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A corresponding Cullen-regularity for split-quaternionic-valued functions

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Abstract

We give a representation of the class of Cullen-regular functions in split-quaternions. We consider each Cullen's form of split-quaternions, which provides corresponding Cauchy-Riemann equations for split-quaternionic variables. Using Cullen's form, we research hyperholomorphy and the properties of functions of split-quaternionic variables which are expressed by hyperbolic coordinates.

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1 Introduction

The skew field of real quaternions, denoted by $\mathbb{H}$, has the form

$$\mathbb{H} = \{q \mid q = x_0 + ix_1 + jx_2 + kx_3, x_r \in \mathbb{R}\ (r = 0,1,2,3)\},$$

where $\mathbb{R}$ is the set of real numbers and i, j, and k are imaginary units with

$$i^2 = j^2 = k^2 = -1,$$
$$ij = -ji = k, \qquad jk = -kj = i, \qquad ki = -ik = j.$$

Theories and applications of functions of a quaternionic variable have been led by holomorphic functions of one complex variable. Quaternions were introduced by Hamilton [1] in 1843 and generalized by Clifford [2] in 1873. After then, Fueter [3, 4] defined differential operators, called the Cauchy-Fueter operators, and regular functions in the space of solutions of the equation with these operators. Kim et al. [5, 6] obtained the regularity of functions on the form of reduced quaternions in Clifford analysis.

Split-quaternions are elements of four-dimensional algebra introduced by Cockle [7] in 1849. Similar to quaternions, they form a four-dimensional real vector space which is a general associative, but non-commutative, form of multiplication. Unlike the quaternions, the split-quaternions contain zero divisors, nilpotent elements, and nontrivial idempotents such as $\frac{(1+j)}{2}$ is an idempotent zero-divisor, and $i - j$ is nilpotent. Indeed, Helmut and Kist [8] constructed an algebra over the real numbers, which is isomorphic to the algebra of 2×2 real matrices. Moreover, the modulus which has an isotropic quadratic form for split-quaternions provides hyperbolic motions of the Poincaré disk model of hyperbolic geometry. From this result, the structure of split-quaternion analysis has been developed and applied in four-dimensional physics by Frenkel and Libine [9, 10]. Kim and Shon [11, 12] researched corresponding Cauchy-Riemann systems and the properties of hyperholomorphic functions with values in modified split-quaternions. Kim and Shon [13] gave relations between the properties of split-quaternionic regular functions and their algebraic inverse mapping.

In 1965, Cullen [14] proposed the notion of intrinsic functions and found quaternions convenient to write in the form $\zeta = x_0 + p\mu$, where x_0 and $p = \sqrt{x_1^2 + x_2^2 + x_3^2}$ are real and μ is a unit vector quaternion. For fixed μ, the elements of the form $x_0 + p\mu$ constitute a subspace of $\mathcal{D}$ which is isomorphic to the complex field. Since $\mathcal{D}$ is a generalization of the algebra of complex numbers, $\mathcal{D}$ has properties analogous to the class of analytic functions of a complex variable. From these properties, Cullen-regular functions are tried to be related to a class of functions of the reduced quaternionic variable, studied by Leutwiler in [15]. Gentili and Struppa [16] and Alayón-Solarz [17, 18] gave definitions of regularity for functions of a quaternionic variable and developed representations of the Cullen-regularity of quaternion analysis. By using the analytic properties and calculating processes of Cullen-regularity, Marin has researched thermoelastic materials in various points of view. For example, Marin and coauthors studied the asymptotic partition of total energy for the solutions of the mixed initial boundary value problem within the thermoelasticity of initially stressed bodies. They obtained a spatial decay estimate which is considered a right cylinder composed of physically micropolar thermoelastic material for which one plane end is subjected to an excitation harmonic in time. Also, they extended the concept of domain of influence in order to cover the elasticity of microstretch materials and studied it for the displacement field, the microrotation field, and the microstretch function (see [19–21]).

Based on these studies, we consider a general type of Cullen-regularity for functions of split-quaternionic variables. First, we give the notions and some properties of split-quaternion-valued functions by using Cullen's form. Also, we investigate the structure of a Cullen-regular function and corresponding split-Cauchy-Riemann systems for Cullen-variables and research properties of hyperholomorphic functions, represented by Cullen-regular functions.

2 Preliminaries

Let $\mathbb{S}$ denote the skew field of real split-quaternions which has elements of the form

$$p = x_0 + ix_1 + jx_2 + kx_3,$$

where x_r $(r = 0, 1, 2, 3)$ are real, i, j and k are imaginary units such that

$$i^2 = -1, \qquad j^2 = k^2 = 1,$$
$$ij = -ji = k, \qquad kj = -jk = i, \qquad ki = -ik = j,$$

which is isomorphic to $\mathbb{R}^4$. By the properties of the imaginary units of split-quaternions, we have the following rules for addition and multiplication:

$$p + q = (x_0 + y_0) + i(x_1 + y_1) + j(x_2 + y_2) + k(x_3 + y_3)$$

and

$$pq = (x_0y_0 - x_1y_1 + x_2y_2 + x_3y_3) + i(x_1y_0 + x_0y_1 + x_3y_2 - x_2y_3)$$
$$+ j(x_2y_0 + x_3y_1 + x_0y_2 - x_1y_3) + k(x_3y_0 - x_2y_1 + x_1y_2 + x_0y_3),$$

respectively. We give the conjugation of a split-quaternion as follows:

$$p^* = x_0 - ix_1 - jx_2 - kx_3.$$

Then we have a modulus, denoted by $\mathcal{N}(p)$, and an inverse element, denoted by p^{-1} of $p \in \mathbb{S}$,

$$\mathcal{N}(p) := pp^* = x_0^2 + x_1^2 - x_2^2 - x_3^2$$

and

$$p^{-1} = \frac{p^*}{\mathcal{N}(p)} \quad \left(x_0^2 + x_1^2 \neq x_2^2 + x_3^2\right),$$

respectively. By the non-commutative property of basis vectors i, j and k, the product of a split-quaternion with its conjugate is given in an isotropic quadratic form. Given two split-quaternions p and q, the following holds:

$$N(pq) = N(p)N(q).$$

For any $p \neq 0$ with $N(p) = 0$, p is a null vector, and when the modulus is non-zero, then p has a multiplicative inverse.

For a split-quaternion p, it can be written as $p = S(p) + V(p)$, where $S(p)$ is the scalar part and $V(p)$ is the vector part of p. Specially, if $S(p) = 0$, then p is called pure split-quaternion. We focus the attention on the pure split-quaternions to configure Cullen's form in $\mathbb{S}$. For a pure split-quaternion p, we consider $\mathcal{T} = \{p = ix + jy + kz \mid -x^2 + y^2 + z^2 = 1\}$. Since points (x, y, z), which satisfy $-x^2 + y^2 + z^2 = 1$, compose Figure 1 in practice, Figure 1 shows that $\mathcal{T}$

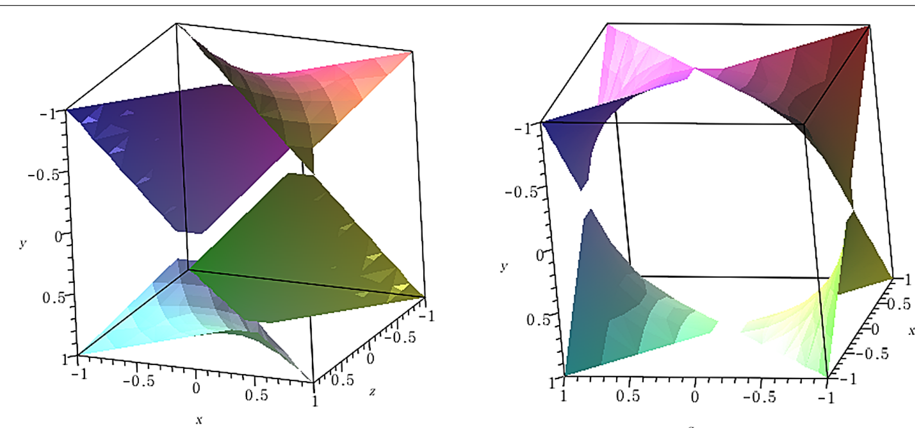
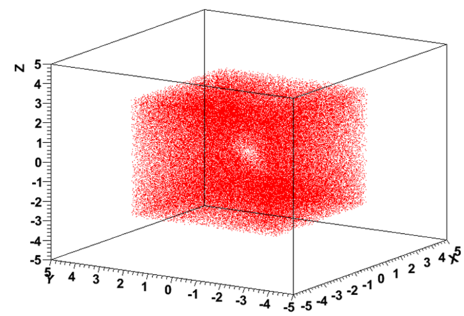
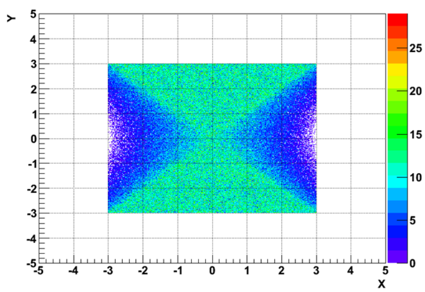


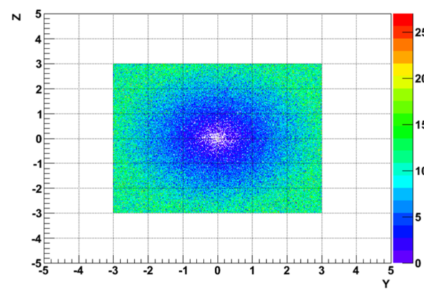
Figure 1 Figures of $-x^2 + y^2 + z^2 = 1$.

Figure 2 $x^2 < y^2 + z^2$.

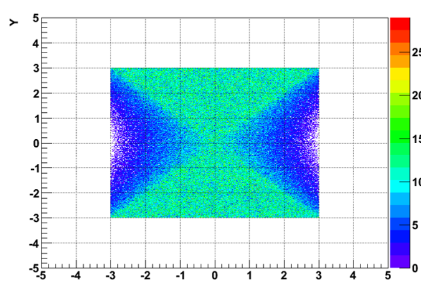




(a) Onto the *xy*-plane



(b) Onto the *yz*-plane



(c) Onto the *xz*-plane

Figure 3 The projection of Figure 2.

can exist and be non-empty in $\mathbb{S}$. So, we can represent the following processes and results. If we let

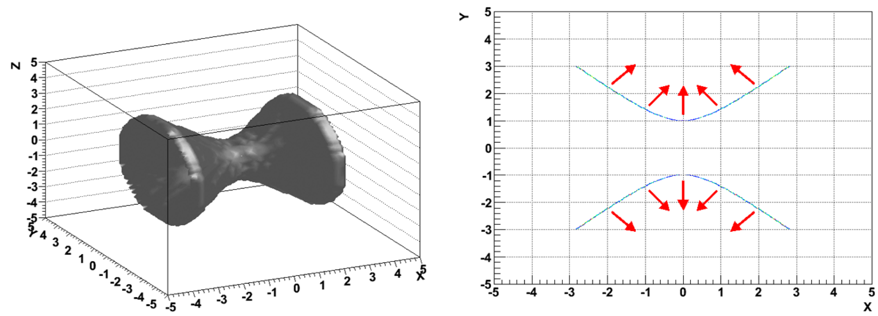
$$J = \frac{ix_1 + jx_2 + kx_3}{\sqrt{-x_1^2 + x_2^2 + x_3^2}} \quad \left(x_1^2 < x_2^2 + x_3^2\right),$$

then $J^2 = 1$ and $J \in \mathcal{T}$. Indeed, the existence of the element J of $\mathcal{T}$ is guaranteed by Figures 2-4.

Let Ω be an open set in $\mathbb{S}$. A function $f : \Omega \to \mathbb{S}$ is given by

$$f(p) = f_0 + if_1 + jf_2 + kf_3,$$

where $f_r = f_r(x_0, x_1, x_2, x_3)$ $(r = 0, 1, 2, 3)$ are real-valued functions. We try to express a function using the element J of $\mathcal{T}$. So, we consider the class of split-quaternionic functions



(a) Figure of $\mathcal{T}$ (b) Figure of $n(p)$ on the xy-plane of $\mathcal{T}$

Figure 4 Figure of $n(p)$ on $\mathcal{T}$.

which has an element f written as

$$f = u + Jv, \tag{2.1}$$

by letting $u = f_0$ be a real-valued function and

$$v = \frac{1}{\sqrt{-x_1^2 + x_2^2 + x_3^2}} \{(-x_1 f_1 + x_2 f_2 + x_3 f_3) + i(-x_2 f_3 + x_3 f_2) \\ + j(-x_1 f_3 + x_3 f_1) + k(x_1 f_2 - x_2 f_1)\}$$

be a split-quaternion-valued function. Let $\Omega_J := \Omega \cup L_J$, where $L_J = \mathbb{R} + J\mathbb{R}$. For all $p \in \Omega_J$, let $f_J : \Omega_J \to \mathbb{S}$ be a function

$$f_J(p) = f(x + Jy) = u(x, y) + Jv(x, y), \tag{2.2}$$

where u and v are real-valued functions. The function f_J is called a restriction of f in $\mathbb{S}$.

Remark 2.1 For any split-quaternion $p \in \mathbb{S}$, the identity function maps one split-quaternion onto itself, and there are unique $x, y \in \mathbb{R}$ with $y > 0$, and $J \in \mathcal{T}$ such that $p = x + Jy$, where

$$x = x_0, \qquad y = \sqrt{-x_1^2 + x_2^2 + x_3^2},$$
$$J = \frac{ix_1 + jx_2 + kx_3}{\sqrt{-x_1^2 + x_2^2 + x_3^2}} \quad (x_1^2 < x_2^2 + x_3^2).$$

3 Regularity of split-quaternionic functions

Consider differential operators given by

$$D_l^* := \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} - j\frac{\partial}{\partial x_2} - k\frac{\partial}{\partial x_3} \tag{3.1}$$

and

$$D_r^* := \frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_1} i - \frac{\partial}{\partial x_2} j - \frac{\partial}{\partial x_3} k, \tag{3.2}$$

where the units i, j and k act from the left or right in each case. We call (3.1) the left-differential operator and (3.2) the right-differential operator in $\mathbb{S}$. Since both operators have similar roles, we will describe D_l^* only later.

Definition 3.1 *Let Ω be an open set in $\mathbb{S}$. A function $f : \Omega \to \mathbb{S}$ is said to be J-regular if for every $J \in \mathcal{T}$, its restriction $f_J : \Omega_J \to \mathbb{S}$ is continuously differentiable and satisfies*

$$D_J^* f(x + Jy) = 0,$$

where

$$D_J^* := \frac{1}{2}\left(\frac{\partial}{\partial x} - J\frac{\partial}{\partial y}\right).$$

Remark 3.2 The equation $D_J^* f(x + Jy) = 0$ is equivalent to the following equations:

$$\begin{cases} \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0, \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0. \end{cases} \tag{3.3}$$

So, a function f is J-regular if and only if f satisfies equation (3.3).

Example 1 For $p \in \mathbb{S}$ and $a_n \in \mathbb{S}$ ($n \in \mathbb{N}$, where $\mathbb{N}$ is the set of positive integers), a polynomial $f(p) = p^n a_n$ satisfies

$$D_J^* f(p) = \frac{1}{2}\left(\frac{\partial}{\partial x} - J\frac{\partial}{\partial y}\right)(x + Jy)^n a_n = 0.$$

Hence, the polynomial $p^n a_n$ is J-regular.

Polynomials are used in a wide variety of fields of mathematics and every branch of chemistry and physics. In advanced mathematics, polynomials are used to construct polynomial rings and central concepts in algebra and algebraic geometry. Moreover, if a function is a solution of the equations consisting of differential operators in some systems, the function can be locally expanded as a power series by Taylor's theorem.

Proposition 3.3 *The sum and product of two J-regular functions are J-regular.*

Proof Let $f = u_1(x,y) + Jv_1(x,y)$ and $g = u_2(x,y) + Jv_2(x,y)$ be J-regular. Then $D_J^* f = D_J^* g = 0$. From the rules of addition and product for $p = x + Jy \in \mathbb{S}$, we get

$$\begin{aligned} D_J^*(f \pm g) &= \frac{1}{2}\left(\frac{\partial}{\partial x} - J\frac{\partial}{\partial y}\right)\left\{(u_1 \pm u_2) + J(v_1 \pm v_2)\right\} \\ &= \frac{1}{2}\left\{\left(\frac{\partial u_1}{\partial x} - \frac{\partial v_1}{\partial y}\right) \pm \left(\frac{\partial u_2}{\partial x} - \frac{\partial v_2}{\partial y}\right)\right\} \\ &\quad + \frac{1}{2}J\left\{\left(-\frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x}\right) \mp \left(\frac{\partial u_2}{\partial y} - \frac{\partial v_2}{\partial x}\right)\right\} \end{aligned}$$

and

$$
\begin{aligned}
D_J^*(fg) &= \frac{1}{2}\left(\frac{\partial}{\partial x} - J\frac{\partial}{\partial y}\right)\bigl\{(u_1u_2) + (v_1v_2) + J(u_1v_2 + v_1u_2)\bigr\} \\
&= \frac{1}{2}\left\{\left(\frac{\partial u_1}{\partial x} - \frac{\partial v_1}{\partial y}\right)u_2 + u_1\left(\frac{\partial u_2}{\partial x} - \frac{\partial v_2}{\partial y}\right) + \left(\frac{\partial v_1}{\partial x} - \frac{\partial u_1}{\partial y}\right)v_2\right. \\
&\quad \left. + v_1\left(\frac{\partial v_2}{\partial x} - v_1\frac{\partial u_2}{\partial y}\right)\right\} \\
&\quad + \frac{1}{2}J\left\{\left(-\frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x}\right)u_2 - u_1\left(\frac{\partial u_2}{\partial y} - \frac{\partial v_2}{\partial x}\right) - \left(\frac{\partial v_1}{\partial y} - \frac{\partial u_1}{\partial x}\right)v_2\right. \\
&\quad \left. - v_1\left(\frac{\partial v_2}{\partial y} - \frac{\partial u_2}{\partial x}\right)\right\}
\end{aligned}
$$

respectively. Since u_λ and v_λ ($\lambda = 1,2$) are real-valued functions, by using equation (3.3), we obtain $D_J^*(f \pm g) = 0$ and $D_J^*(fg) = 0$. Therefore, the sum and product of two J-regular functions are J-regular. □

Proposition 3.4 *If a function is J-regular and non-zero, then its algebraic inverse is J-regular.*

Proof Let $f = u(x,y) + Jv(x,y)$ be J-regular. Then its algebraic inverse, denoted by $(f)^{-1}$, is

$$
(f)^{-1} = \frac{f^*}{\mathcal{N}(f)} = \frac{u - Jv}{u^2 - v^2} \quad \bigl(u^2 \neq v^2\bigr).
$$

Since u and v are real-valued functions, we have

$$
\begin{aligned}
D_J^*\bigl((f)^{-1}\bigr) &= \frac{1}{2}\left(\frac{\partial}{\partial x} - J\frac{\partial}{\partial y}\right)\left\{\frac{u - Jv}{u^2 - v^2}\right\} \\
&= \frac{1}{2(u^2 - v^2)^2}\left\{\left(\frac{\partial u}{\partial x} - J\frac{\partial v}{\partial x}\right)\bigl(u^2 - v^2\bigr) - (u - Jv)\left(2u\frac{\partial u}{\partial x} - 2v\frac{\partial v}{\partial x}\right)\right. \\
&\quad \left. - J\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right)\bigl(u^2 - v^2\bigr) + J(u - Jv)\left(2u\frac{\partial u}{\partial y} - 2v\frac{\partial v}{\partial y}\right)\right\} \\
&= \frac{1}{2(u^2 - v^2)^2}\left\{(u - Jv)^2\left(-\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + J(u - Jv)^2\left(-\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right\}.
\end{aligned}
$$

From equation (3.3), we obtain $D_J^*((f)^{-1}) = 0$. Therefore, the function $(f)^{-1}$ is J-regular. □

Definition 3.5 *Let Ω be an open set in $\mathbb{S}$. Let a function $f : \Omega \to \mathbb{S}$ be differentiable. Then $D_J f$ is said to be a J-derivative of f in $\mathbb{S}$ by*

$$
D_J f(x + Jy) = \frac{1}{2}\left(\frac{\partial}{\partial x} + J\frac{\partial}{\partial y}\right)f_J(x + Jy).
$$

4 Hyperholomorphy in the coordinate system

The coordinate system used by the variables (t, r, θ, φ) can be represented by hyperbolic coordinates

$$
J = (\cosh\theta \sinh\varphi, \sinh\theta \sinh\varphi, \cosh\varphi).
$$

For a split-quaternion $p = t + Jr$, that is, $p = t + ir\cosh\theta\sinh\varphi + jr\sinh\theta\sinh\varphi + kr\cosh\varphi \in \mathbb{S}$, a function $f : \Omega \to \mathbb{S}$ is written as follows:

$$f(p) = u(t, r, \theta, \varphi) + Jv(t, r, \theta, \varphi).$$

We let J_θ and J_φ be the derivatives of J with respect to θ and φ, respectively, such that

$$J_\theta = i\sinh\theta\sinh\varphi + j\cosh\theta\sinh\varphi,$$
$$J_\varphi = i\cosh\theta\cosh\varphi + j\sinh\theta\cosh\varphi + k\sinh\varphi,$$
$$J_\theta^{-1} = i(\sinh\varphi)^{-1}\sinh\theta + j(\sinh\varphi)^{-1}\cosh\theta$$

and

$$J_\varphi^{-1} = -i\cosh\theta\cosh\varphi - j\sinh\theta\cosh\varphi - k\sinh\varphi,$$

where J_θ^{-1} and J_φ^{-1} are algebraic inverse elements of J_θ and J_φ, respectively, that is, the equations

$$J_\theta J_\theta^{-1} = J_\theta^{-1} J_\theta = 1$$

and

$$J_\varphi J_\varphi^{-1} = J_\varphi^{-1} J_\varphi = 1$$

hold.

Lemma 4.1 *The left-differential operator* (3.1) *in this coordinate system has the form*

$$D_l^* = \frac{\partial}{\partial t} - \frac{1}{2}\left(J\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial J}\right),$$

where

$$\frac{\partial}{\partial J} = (J_\theta)^{-1}\frac{\partial}{\partial\theta} + (J_\varphi)^{-1}\frac{\partial}{\partial\varphi}.$$

Proof From the representation of J, we have $Jr = ix_1 + jx_2 + kx_3$. Then

$$\frac{\partial}{\partial r} = (-Ji)\frac{\partial}{\partial x_1} + (Jj)\frac{\partial}{\partial x_2} + (Jk)\frac{\partial}{\partial x_3} = (-J)\left(i\frac{\partial}{\partial x_1} - j\frac{\partial}{\partial x_2} - k\frac{\partial}{\partial x_3}\right),$$
$$\frac{\partial}{\partial J} = (-ri)\frac{\partial}{\partial x_1} + (rj)\frac{\partial}{\partial x_2} + (rk)\frac{\partial}{\partial x_3} = (-r)\left(i\frac{\partial}{\partial x_1} - j\frac{\partial}{\partial x_2} - k\frac{\partial}{\partial x_3}\right)$$

and

$$\frac{\partial}{\partial\theta} = (J_\theta)\frac{\partial}{\partial J}, \qquad \frac{\partial}{\partial\varphi} = (J_\varphi)\frac{\partial}{\partial J}.$$

Thus, we obtain

$$i\frac{\partial}{\partial x_1}-j\frac{\partial}{\partial x_2}-k\frac{\partial}{\partial x_3}=\frac{1}{2}\left(-J\frac{\partial}{\partial r}+\frac{1}{-r}\frac{\partial}{\partial J}\right). \qquad \square$$

This coordinate system can be used to show that a function is hyperholomorphic. Let $p = t + Jr \in \mathbb{S}$ $(t, r \in \mathbb{R})$ and $f(p) = u(t, r, \theta, \varphi) + Jv(t, r, \theta, \varphi)$, where u and v are real-valued functions.

Definition 4.2 *Let Ω be an open set. Then a function $f = u(t, r, \theta, \varphi) + Jv(t, r, \theta, \varphi)$ is said to be hyperholomorphic if $u, v \in \mathcal{C}^1(\Omega)$ and f satisfies $D_l^* f = 0$.*

Theorem 4.3 *A function $f(p) = u(t, r, \theta, \varphi) + Jv(t, r, \theta, \varphi)$ is hyperholomorphic if and only if u and v satisfy the following equations*:

$$\begin{cases} \frac{\partial u}{\partial t}-\frac{1}{2}\frac{\partial v}{\partial r}=0,\\ \frac{\partial v}{\partial t}-\frac{1}{2}\frac{\partial u}{\partial r}=0,\\ \frac{\partial u}{\partial J}+\frac{\partial (Jv)}{\partial J}=0. \end{cases} \tag{4.1}$$

More precisely,

$$\begin{cases} \frac{\partial u}{\partial t}-\frac{1}{2}\frac{\partial v}{\partial r}=0,\\ \frac{\partial v}{\partial t}-\frac{1}{2}\frac{\partial u}{\partial r}=0,\\ \begin{cases} (\sinh\varphi)^{-1}\frac{\partial u}{\partial \theta}+\frac{\partial v}{\partial \varphi}=0 \quad (or\ \frac{\partial u}{\partial \theta}+\sinh\varphi\frac{\partial v}{\partial \varphi}=0),\\ (\sinh\varphi)^{-1}\frac{\partial v}{\partial \theta}+\frac{\partial u}{\partial \varphi}=0 \quad (or\ \frac{\partial v}{\partial \theta}+\sinh\varphi\frac{\partial u}{\partial \varphi}=0). \end{cases} \end{cases}$$

Proof From the definition of a hyperholomorphic function in split-quaternions, we have

$$\begin{aligned} D_l^* f &= \left(\frac{\partial}{\partial t}-\frac{1}{2}J\frac{\partial}{\partial r}-\frac{1}{2r}\frac{\partial}{\partial J}\right)(u+Jv)\\ &= \frac{\partial u}{\partial t}-\frac{1}{2}J\frac{\partial u}{\partial r}-\frac{1}{2r}\frac{\partial u}{\partial J}+J\frac{\partial v}{\partial t}-\frac{1}{2}\frac{\partial v}{\partial r}-\frac{1}{2r}\frac{\partial (Jv)}{\partial J}\\ &= \left(\frac{\partial u}{\partial t}-\frac{1}{2}\frac{\partial v}{\partial r}\right)+J\left(\frac{\partial v}{\partial t}-\frac{1}{2}\frac{\partial u}{\partial r}\right)-\frac{1}{2r}\left(\frac{\partial u}{\partial J}+\frac{\partial (Jv)}{\partial J}\right). \end{aligned}$$

Since $D_l^* f = 0$, we obtain equation (4.1). Specially, the equation

$$\frac{\partial u}{\partial J}+\frac{\partial (Jv)}{\partial J}=0$$

can be more specifically written as follows.

Since we have

$$\begin{aligned} (J_\theta)^{-1}J &= (\sinh\varphi)^{-1}(i\sinh\theta+j\cosh\theta)\\ &\quad\times(i\cosh\theta\sinh\varphi+j\sinh\theta\sinh\varphi+k\cosh\varphi)\\ &= (\sinh\varphi)^{-1}(-i\cosh\theta\cosh\varphi-j\sinh\theta\cosh\varphi-k\sinh\varphi)\\ &= (\sinh\varphi)^{-1}(J_\varphi)^{-1} \end{aligned}$$

and

$$\begin{aligned}
(J_\varphi)^{-1}J &= (-i\cosh\theta\cosh\varphi - j\sinh\theta\cosh\varphi - k\sinh\varphi) \\
&\quad \times (i\cosh\theta\sinh\varphi + j\sinh\theta\sinh\varphi + k\cosh\varphi) \\
&= \left(\cosh^2\varphi - \sinh^2\varphi\right)(i\sinh\theta + j\cosh\theta) \\
&= \sinh\varphi (J_\theta)^{-1},
\end{aligned}$$

we get

$$\begin{aligned}
\frac{\partial u}{\partial J} + \frac{\partial (Jv)}{\partial J} &= (J_\theta)^{-1}\frac{\partial u}{\partial \theta} + (J_\varphi)^{-1}\frac{\partial u}{\partial \varphi} + (J_\theta)^{-1}J\frac{\partial v}{\partial \theta} + (J_\varphi)^{-1}J\frac{\partial v}{\partial \varphi} \\
&= (\sinh\varphi)^{-1}(J_\varphi)^{-1}J\frac{\partial u}{\partial \theta} + (J_\varphi)^{-1}\frac{\partial u}{\partial \varphi} \\
&\quad + (\sinh\varphi)^{-1}(J_\varphi)^{-1}\frac{\partial v}{\partial \theta} + (J_\varphi)^{-1}J\frac{\partial v}{\partial \varphi} \\
&= (J_\varphi)^{-1}J\left((\sinh\varphi)^{-1}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial \varphi}\right) \\
&\quad + (J_\varphi)^{-1}\left(\frac{\partial u}{\partial \varphi} + (\sinh\varphi)^{-1}\frac{\partial v}{\partial \theta}\right) \\
&= 0.
\end{aligned}$$

Thus, we obtain

$$\begin{cases} (\sinh\varphi)^{-1}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial \varphi} = 0, \\ (\sinh\varphi)^{-1}\frac{\partial v}{\partial \theta} + \frac{\partial u}{\partial \varphi} = 0. \end{cases}$$

□

Proposition 4.4 *The sum and product of two hyperholomorphic functions are hyperholomorphic.*

Proof Let $f = u_1(t,r,\theta,\varphi) + Jv_1(t,r,\theta,\varphi)$ and $g = u_2(t,r,\theta,\varphi) + J(t,r,\theta,\varphi)$ be hyperholomorphic. Then $D_l^* f = D_l^* g = 0$. From the rules of addition and product for $p = t + Jr \in \mathbb{S}$ with the coordinate system, we get

$$\begin{aligned}
D_l^*(f \pm g) &= \left(\frac{\partial}{\partial t} - \frac{1}{2}J\frac{\partial}{\partial r} - \frac{1}{2r}\frac{\partial}{\partial J}\right)\left\{(u_1 \pm u_2) + J(v_1 \pm v_2)\right\} \\
&= \left(\frac{\partial}{\partial t} - \frac{1}{2}J\frac{\partial}{\partial r}\right)\left\{(u_1 \pm u_2) + J(v_1 \pm v_2)\right\} - \frac{1}{r}\frac{\partial}{\partial J}\left\{(u_1 \pm u_2) + J(v_1 \pm v_2)\right\} \\
&= -\frac{1}{2r}\left(\frac{\partial u_1}{\partial J} \pm \frac{\partial u_2}{\partial J} + \frac{\partial (Jv_1)}{\partial J} \pm \frac{\partial (Jv_2)}{\partial J}\right)
\end{aligned}$$

and

$$\begin{aligned}
D_l^*(fg) &= \left(\frac{\partial}{\partial t} - \frac{1}{2}J\frac{\partial}{\partial r} - \frac{1}{2r}\frac{\partial}{\partial J}\right)\left\{(u_1u_2) + (v_1v_2) + J(u_1v_2 + v_1u_2)\right\} \\
&= \left(\frac{\partial}{\partial t} - \frac{1}{2}J\frac{\partial}{\partial r}\right)\left\{(u_1u_2) + (v_1v_2) + J(u_1v_2 + v_1u_2)\right\}
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2r}\frac{\partial}{\partial J}\bigl\{(u_1u_2)+(v_1v_2)+J(u_1v_2+v_1u_2)\bigr\}\\
&=-\frac{1}{2r}\biggl((J_\theta)^{-1}\frac{\partial}{\partial\theta}\bigl\{(u_1u_2)+(v_1v_2)\bigr\}+(J_\varphi)^{-1}\frac{\partial}{\partial\varphi}\bigl\{(u_1u_2)+(v_1v_2)\bigr\}\\
&\quad+(\sinh\varphi)^{-1}(J_\varphi)^{-1}\frac{\partial}{\partial\theta}\bigl\{(u_1v_2)+(v_1u_2)\bigr\}\\
&\quad+(\sinh\varphi)(J_\theta)^{-1}\frac{\partial}{\partial\varphi}\bigl\{(u_1v_2)+(v_1u_2)\bigr\}\biggr)\\
&=-\frac{1}{2r}(J_\theta)^{-1}\biggl\{u_2\biggl(\frac{\partial u_1}{\partial\theta}+\sinh\varphi\frac{\partial v_1}{\partial\varphi}\biggr)+u_1\biggl(\frac{\partial u_2}{\partial\theta}+\sinh\varphi\frac{\partial v_2}{\partial\varphi}\biggr)\\
&\quad+v_2\biggl(\frac{\partial v_1}{\partial\theta}+\sinh\varphi\frac{\partial u_1}{\partial\varphi}\biggr)+v_1\biggl(\frac{\partial v_2}{\partial\theta}+\sinh\varphi\frac{\partial u_2}{\partial\varphi}\biggr)\biggr\}\\
&\quad-\frac{1}{2r}(J_\varphi)^{-1}\biggl\{u_2\biggl(\frac{\partial u_1}{\partial\varphi}+(\sinh\varphi)^{-1}\frac{\partial v_1}{\partial\theta}\biggr)\\
&\quad+u_1\biggl(\frac{\partial u_2}{\partial\varphi}+(\sinh\varphi)^{-1}\frac{\partial v_2}{\partial\theta}\biggr)+v_2\biggl(\frac{\partial v_1}{\partial\varphi}+(\sinh\varphi)^{-1}\frac{\partial u_1}{\partial\theta}\biggr)\\
&\quad+v_1\biggl(\frac{\partial v_2}{\partial\varphi}+(\sinh\varphi)^{-1}\frac{\partial u_2}{\partial\theta}\biggr)\biggr\},
\end{aligned}
$$

respectively. Since u_λ and v_λ ($\lambda=1,2$) are real-valued functions, by using equation (4.1), we obtain $D_l^*(f\pm g)=0$ and $D_l^*(fg)=0$. Therefore, the sum and product of two hyperholomorphic functions are hyperholomorphic. □

Proposition 4.5 *If a function is hyperholomorphic and non-zero, then its algebraic inverse is hyperholomorphic.*

Proof Let $f=u(t,r,\theta,\varphi)+Jv(t,r,\theta,\varphi)$ be hyperholomorphic. Then its algebraic inverse

$$
(f)^{-1}=\frac{f^*}{\mathcal{N}(f)}=\frac{u-Jv}{u^2-v^2}\qquad\bigl(u^2\neq v^2\bigr),
$$

where u and v are real-valued functions, satisfies

$$
\begin{aligned}
D_l^*\bigl((f)^{-1}\bigr)&=\biggl(\frac{\partial}{\partial t}-\frac{1}{2}J\frac{\partial}{\partial r}-\frac{1}{2r}\frac{\partial}{\partial J}\biggr)\frac{u-Jv}{u^2-v^2}\\
&=\biggl(\frac{\partial}{\partial t}-\frac{1}{2}J\frac{\partial}{\partial r}\biggr)\frac{u-Jv}{u^2-v^2}-\frac{1}{2r}\frac{\partial}{\partial J}\frac{u-Jv}{u^2-v^2}\\
&=-\frac{1}{2r}\biggl\{\frac{\partial}{\partial J}\biggl(\frac{u}{u^2-v^2}\biggr)-\frac{\partial}{\partial J}\biggl(\frac{Jv}{u^2-v^2}\biggr)\biggr\}\\
&=-\frac{1}{2r}(J_\theta)^{-1}\biggl\{\frac{\partial u}{\partial\theta}\bigl(u^2-v^2\bigr)-2u^2\frac{\partial u}{\partial\theta}+2uv\frac{\partial v}{\partial\theta}-\sinh\varphi\frac{\partial v}{\partial\varphi}u^2\\
&\quad-\sinh\varphi\frac{\partial v}{\partial\varphi}v^2+\sinh\varphi\frac{\partial u}{\partial\varphi}2uv\biggr\}\frac{1}{(u^2-v^2)^2}\\
&\quad-\frac{1}{2r}(J_\varphi)^{-1}\biggl\{\frac{\partial u}{\partial\varphi}\bigl(u^2-v^2\bigr)-2u^2\frac{\partial u}{\partial\varphi}+2uv\frac{\partial v}{\partial\varphi}-(\sinh\varphi)^{-1}\frac{\partial v}{\partial\theta}u^2\\
&\quad-(\sinh\varphi)^{-1}\frac{\partial v}{\partial\theta}v^2+(\sinh\varphi)^{-1}\frac{\partial u}{\partial\theta}2uv\biggr\}\frac{1}{(u^2-v^2)^2}
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{2r}(J_\theta)^{-1}\left\{\left(-u^2\right)\left(\frac{\partial u}{\partial\theta} + (\sinh\varphi)\frac{\partial v}{\partial\varphi}\right)\right. \\
&\quad \left. + \left(-v^2\right)\left(\frac{\partial u}{\partial\theta} + (\sinh\varphi)\frac{\partial v}{\partial\varphi}\right) + (2uv)\left(\frac{\partial v}{\partial\theta} + (\sinh\varphi)\frac{\partial u}{\partial\varphi}\right)\right\} \\
&\quad - \frac{1}{2r}(J_\varphi)^{-1}\left\{\left(-u^2\right)\left(\frac{\partial u}{\partial\varphi} + (\sinh\varphi)^{-1}\frac{\partial v}{\partial\theta}\right)\right. \\
&\quad + \left(-v^2\right)\left(\frac{\partial u}{\partial\varphi} + (\sinh\varphi)^{-1}\frac{\partial v}{\partial\theta}\right) \\
&\quad \left. + (2uv)\left(\frac{\partial v}{\partial\varphi} + (\sinh\varphi)^{-1}\frac{\partial u}{\partial\theta}\right)\right\}\frac{1}{(u^2 - v^2)^2}.
\end{aligned}
$$

From equation (4.1), we obtain $D_l^*((f)^{-1}) = 0$. Therefore, the function $(f)^{-1}$ is hyperholomorphic. □

Lemma 4.6 *Let Ω be an open set in $\mathbb{S}$. For $p \in \Omega$, if a split-quaternionic function $f(x + Jy) = u(x,y) + Jv(x,y)$ is hyperholomorphic, then f satisfies*

$$
D_l^* f = \frac{-v}{r}. \tag{4.2}
$$

Proof Since $u(x,y)$ and $v(x,y)$ are functions with respect to x and y, the calculation of $D_l^* f$ is

$$
\begin{aligned}
D_l^* f &= \left(\frac{\partial f}{\partial t} - \frac{1}{2}J\frac{\partial f}{\partial r}\right) - \frac{1}{2r}\frac{\partial u}{\partial J} - \frac{1}{2r}\frac{\partial (Jv)}{\partial J} \\
&= -\frac{1}{2r}\left((J_\theta)^{-1}\frac{\partial (J)}{\partial\theta}v + (J_\varphi)^{-1}\frac{\partial (J)}{\partial\varphi}v\right) \\
&= -\frac{v}{r}.
\end{aligned}
$$

Therefore, f satisfies equation (4.2). □

Theorem 4.7 *For $p \in \Omega_J$, if a split-quaternionic function $f(x + Jy) = u(x,y) + Jv(x,y)$ is hyperholomorphic, then f satisfies*

$$
D_l^*\left(\frac{f}{r^n}\right) = \frac{1}{r^n}D_l^* f - nJ\frac{f}{r^{n+1}}.
$$

Proof Since $u(x,y)$ and $v(x,y)$ are functions with respect to x and y, the calculation of $D_l^*(\frac{f}{r^n})$ is

$$
\begin{aligned}
D_l^*\left(\frac{f}{r^n}\right) &= \left\{\frac{\partial}{\partial t}\left(\frac{f}{r^n}\right) - \frac{1}{2}J\frac{\partial}{\partial r}\left(\frac{f}{r^n}\right)\right\} - \frac{1}{2r}\frac{\partial}{\partial J}\left(\frac{f}{r^n}\right) \\
&= \frac{\partial}{\partial t}\left(\frac{f}{r^n}\right) - \frac{1}{2}J\frac{1}{r^{n+1}}\left\{\frac{\partial u}{\partial r}r - un + J\left(\frac{\partial v}{\partial r}r - vn\right)\right\} - \frac{1}{r^{n+1}}v \\
&= \frac{1}{r^n}\left(\frac{\partial u}{\partial t} - \frac{1}{2}\frac{\partial v}{\partial r}\right) + J\frac{1}{r^n}\left(\frac{\partial v}{\partial t} - \frac{1}{2}\frac{\partial u}{\partial r}\right) + \frac{n}{2}J\frac{u + Jv}{r^{n+1}} - \frac{1}{r^{n+1}}v \\
&= \frac{n}{2}J\frac{f}{r^{n+1}} + \frac{1}{r^n}D_l^* f.
\end{aligned}
$$

Therefore, we obtain the result. □

By referring to [14, 18] and observing Figures 1-4, we consider the corresponding Gauss theorem in four dimensions for the components of f in $\mathbb{S}$. Specially, if the electric field is known, the Gauss theorem can be of help to find the division of electric charge which is inferred by integrating the electric field. For some symmetry, like cylindrical symmetry, planar symmetry, and spherical symmetry, the electric field passes through the surfaces. As shown in Figure 4, the Gauss theorem can be defined in the class containing the split-quaternionic variables with the cylindrical symmetry as a constituent. From Lemma 4.6 and Theorem 4.7, the Gauss theorem is described as follows.

Theorem 4.8 *Let $f = u(x,y) + J(x,y)$ be a hyperholomorphic function, and let K be any smooth and simple closed hypersurface on $\mathcal{T}$ in $\mathbb{S}$, disjoint from the real axis, K^* being the interior of K. Let $n(p) = n_0 + n_1 i + n_2 j + n_3 k$, where (n_0, n_1, n_2, n_3) is the unit outer normal to K at p. Then*

$$\int_K n(p) f(p) \frac{1}{r^n}\, dS_K = -\frac{n}{2} \int_{K^*} J \frac{u}{r^{n+1}}\, dV,$$

where dS_k is the element of surface area on K.

Proof Suppose that f is hyperholomorphic and $n(p) = n_0 + n_1 i + n_2 j + n_3 k$, where (n_0, n_1, n_2, n_3) is the unit outer normal to K at p (such as Figure 4). From Theorem 4.7, we have

$$\begin{aligned}
\int_K n(p) f(p) \frac{1}{r^n}\, dS_K &= \int_K \frac{1}{r^n} (n_0 f + i n_1 f + j n_2 f + k n_3 f)\, dS_K \\
&= \int_{K^*} D_l \left(\frac{f}{r^n} \right) dV = \int_{K^*} \left(\frac{1}{r^n} D_l f - \frac{n}{2} J \frac{f}{r^{n+1}} \right) dV \\
&= \int_{K^*} \left(-\frac{n}{2} J \frac{u}{r^{n+1}} + \frac{-n}{2r^n} \frac{-v}{r} \right) dV \\
&= \int_{K^*} \left(-\frac{n}{2} J \frac{u}{r^{n+1}} \right) dV.
\end{aligned}$$

Thus, we can obtain the result. □

Competing interests
The author declares that they have no competing interests.

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