigl(m,u(m)\bigr)\bigr|\,dm \right) dx \right) ds \\
&\qquad + \bigl|\nu_3(t_2) - \nu_3(t_1)\bigr| \\
&\qquad \times \int_0^{T} (T - s)^{2-q} \left(\int_0^s e^{-k(s-x)} \left(\int_0^x \frac{(x-m)^{q-2}}{\Gamma(q-1)} \bigl|f\bigl(m,u(m)\bigr)\bigr|\,dm \right) dx \right) ds \\
&\qquad + \bigl|\nu_4(t_2) - \nu_4(t_1)\bigr| \int_0^{\eta} (\eta - s)^{2-q} \left(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} \bigl|f\bigl(x,u(x)\bigr)\bigr|\,dx \right) ds \\
&\qquad + \bigl|\nu_5(t_2) - \nu_5(t_1)\bigr| \int_0^{T} (T - s)^{2-q} \left(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} \bigl|f\bigl(x,u(x)\bigr)\bigr|\,dx \right) ds \\
&\qquad + \bigl|\nu_6(t_2) - \nu_6(t_1)\bigr| \int_0^{\eta} (\eta - s)^{2-q} \left(\int_0^s \frac{(s-x)^{q-3}}{\Gamma(q-2)} \bigl|f\bigl(x,u(x)\bigr)\bigr|\,dx \right) ds \\
&\qquad + \bigl|\nu_7(t_2) - \nu_7(t_1)\bigr| \int_0^{T} (T - s)^{2-q} \left(\int_0^s \frac{(s-x)^{q-3}}{\Gamma(q-2)} \bigl|f\bigl(x,u(x)\bigr)\bigr|\,dx \right) ds \\
&\quad \le \left| \frac{\Gamma(3-q)b}{k^2 N\sigma} \bigl(N\bigl(e^{-kt_2} - e^{-kt_1}\bigr) + k(t_2 - t_1)\vartheta \bigr) + \frac{\Gamma(3-q)c}{N}(t_2 - t_1) \right| \\
&\qquad + \|p\|\psi\bigl(\|u\|\bigr) \left(\bigl|e^{-kt_2} - e^{-kt_1}\bigr| \int_0^{t_1} e^{ks} \left(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)}\,dx \right) ds \right. \\
&\qquad \left. + \int_{t_1}^{t_2} e^{-k(t_2-s)} \left(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)}\,dx \right) ds \right) \\
&\qquad + \left| \frac{-\alpha_2}{\sigma}\bigl(e^{-kt_2} - e^{-kt_1}\bigr) + \left(\frac{k(t_2 - t_1)}{N} \right) \left(\alpha_3 - \frac{\alpha_2 \vartheta}{\sigma} \right) \right| \\
&\qquad \times \int_0^{\eta} (\eta - s)^{2-q} \left(\int_0^s e^{-k(s-x)} \left(\int_0^x \frac{(x-m)^{q-2}}{\Gamma(q-1)} \bigl|f\bigl(m,u(m)\bigr)\bigr|\,dm \right) dx \right) ds \\
&\qquad + \left| \frac{-\beta_2}{\sigma}\bigl(e^{-kt_2} - e^{-kt_1}\bigr) + \left(\frac{k(t_2 - t_1)}{N} \right) \left(\beta_3 - \frac{\beta_2 \vartheta}{\sigma} \right) \right|
\end{aligned}
$$

$$
\begin{aligned}
&\times \int_0^T (T-s)^{2-q}\left(\int_0^s e^{-k(s-x)}\left(\int_0^x \frac{(x-m)^{q-2}}{\Gamma(q-1)}\left|f\bigl(m,u(m)\bigr)\right| dm\right) dx\right) ds \\
&+ \left|\frac{\alpha_2}{k\sigma}\bigl(e^{-kt_2}-e^{-kt_1}\bigr) - \left(\frac{(t_2-t_1)}{N}\right)\left(\alpha_3 - \frac{\alpha_2\vartheta}{\sigma}\right)\right| \\
&\times \int_0^\eta (\eta-s)^{2-q}\left(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)}\left|f\bigl(x,u(x)\bigr)\right| dx\right) ds \\
&+ \left|\frac{\beta_2}{k\sigma}\bigl(e^{-kt_2}-e^{-kt_1}\bigr) - \left(\frac{(t_2-t_1)}{N}\right)\left(\beta_3 - \frac{\beta_2\vartheta}{\sigma}\right)\right| \\
&\times \int_0^T (T-s)^{2-q}\left(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)}\left|f\bigl(x,u(x)\bigr)\right| dx\right) ds \\
&+ \left|\frac{-\alpha_2}{Nk^2\sigma}\bigl[N\bigl(e^{-kt_2}-e^{-kt_1}\bigr) + k\vartheta(t_2-t_1)\bigr]\right| \\
&\times \int_0^\eta (\eta-s)^{2-q}\left(\int_0^s \frac{(s-x)^{q-3}}{\Gamma(q-2)}\left|f\bigl(x,u(x)\bigr)\right| dx\right) ds \\
&+ \left|\frac{-\beta_2}{Nk^2\sigma}\bigl[N\bigl(e^{-kt_2}-e^{-kt_1}\bigr) + k\vartheta(t_2-t_1)\bigr]\right| \\
&\times \int_0^T (T-s)^{2-q}\left(\int_0^s \frac{(s-x)^{q-3}}{\Gamma(q-2)}\left|f\bigl(x,u(x)\bigr)\right| dx\right) ds.
\end{aligned}
$$

Obviously, the right-hand side of the above inequality tends to zero independently of $u \in B_r$ as $t_2 - t_1 \to 0$. As $\mathcal{H}$ satisfies the above assumptions, it follows by the Arzelá-Ascoli theorem that $\mathcal{H} : C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$ is completely continuous.

The result will follow from the Leray-Schauder nonlinear alternative (Lemma 3.4) once we establish the boundedness of the set of all solutions to equations $u = \lambda\mathcal{H}u$ for $\lambda \in [0,1]$.

Let u be a solution. Then, for $t \in [0,T]$, using the method of computations in proving that $\mathcal{H}$ is bounded, we can obtain

$$
\bigl|u(t)\bigr| = \bigl|\lambda(\mathcal{H}u)(t)\bigr| \leq \bigl\|\nu_1(t)\bigr\| + \|p\|\psi(r)Q,
$$

which implies that

$$
\|u\|/\bigl(|\nu_1| + \|p\|\psi(r)Q\bigr) \leq 1.
$$

In view of (A_3), there exists $\overline{M}$ such that $\|u\| \neq \overline{M}$. Let us set

$$
U = \bigl\{u \in C\bigl([0,T],\mathbb{R}\bigr) : \|u\| < \overline{M}\bigr\}.
$$

Note that the operator $\mathcal{H} : \overline{U} \to C([0,T],\mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $u \in \partial U$ such that $u = \lambda\mathcal{H}(u)$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 3.4), we deduce that $\mathcal{H}$ has a fixed point $u \in \overline{U}$ which is a solution of problem (1.1). This completes the proof. □

Our final result is based on Leray-Schauder degree theory.

Theorem 3.6 *Let $f:[0,T]\times\mathbb{R}\to\mathbb{R}$. Assume that there exist constants $0\le\overline{k}<\frac{1}{Q}$, where Q is given by* (3.3) *and $\overline{\overline{M}}>0$ such that $|f(t,u(t))|\le\overline{k}\|u\|+\overline{\overline{M}}$ for all $t\in[0,T]$, $u\in\mathbb{R}$. Then boundary value problem* (1.1) *has at least one solution*.

Proof In view of fixed-point problem (3.1), we just need to prove the existence of at least one solution $u\in\mathbb{R}$ satisfying (3.1). Define a suitable ball $B_R\subset C[0,T]$ with radius $R>0$ as

$$B_R=\{u\in C[0,T]:\|u\|\le R\},$$

where R will be fixed later. Then it is sufficient to show that $\mathcal{H}:\overline{B_R}\to C$ satisfies

$$u\neq\lambda\mathcal{H}u,\quad\forall u\in\partial B_R\quad\text{and}\quad\forall\lambda\in[0,1].\tag{3.6}$$

Let us set

$$\Phi(\lambda,u)=\lambda\mathcal{H}u,\quad u\in C,\lambda\in[0,1].$$

Then, by the Arzelá-Ascoli theorem, $\omega_\lambda(u)=u-\Phi(\lambda,u)=u-\mathcal{H}u$ is completely continuous. If (3.6) is true, then the following Leray-Schauder degrees are well defined and, by the homotopy invariance of topological degree, it follows that

$$\begin{aligned}\deg(\omega_\lambda,B_R,0)&=\deg(I-\lambda\mathcal{H},B_R,0)=\deg(\omega_1,B_R,0)\\&=\deg(\omega_0,B_R,0)=\deg(I,B_R,0)=1\neq0,\quad 0\in B_R,\end{aligned}\tag{3.7}$$

where I denotes the unit operator. By the nonzero property of Leray-Schauder degree, $\omega_1(t)=u-\lambda\mathcal{H}u=0$ for at least one $u\in B_R$. To prove (3.6), we assume that $u=\lambda\mathcal{H}u=0$ for some $\lambda\in[0,1]$ and for all $t\in[0,T]$. Then, as in the preceding results, one can obtain

$$\|u\|\le\frac{\|\nu_1\|+\overline{\overline{M}}Q}{1-\overline{k}Q}.$$

Letting $R=\frac{\|\nu_1\|+\overline{\overline{M}}Q}{1-\overline{k}Q}+0.5$, (3.6) holds. This completes the proof. □

Example 3.7 Consider the following anti-periodic fractional boundary value problem:

$$\begin{cases}({}^cD^{5/2}+2{}^cD^{3/2})u(t)=f(t,u(t)),\quad t\in[0,4],\\u(1)+2u(4)=1,\\-2{}^cD^{3/2}u(1)-{}^cD^{3/2}u(4)=1.5,\\{}^cD^{1/2}u(1)+{}^cD^{1/2}u(4)=-1.\end{cases}\tag{3.8}$$

$T=4$, $\eta=1$, $k=2$, $\alpha_1=1$, $\beta_1=2$, $\alpha_2=-2$, $\beta_2=-1$, $\alpha_3=1$, $\beta_3=1$, $a=1$, $b=1.5$, $c=-1$. With the given values, it is found that $Q\approx16.51234$ and $Q_1\approx13.50434$, where Q and Q_1 are respectively defined by (3.3) and (3.4).

- For the applicability of Theorem 3.1, let us take $f(t,u(t))=\frac{\sin u}{30}+e^{-t}\cos t$ in (3.8). Then $\ell=1/30$ as $|f(t,u)-f(t,v)|\le\frac{1}{30}|u-v|$ and $\ell Q\approx0.55041<1$. Thus all the conditions of Theorem 3.1 are satisfied. Hence we deduce by the conclusion of Theorem 3.1 that there exists a unique solution for problem (3.8) on $[0,4]$. To illustrate Theorem 3.3, we find that $|f(t,u)| \leq g(t) = \frac{1}{30} + e^{-t}\cos t$ with $\|g\| = \frac{31}{30}$ and $\ell Q_1 \approx 0.45014 < 1$. In consequence, by Theorem (3.3), problem (3.8) has at least one solution on $[0,4]$.

- For the illustration of Theorem 3.5, we consider $f(t,u(t)) = \frac{4e^{-t}\tan^{-1}(u(t))}{\pi\sqrt{t^2+225}}$ in (3.8). Then $|f(t,u)| \leq p(t)\psi(\|u\|)$ with $\psi(\|u\|) = 2, p(t) = \frac{e^{-t}}{\sqrt{t^2+225}}$, $\overline{Q} = \|\nu_1\| + \|p\|\psi(\|\overline{M}\|)Q \approx 3.84039$ ($\overline{Q}$ is given by (3.5)) and $\overline{M} > 3.84039$. Thus all the conditions of Theorem 3.5 are satisfied, and the conclusion of Theorem 3.5 applies to problem (3.8).
- For demonstrating the applicability of Theorem 3.6, let $f(t,u(t)) = \frac{sin(\pi u)}{30\pi} + \frac{2}{1+u^2}$ in (3.8). Obviously, $|f(t,u)| \leq \|u\|/30 + 2$ with $\overline{k} = \frac{1}{30}, \overline{\overline{M}} = 2$. Clearly, $\frac{1}{30} = \overline{k} < \frac{1}{Q} = \frac{1}{16.51234}$. Thus the hypothesis of Theorem 3.6 is satisfied. Hence, it follows by the conclusion of Theorem 3.6 that problem (3.8) has at least one solution on $[0,4]$.

4 Concluding remarks

We have discussed the existence of solutions for sequential fractional differential equations of order $q \in (2,3]$ equipped with nonlocal non-separated boundary conditions involving lower-order fractional derivatives. The uniqueness result relies on Banach's contraction mapping principle, while the existence of solutions is established by applying Krasnoselskii's fixed-point theorem, the Leray-Schauder nonlinear alternative and Leray-Schauder degree theory. Though we make use of the standard tools of fixed-point theory, our results are new and significantly contribute to the existing literature on fractional-order boundary value problems with separated boundary conditions. Now we enlist some interesting observations.

(a) For $\eta = 0$, our results correspond to the boundary conditions

$$\alpha_1 u(0) + \beta_1 u(T) = a, \qquad {}^cD^{q-1}u(T) = b/\beta_2, \qquad {}^cD^{q-2}u(T) = c/\beta_3$$

as ${}^cD^{q-1}u(0) = 0$, ${}^cD^{q-2}u(0) = 0$. This situation is in contrast to classical boundary conditions involving $u'(\eta)$ and $u''(\eta)$, which do not vanish when $\eta = 0$. This is equivalent to saying that the reduced results remain the same whether we take $\eta = 0$ or $\alpha_2 = 0 = \alpha_3$. From the preceding discussion, one can easily infer that the nonlocal boundary conditions involving lower-order fractional derivatives considered in problem (1.1) do not reduce to classical anti-periodic boundary conditions of the form $u(0) + u(T) = 0$, $u'(0) + u'(T) = 0$, $u''(0) + u''(T) = 0$ when we take $\eta = 0$, $\alpha_i = 1 = \beta_i$ $(i = 1,2,3)$, $a = b = c = 0$.

(b) By taking $\beta_i = 0$, $i = 1,2,3$, we obtain new results associated with one-point nonlocal conditions involving lower-order derivative:

$$u(\eta) = a/\alpha_1, \qquad {}^cD^{q-1}u(\eta) = b/\alpha_2, \qquad {}^cD^{q-2}u(\eta) = c/\alpha_3, \quad \alpha_i \neq 0, i = 1,2,3, 0 < \eta < T.$$

(c) In case we take $\alpha_i = 1 = \beta_i$ $(i = 1,2,3)$, $a = b = c = 0$ with $0 < \eta \ll T$, our results become the ones supplemented with perturbed anti-periodic boundary conditions involving lower-order derivatives:

$$u(\eta) = -u(T), \qquad {}^cD^{q-1}u(\eta) = -{}^cD^{q-1}u(T), \qquad {}^cD^{q-2}u(\eta) = -{}^cD^{q-2}u(T).$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Each of the authors, MSA, AA, BA, and MHA, contributed to each part of this work equally and read and approved the final version of the manuscript.

Acknowledgements
This project was funded by the Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah, under grant No. (59-130-35-RG). The authors, therefore, acknowledge with thanks DSR for technical and financial support.

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Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 3 January 2017 Accepted: 21 March 2017 Published online: 05 April 2017

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