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A graph-theoretic approach to global input-to-state stability for coupled control systems

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Abstract

In this paper, the input-to-state stability for coupled control systems is investigated. A systematic method of constructing a global Lyapunov function for the coupled control systems is provided by combining graph theory and the Lyapunov method. Consequently, some novel global input-to-state stability principles are given. As an application to this result, a coupled Lurie system is also discussed. By constructing an appropriate Lyapunov function, a sufficient condition ensuring input-to-state stability of this coupled Lurie system is established. Two examples are provided to demonstrate the effectiveness of the theoretical results.

Keywords: input-to-state stability; coupled control system; Lyapunov function

1 Introduction

In recent years, coupled control systems (CCSs) have received considerable attention for their interesting characteristics from the mathematical point of view. The main interest has been focused on the investigation of the global dynamics of the systems, with a special emphasis on the study of stability. Meanwhile, input-to-state stability (ISS) for control systems has been extensively studied due to a wide range of applications in physics, biology, social science, neural networks, engineering fields, and artificial complex dynamical systems. For example, Sontag and Wang [1] showed the importance of the well-known Lyapunov sufficient condition for ISS and provided additional characterizations of the ISS property, including one in terms of nonlinear stability margins. Grüne [2] presented a new variant of the ISS property which is based on a one-dimensional dynamical system, showed the relation to the original ISS formulation, and described the characterizations by means of suitable Lyapunov functions. In [3], Angeli presented a framework for understanding such questions fully compatible with the well-known ISS approach and discussed applications of the newly introduced stability notions. In [4], Arcak and Teel analyzed ISS for the feedback interconnection of a linear block and a nonlinear element.

As far as we know, there are a lot of papers dealing with the ISS of individual control systems but few papers dealing with the ISS of CCSs. In general, the study of ISS for CCSs is complex, because it is very difficult to straightly construct an appropriate Lyapunov function for CCSs. However, in [5], Li and Shuai studied the global-stability problem of equilibrium and developed a systematic approach that allows one to construct global Lyapunov functions for large-scale coupled systems from building blocks of individual vertex systems. Later, this technique was appropriately developed and extended to some other coupled systems. In [6–8] several delayed coupled systems were discussed, and some sufficient conditions were obtained. Li et al. in [9–12] investigated the stochastic stability of coupled systems with both white noise and color noise. Moreover, by using this technique, Su et al. derived sufficient conditions ensuring global stability of discrete-time coupled systems [13, 14], and Zhang et al. extended this technique to multi-dispersal coupled systems [15]. Besides, this technique is also applied to many practical applications, such as biological systems [16–18], neural networks [19, 20], and mechanical systems [20–23]. Hence, the graph theory is a great method in the study of coupled systems.

Motivated by the above discussions, in this paper, we investigate the ISS of CCSs. A systematic method of constructing a global Lyapunov function for the CCSs is provided by combining graph theory and the Lyapunov method. Consequently, some novel global stability principles are given. As an application to this result, a coupled Lurie system is also discussed. By constructing an appropriate Lyapunov function, a sufficient condition ensuring the ISS of this coupled Lurie system is established. Finally, two examples and their numerical simulations are provided to demonstrate the effectiveness and correctness of the theoretical results.

The rest of the paper is organized as follows. In Section 2, some preliminaries and the problem description are presented. In Section 3, the main theorems and their rigorous proofs are described. Finally, in Section 4, an application to a coupled Lurie system is given, and the respective simulations are also given to demonstrate the effectiveness of our results.

2 Preliminaries and model formulation

Throughout the paper, unless otherwise specified, the following notations will be used. As we usually use, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. Notations $\mathbb{R}^1_+ = [0, +\infty)$, $\mathbb{Z}^+ = \{1, 2, \ldots\}$, $\mathbb{L} = \{1, 2, \ldots, l\}$, $n = \sum_{i=1}^{l} n_i$, and $m = \sum_{i=1}^{l} m_i$ for $n_i, m_i \in \mathbb{Z}^+$ are used. For any $x \in \mathbb{R}^n$, x^{T} is its transpose and $|x|$ is its Euclidean norm. Let $\mathbb{R}^{n\times n}$ denote the set of $n \times n$ real matrix space. For a matrix P, $P \geq 0$ (≤ 0) means that P is positive semi-definite (negative semi-definite). The symbol $\psi_1 \circ \psi_2$ stands for the composition of two functions ψ_1 and ψ_2. The gradient function of a function f is indicated by $\bigtriangledown f$. In an m-dimensional space, the symbol L^m_∞ indicates the set of all the functions which are endowed with essential supremum norm $\|u\| = \sup\{|u(t)| \mid t \geq 0\} \leq \infty$.

We recall some knowledge of graph theory that will be used in the rest of the paper. Define a weighted digraph $\mathcal{G} = \{V, E, A\}$, in which set $V = \{v_1, v_2, \ldots, v_l\}$ denotes l vertices of the graph, element e_{ij} of E denotes the arc leading from initial vertex j to terminal vertex i, and the element a_{ij} of a weighted adjacency matrix A denotes the weight of arc e_{ij}. We denote $a_{ij} > 0$ if and only if there exists an arc from vertex i to vertex j in $\mathcal{G}$, otherwise $a_{ij} = 0$, and we denote $a_{ii} = 0$ for all $i \in \mathbb{L}$. Denote the digraph with weight matrix A as $(\mathcal{G}, A)$. If a graph $\mathcal{S}$ has the same vertex as $\mathcal{G}$, we call it a subgraph of $\mathcal{G}$. The weight $W(\mathcal{S})$ of a subgraph $\mathcal{S}$ is the product of the weights on all its arcs. If a connected subgraph has no cycle, it is a tree. We call v_i the root of the tree if vertex i of the tree is not a terminal vertex of any arcs and each of the remaining vertices is a terminal vertex of one arc. A subgraph Q is unicyclic when it is a disjoint union of rooted trees whose roots form a directed cycle. The

Laplacian matrix of $\mathcal{G}$ is defined as $L = (b_{ij})_{l\times l}$, where $b_{ij} = -a_{ij}$ for $i \neq j$ and $b_{ij} = \sum_{k\neq i} a_{ik}$ for $i = j$.

The following lemma will be used in the proof of our main results.

Lemma 1 ([5]) *Assume $l \geq 2$. Then the following identity holds*:

$$\sum_{i,j=1}^{l} c_i a_{ij} F_{ij}(x_i, x_j) = \sum_{\mathcal{Q}\in\mathbb{Q}} W(\mathcal{Q}) \sum_{(s,r)\in E(C_{\mathcal{Q}})} F_{rs}(x_r, x_s).$$

Here $F_{ij}(x_i, x_j)$ are arbitrary functions for any $1 \leq i, j \leq l$, a_{ij} are elements of matrix A, $\mathbb{Q}$ is the set of all spanning unicyclic graphs of $(\mathcal{G}, A)$, $W(\mathcal{Q})$ is the weight of $\mathcal{Q}$, and $C_{\mathcal{Q}}$ denotes the directed cycle of $\mathcal{Q}$. And c_i denotes the cofactor of the ith diagonal element of L, in particular, if $(\mathcal{G}, A)$ is strongly connected, then $c_i > 0$ for $1 \leq i \leq l$.

In the remainder of this section, we shall give the model formulation and state some definitions that will be used in the main results.

Given a digraph $(\mathcal{G}, A)$ with l vertices ($l \geq 2$) and $A = (a_{ij})_{l\times l}$. A coupled control system can be constructed on $(\mathcal{G}, A)$ by assigning each vertex its own dynamics and then coupling these vertex dynamics based on directed arcs in $(\mathcal{G}, A)$. The details are as follows. Assume that the ith vertex dynamic is described by the control system

$$\dot{x}_i(t) = f_i\bigl(x_i(t), u_i\bigr), \quad t \geq 0,$$

where $x_i \in \mathbb{R}^{n_i}$ denotes the value of vertex i, $f_i : \mathbb{R}^{n_i} \times \mathbb{R}^{m_i} \to \mathbb{R}^{n_i}$ is continuously differentiable and satisfies $f_i(0,0) = 0$, function $u_i : \mathbb{R}^1_+ \to \mathbb{R}^{m_i}$ denotes the input of vertex i and it is measurable and locally essentially bounded. Assume that $a_{ij} \geq 0$ represents the effect factor from vertex j to vertex i and $a_{ij} = 0$ iff there exists no arc from j to i. Then we use function P_{ij} to describe the effect that subsystem j has on i and $P_{ij} : \mathbb{R}^{n_i} \times \mathbb{R}^{n_j} \times \mathbb{R}^{m_j} \to \mathbb{R}^{n_i}$ is continuously differentiable and satisfies $P_{ij}(0,0,0) = 0$. For example, in a digraph with six vertices, we show the interaction in vertex j and vertex i (see Figure 1).

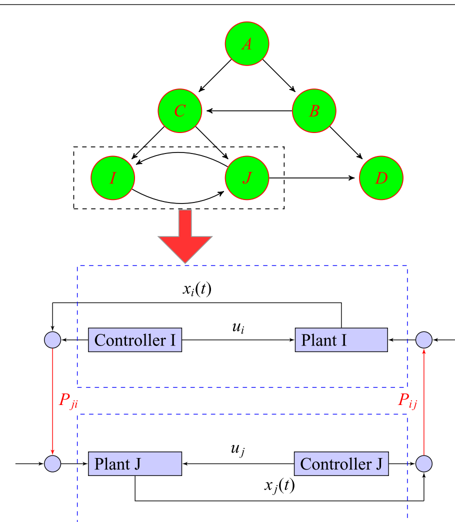


Figure 1 A coupled control system on a digraph with six vertices.

Then coupling the vertex systems together, we obtain the following coupled control system:

$$\dot{x}_i(t) = f_i\big(x_i(t), u_i\big) + \sum_{j=1}^{l} a_{ij} P_{ij}\big(x_i(t), x_j(t), u_j\big), \quad i \in \mathbb{L}, t \geq 0. \tag{1}$$

Here we use $x = (x_1^{\mathrm{T}}, x_2^{\mathrm{T}}, \ldots, x_l^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^n$ to stand for the vector of state variables of (1), and denote by $x(t) = x(t, x_0, u)$ the solution of CCS (1) with initial state $x_0 = x(0)$ and input $u = (u_1^{\mathrm{T}}, u_2^{\mathrm{T}}, \ldots, u_l^{\mathrm{T}})^{\mathrm{T}} \in L_\infty^m$.

To be more precise, we recall some definitions on the ISS of CCS (1). We refer to [1, 2] for definitions as follows.

Definition 1 A function $\gamma : \mathbb{R}_+^1 \to \mathbb{R}_+^1$ is a $\mathcal{K}$-function if it is continuous, strictly increasing, and $\gamma(0) = 0$. If a $\mathcal{K}$-function satisfies $\gamma(s) \to \infty$ as $s \to \infty$, we call it $\mathcal{K}_\infty$-function. A function $\beta : \mathbb{R}_+^1 \times \mathbb{R}_+^1 \to \mathbb{R}_+^1$ is a $\mathcal{K}\phi$-function if the function $\beta(\cdot, t)$ is a $\mathcal{K}$-function for each fixed $t \geq 0$, and for each fixed $s \geq 0$, $\beta(s, t)$ is decreasing to zero as $t \to \infty$.

Definition 2 CCS (1) is called ISS if there exist a $\mathcal{K}\phi$-function $\beta : \mathbb{R}_+^1 \times \mathbb{R}_+^1 \to \mathbb{R}_+^1$ and a $\mathcal{K}$-function γ such that for each input $u \in L_\infty^m$ and $x_0 \in \mathbb{R}^n$, it holds that

$$\big|x(t, x_0, u)\big| \leq \beta\big(|x_0|, t\big) + \gamma\big(\|u\|\big). \tag{2}$$

In the proof of our main results, we need to find a global ISS-Lyapunov function for CCS (1). For the convenience of the proof, we now define vertex ISS-Lyapunov functions for CCS (1).

Definition 3 Set $\{V_i(x_i), i \in \mathbb{L}\}$ is called a vertex ISS-Lyapunov function set for CCS (1) if every $V_i(x_i)$ is smooth and satisfies the following conditions:

Q1. There exist positive constants α_i, δ_i, $p \geq 2$, such that

$$\alpha_i |x_i|^p \leq V_i(x_i) \leq \delta_i |x_i|^p, \quad x_i \in \mathbb{R}^{n_i}.$$

Q2. There exist constants $\xi_i, d_{ij} \geq 0$, functions $F_{ij}(x_i, x_j)$, and $\mathcal{K}$-function χ_i such that for any $x_i \in \mathbb{R}^{n_i}$ and $\mu_i \in \mathbb{R}^{m_i}$ satisfying $\sum_{i=1}^{l} c_i \delta_i |x_i|^p \geq \sum_{i=1}^{l} c_i \delta_i |\chi_i(|\mu_i|)|^p$, where $D = (d_{ij})_{l \times l}$ and c_i is the cofactor of the ith diagonal element of Laplacian matrix of $(\mathcal{G}, D)$. Then we have

$$\dot{V}_i\big(x_i(t)\big) \leq -\xi_i \big|x_i(t)\big|^p + \sum_{j=1}^{l} d_{ij} F_{ij}\big(x_i(t), x_j(t)\big),$$

in which

$$\dot{V}_i\big(x_i(t)\big) = \triangledown V_i\big(x_i(t)\big) \left[f_i\big(x_i(t), u_i\big) + \sum_{j=1}^{l} a_{ij} P_{ij}\big(x_i(t), x_j(t), u_j\big) \right].$$

Q3. Along each directed cycle C_Q of weighted digraph $(\mathcal{G}, D)$, there is

$$\sum_{(s,r) \in E(C_Q)} F_{rs}(x_r, x_s) \leq 0.$$

3 Main results

In this section, the ISS of CCS (1) will be investigated. The approaches used in the proof of the main results are motivated by [1, 5].

Theorem 1 *If CCS* (1) *admits a vertex ISS-Lyapunov function set* $\{V_i(x_i), i \in \mathbb{L}\}$, *and digraph* $(\mathcal{G}, D)$ *is strongly connected, then the solution of CCS* (1) *is ISS.*

Proof In order to prove the conclusion, we need to find a $\mathcal{K}$-function $\gamma(\cdot)$ and a $\mathcal{K}\phi$ function $\beta(\cdot,\cdot)$ satisfying

$$\bigl|x(t,x_0,u)\bigr| \leq \beta\bigl(|x_0|,t\bigr) + \gamma\bigl(\|u\|\bigr) \tag{3}$$

for $x_0 \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$.

Let

$$V(x) = \sum_{i=1}^{l} c_i V_i(x_i),$$

in which c_i is the cofactor of the ith diagonal element of Laplacian matrix of $(\mathcal{G}, D)$.

Consider a set: $S = \{\eta : V(\eta) \leq b\}$, where $b = \sum_{i=1}^{l} c_i\delta_i|\chi_i(|u_i|)|^p$. We can assert that if there exists $t_0 \geq 0$ making $x(t_0) \in S$, then $x(t) \in S$ for all $t \geq t_0$. Suppose that this is not true, then there exist $t > t_0$ and $\varepsilon > 0$ such that $V(x(t)) > b + \varepsilon$. We observe from condition Q1 that

$$\sum_{i=1}^{l} c_i\delta_i\bigl|\chi_i\bigl(|u_i|\bigr)\bigr|^p \leq V\bigl(x(t)\bigr) \leq \sum_{i=1}^{l} c_i\delta_i\bigl|x_i(t)\bigr|^p. \tag{4}$$

Let $\tau = \inf\{t \geq t_0 : V(x(t)) \geq b + \varepsilon\}$. From conditions Q2 and Q3, we can obtain

$$\begin{aligned}
\dot{V}\bigl(x(t)\bigr) &= \sum_{i=1}^{l} c_i \dot{V}_i\bigl(x_i(t)\bigr) \\
&\leq -\sum_{i=1}^{l} c_i\xi_i\bigl|x_i(t)\bigr|^p + \sum_{i,j=1}^{l} c_i d_{ij} F_{ij}\bigl(x_i(t), x_j(t)\bigr) \\
&= -\sum_{i=1}^{l} c_i\xi_i\bigl|x_i(t)\bigr|^p + \sum_{Q\in Q} W(Q) \sum_{(s,r)\in E(C_Q)} F_{rs}\bigl(x_r(t), x_s(t)\bigr) \\
&\leq -\sum_{i=1}^{l} c_i\xi_i\bigl|x_i(t)\bigr|^p \\
&\leq 0.
\end{aligned}$$

Therefore, $V(x(t)) \geq V(x(\tau))$ for some t in (t_0, τ). This contradicts the minimality of τ, and hence $x(t) \in S$ for all $t \geq t_0$.

Now let $t_1 = \inf\{t \geq 0; x(t) \in S\} \leq \infty$, then it follows from the above argument that

$$V\bigl(x(t)\bigr) \leq \sum_{i=1}^{l} c_i\delta_i\bigl|\chi_i\bigl(|u_i|\bigr)\bigr|^p, \quad t \geq t_1.$$

This implies that

$$\sum_{i=1}^{l} c_i\alpha_i\big|x_i(t)\big|^p \le V\big(x(t)\big) \le \sum_{i=1}^{l} c_i\delta_i\big|\chi_i\big(|u_i|\big)\big|^p.$$

Denote

$$\chi\big(\|u\|\big) = \left(\frac{1}{\delta}\sum_{i=1}^{l} c_i\delta_i\big|\chi_i\big(\|u\|\big)\big|^p\right)^{\frac{1}{p}},$$

in which $\delta > 0$ is a certain constant. For simplicity, we write

$$\alpha = \left\{\sum_{i=1}^{l}\alpha_i\right\}^{1-\frac{p}{2}}\Big(\min_{i\in\mathbb{L}}\{c_i\alpha_i\}\Big)^{\frac{p}{2}},$$

then it is easy to see from condition Q1 that for $t \ge t_1$,

$$\begin{aligned} V\big(x(t)\big) &= \sum_{i=1}^{l} c_i V_i\big(x_i(t)\big) \\ &\le \sum_{i=1}^{l} c_i\delta_i\big|\chi_i\big(\|u_i\|\big)\big|^p \\ &\le \sum_{i=1}^{l} c_i\delta_i\big|\chi_i\big(\|u\|\big)\big|^p \\ &= \delta\big|\chi\big(\|u\|\big)\big|^p \end{aligned}$$

and

$$\begin{aligned} V\big(x(t)\big) &= \sum_{i=1}^{l} c_i V_i\big(x_i(t)\big) \\ &\ge \sum_{i=1}^{l} c_i\alpha_i\big|x_i(t)\big|^p \\ &= \sum_{j=1}^{l} c_j\alpha_j\sum_{i=1}^{l}\left[\frac{c_i\alpha_i}{\sum_{k=1}^{l} c_k\alpha_k}\big|x_i(t)\big|^{\frac{p}{2}}\right] \\ &\ge \sum_{j=1}^{l} c_j\alpha_j\left[\sum_{i=1}^{l}\frac{c_i\alpha_i}{\sum_{k=1}^{l} c_k\alpha_k}\big|x_i(t)\big|^2\right]^{\frac{p}{2}} \\ &\ge \left\{\sum_{i=1}^{l}\alpha_i\right\}^{1-\frac{p}{2}}\Big(\min_{i\in\mathbb{L}}\{c_i\alpha_i\}\Big)^{\frac{p}{2}}\big|x(t)\big|^p. \end{aligned}$$

Hence $\alpha|x(t)|^p \le V(x(t)) \le \delta|\chi(\|u\|)|^p$.

Since the digraph is strongly connected, it implies that $c_i > 0$, and then $\alpha > 0$, $\delta > 0$. Thus if we let $\gamma(\|u\|) = (\delta/\alpha)^{\frac{1}{p}}\chi(\|u\|)$, we can obtain

$$\left|x(t)\right| \leq \gamma\left(\|u\|\right), \quad t \geq t_1. \tag{5}$$

For $t < t_1$, we have $x(t) \bar{\in} S$, which implies that $\sum_{i=1}^{l} c_i\delta_i|x_i(t)|^p \geq \sum_{i=1}^{l} c_i\delta_i|\chi_i(|\mu_i|)|^p$. Consequently, from condition Q2, we can derive

$$\begin{aligned}
\dot{V}\left(x(t)\right) &\leq -\sum_{i=1}^{l} c_i\xi_i\left|x_i(t)\right|^p + \sum_{i,j=1}^{l} c_i d_{ij}F_{ij}\left(x_i(t), x_j(t)\right) \\
&\leq -\sum_{i=1}^{l} c_i\xi_i\left(\frac{V_i(x_i(t))}{\alpha_i}\right) \\
&\leq -\min_{i\in\mathbb{L}}\left\{\frac{c_i\xi_i}{\alpha_i}\right\}\sum_{i=1}^{l} V_i\left(x_i(t)\right) \\
&= -\xi V\left(x(t)\right),
\end{aligned}$$

where $\xi = \min_{i\in\mathbb{L}}\{c_i\xi_i/\alpha_i\}$. The proof of Theorem 1 in [24] implies that there exists some $\mathcal{K}\phi$-function β_0 such that $V(x(t)) \leq \beta_0(V(x_0), t)$ for all $t \leq t_1$. And letting $\delta_0 = \sum_{i=1}^{l} c_i\delta_i$, we can obtain from condition Q1 that

$$V(x_0) = \sum_{i=1}^{l} c_i V_i\left(x_i(0)\right) \leq \sum_{i=1}^{l} c_i\delta_i\left|x_i(0)\right|^p \leq \sum_{i=1}^{l} c_i\delta_i|x_0|^p = \delta_0|x_0|^p.$$

Therefore

$$\left|x(t)\right| \leq \beta\left(|x_0|, t\right), \tag{6}$$

where $\beta(r,t) = (\beta_0(\delta_0|r|^p, t)/\alpha)^{\frac{1}{p}}$.

From (5) and (6), we can obtain $|x(t)| \leq \beta(|x_0|, t) + \gamma(\|u\|)$ for all $t \geq 0$, that is, CCS (1) is ISS. □

In [1], the ISS for individual nonlinear control system was investigated by Sontag and Wang. Some classes of stability, like robust stability and weak robust stability for control systems, were investigated and some sufficient conditions were established to guarantee these stabilities. Motivated by [1], we have the following results.

Theorem 2 *Let the conditions in Theorem* 1 *hold. Then*:

(1) *CCS* (1) *is robustly stable.*

(2) *There exist $\mathcal{K}\phi$-functions β_1, β_2 and a $\mathcal{K}$-function γ such that, for any $x_0 \in \mathbb{R}^n$ and any input $u \in L_\infty^m$, it holds that*

$$\left|x(t, x_0, u)\right| \leq \beta_1\left(|x_0|, t\right) + \beta_2\left(\|u_T\|, t-T\right) + \gamma\left(\left\|u^{\mathrm{T}}\right\|\right)$$

for any $0 \leq T \leq t$, where u_T denotes the input for CCSs (1) *when $t = T$ and u^{T} is defined by $u^{\mathrm{T}} = u - u_T$.*

(3) *For each* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that* $|x(t, x_0, u)| \leq \varepsilon$ *for all inputs* $u \in L_\infty^m$ *and initial states* x_0 *with* $|x_0| \leq \delta$ *and* $\|u\| \leq \delta$.

(4) *There exists a* $\mathcal{K}$*-function* γ *such that, for any* $r, \varepsilon > 0$, *there is* $T > 0$ *so that for every input* $u \in L_\infty^m$, *it holds that* $|x(t, x_0, u)| \leq \varepsilon + \gamma(\|u\|)$, *whenever* $|x_0| \leq r$ *and* $t \geq T$.

(5) *CCS* (1) *is weakly robustly stable.*

4 An application to a coupled Lurie system

Now in order to illustrate the result of Theorem 1, let us apply this result to a coupled Lurie system (CLS). The absolute stability problem, formulated by Lurie and coworkers in the 1940s, has been a well-studied and fruitful area of research.

Assume that each vertex dynamic is described by a feedback interconnection of a linear block and a nonlinear element. To be simplified, $x_i(t)$ and $y_i(t)$ are denoted by x_i and y_i, $i = 1, 2, \ldots, l$. When a bounded input is set to every vertex system, it can be described as

$$\begin{aligned} \dot{x}_i &= A_i x_i + B_i\bigl(-\alpha_i(y_i) + u_i\bigr), \\ y_i &= K_i x_i, \end{aligned}$$

where $x_i \in \mathbb{R}^{n_i}$, $y_i \in \mathbb{R}^{m_i}$, $K_i \in \mathbb{R}^{m_i \times n_i}$, $A_i \in \mathbb{R}^{n_i \times n_i}$ is the personal state alteration matrix for the ith vertex system, $B_i \in \mathbb{R}^{n_i \times m_i}$ is the feedback and input effect matrix, u_i denotes the input of vertex i, and $\alpha_i(\cdot) : \mathbb{R}^{m_i} \to \mathbb{R}^{m_i}$ is a feedback function. Let $D_j \in \mathbb{R}^{n_i \times n_j}$ describe the effect that vertex system j has on i. Thus a CLS is obtained as follows:

$$\begin{aligned} \dot{x}_i &= A_i x_i + B_i\bigl(-\alpha_i(y_i) + u_i\bigr) + \sum_{j=1}^{l} D_j x_j, \\ y_i &= K_i x_i. \end{aligned} \tag{7}$$

Before the main theorem, let us present some assumptions and two lemmas. The following fundamental assumptions for CLS (7) are given:

A1: If (A_i, K_i) is detectable and there exists matrix $P_i = P_i^{\mathrm{T}} \geq 0$ satisfying

$$A_i^{\mathrm{T}} P_i + P_i A_i + l\bigl(P_i^{\mathrm{T}} P_i + D_i^{\mathrm{T}} D_i\bigr) \leq 0, \qquad K_i^{\mathrm{T}} = P_i B_i. \tag{8}$$

A2: If φ_i is a $\mathcal{K}_\infty$-function, and for all $y_i \in \mathbb{R}^{m_i}$,

$$|y_i| \varphi_i\bigl(|y_i|\bigr) \leq y_i^{\mathrm{T}} \alpha_i(y_i). \tag{9}$$

A3: When $|y_i| \geq \mu_i$, where $\mu_i > 0$

$$\bigl|\alpha_i(y_i)\bigr| \leq y_i^{\mathrm{T}} \alpha_i(y_i). \tag{10}$$

The results in this section and their proofs are motivated by [4].

Lemma 2 *For CLS* (7), *there exist a constant* $\theta_i > 0$ *and a* $\mathcal{K}_\infty$*-function* $\eta_i(\cdot)$ *satisfying*

$$\theta_i\bigl(\bigl|\alpha_i(y_i)\bigr| + |y_i|\bigr) \leq y_i^{\mathrm{T}} \alpha_i(y_i) \tag{11}$$

when $|y_i| \geq \mu_i$,

$$\eta_i\big(|y_i|\big)|y_i|^2 + \eta_i\big(\big|\alpha_i(y_i)\big|\big)\big|\alpha_i(y_i)\big|^2 \leq y_i^{\mathrm{T}}\alpha_i(y_i) \tag{12}$$

when $|y_i| \leq \mu_i$.

The proof of Lemma 2 can be seen in [4].

Since it is complex to construct a vertex ISS-Lyapunov function for CLS (7), we firstly construct a section of the vertex ISS-Lyapunov function in the following lemma. And then we give the entire vertex ISS-Lyapunov function for CLS (7) in the main theorem.

Lemma 3 *Suppose that* (7) *satisfies assumptions* A1-A3. *Definite a function*

$$\sigma_i(r) = \varepsilon_i \int_0^r \min\left\{1, \frac{1}{\sqrt{z}}, \pi_i(z)\right\} \mathrm{d}z,$$

where the constant $\varepsilon_i > 0$ *and the* $\mathcal{K}$*-function* $\pi_i(\cdot)$ *is to be specified. Let* $S_i(x_i) = \sigma_i(x_i^{\mathrm{T}}Q_ix_i)$, *in which matrix* $Q_i^{\mathrm{T}} = Q_i > 0$ *satisfying*

$$(A_i - J_iK_i)^{\mathrm{T}}Q_i + Q_i(A_i - J_iK_i) + l\big(Q_i^{\mathrm{T}}Q_i + D_i^{\mathrm{T}}D_i\big) \leq -I, \tag{13}$$

then there exists a $\mathcal{K}$*-function* $\gamma_i(\cdot)$ *satisfying*

$$\dot{S}_i(x_i) \leq -\gamma_i\big(|x_i|\big) + y_i^{\mathrm{T}}\alpha_i(y_i) + \theta_i|u_i| + \sum_{j=1}^{l} F_{ij}(x_i, x_j),$$

where $F_{ij}(x_i, x_j) = x_j^{\mathrm{T}}D_j^{\mathrm{T}}D_jx_j - x_i^{\mathrm{T}}D_i^{\mathrm{T}}D_ix_i$.

Proof Rewrite CLS (7) as

$$\dot{x}_i = (A_i - J_iK_i)x_i + J_iy_i + B_i\big(-\alpha_i(y_i) + u_i\big) + \sum_{j=1}^{l} D_jx_j, \tag{14}$$

where J_i is chosen so that $A_i - J_iK_i$ is a Hurwitz matrix. By the construction of $S_i(\cdot)$, it is easy to see that $S_i(\cdot)$ is positive definite and radially unbounded. Then we let $k > 0$ satisfy

$$2\max\big\{\big|B_i^{\mathrm{T}}Q_ix_i\big|, \big|J_i^{\mathrm{T}}Q_ix_i\big|\big\} \leq k|x_i|$$

for all $1 \leq i \leq l$, and note from (14) that

$$\dot{S}_i(x_i) \leq \sigma_i'\big(x_i^{\mathrm{T}}Q_ix_i\big)\left[2x_i^{\mathrm{T}}Q_i\left((A_i - J_iK_i)x_i + \sum_{j=1}^{l} D_jx_j\right) + k|x_i|\big(\big|\alpha_i(y_i)\big| + |y_i| + |u_i|\big)\right]. \tag{15}$$

Because $\sigma_i'(z) \leq \varepsilon_i/\sqrt{z}$, we can find a constant $c_i > 0$, independent of ε_i, so that $\sigma_i'(x_i^{\mathrm{T}}Q_ix_i) \times k|x_i| \leq c_i\varepsilon_i$ for all $x_i \in \mathbb{R}^{n_i}$. And then we can obtain

$$\dot{S}_i(x_i) \leq \sigma_i'\big(x_i^{\mathrm{T}}Q_ix_i\big)2x_i^{\mathrm{T}}Q_i\left((A_i - J_iK_i)x_i + \sum_{j=1}^{l} D_jx_j\right) + c_i\varepsilon_i\big(\big|\alpha_i(y_i)\big| + |y_i| + |u_i|\big).$$

Because of

$$\begin{aligned}
&2\sum_{j=1}^{l} x_i^{\mathrm{T}} Q_i D_j x_j \\
&\quad\le \sum_{j=1}^{l}\bigl(Q_i^{\mathrm{T}}Q_i|x_i|^2 + D_j^{\mathrm{T}}D_j|x_j|^2\bigr) \\
&\quad= lQ_i^{\mathrm{T}}Q_i|x_i|^2 + lD_i^{\mathrm{T}}D_i|x_i|^2 + \sum_{j=1}^{l}\bigl(D_j^{\mathrm{T}}D_j|x_j|^2 - D_i^{\mathrm{T}}D_i|x_i|^2\bigr) \\
&\quad= l\bigl(Q_i^{\mathrm{T}}Q_i + D_j^{\mathrm{T}}D_j\bigr)|x_i|^2 + \sum_{j=1}^{l}\bigl(x_j^{\mathrm{T}}D_j^{T}D_jx_j - x_i^{\mathrm{T}}D_i^{\mathrm{T}}D_ix_i\bigr) \\
&\quad= l\bigl(x_i^{\mathrm{T}}Q_i^{\mathrm{T}}Q_ix_i + x_i^{\mathrm{T}}D_i^{\mathrm{T}}D_ix_i\bigr) + \sum_{j=1}^{l}F_{ij}(x_i,x_j),
\end{aligned}$$

where $F_{ij}(x_i,x_j) = x_j^{\mathrm{T}}D_j^{\mathrm{T}}D_jx_j - x_i^{\mathrm{T}}D_i^{\mathrm{T}}D_ix_i$, we can get according to (13) that

$$\begin{aligned}
\dot{S}_i(x_i) &\le \sigma_i'\bigl(x_i^{\mathrm{T}}Q_ix_i\bigr)x_i^{\mathrm{T}}\bigl((A_i - J_iK_i)^{\mathrm{T}}Q_i + Q_i(A_i - J_iK_i) + l\bigl(Q_i^{\mathrm{T}}Q_i + D_i^{\mathrm{T}}D_i\bigr)\bigr)x_i \\
&\qquad + c_i\varepsilon_i\bigl(\bigl|\alpha_i(y_i)\bigr| + |y_i| + |u_i|\bigr) + \sum_{j=1}^{l}F_{ij}(x_i,x_j) \\
&\le -\sigma_i'\bigl(x_i^{\mathrm{T}}Q_ix_i\bigr)|x_i|^2 + c_i\varepsilon_i\bigl(\bigl|\alpha_i(y_i)\bigr| + |y_i| + |u_i|\bigr) + \sum_{j=1}^{l}F_{ij}(x_i,x_j).
\end{aligned} \tag{16}$$

- When $|y_i| \ge \mu_i$, choosing $\varepsilon_i = \theta_i/c_i$ and using (11), we have

$$\dot{S}_i(x_i) \le -\sigma_i'\bigl(x_i^{\mathrm{T}}Q_ix_i\bigr)|x_i|^2 + y_i^{\mathrm{T}}\alpha_i(y_i) + \theta_i|u_i| + \sum_{j=1}^{l}F_{ij}(x_i,x_j). \tag{17}$$

- When $|y_i| \le \mu_i$, we denote by λ_i the maximum eigenvalue of Q_i. Considering the two cases $|\alpha_i(y_i)| \le |x_i|/4k$ and $|x_i| \le 4k|\alpha_i(y_i)|$, and using $\sigma_i'(z) \le \pi_i(z)$, we can obtain

$$\sigma_i'\bigl(x_i^{\mathrm{T}}Q_ix_i\bigr)k|x_i|\bigl|\alpha_i(y_i)\bigr| \le \frac{1}{4}\sigma_i'\bigl(x_i^{\mathrm{T}}Q_ix_i\bigr)|x_i|^2 + 4k^2\bigl|\alpha_i(y_i)\bigr|^2\pi_i\bigl(16\lambda_ik^2\bigl|\alpha_i(y_i)\bigr|^2\bigr). \tag{18}$$

Considering the two cases $|y_i| \le |x_i|/4k$ and $|x_i| \le 4k|y_i|$, we can denote

$$\sigma_i'\bigl(x_i^{\mathrm{T}}Q_ix_i\bigr)k|x_i||y_i| \le \frac{1}{4}\sigma_i'\bigl(x_i^{\mathrm{T}}Q_ix_i\bigr)|x_i|^2 + 4k^2|y_i|^2\pi_i\bigl(16\lambda_ik^2|y_i|^2\bigr). \tag{19}$$

We choose

$$\pi_i(z) = \frac{1}{4k^2}\eta_i\left(\sqrt{\frac{z}{16\lambda_ik^2}}\right),$$

then substituting (18) and (19) into (16) yields

$$\begin{aligned}\dot{S}_i(x_i) \leq &-\frac{1}{2}\sigma_i'\big(x_i^{\mathrm{T}}Q_ix_i\big)|x_i|^2 + \eta_i\big(|y_i|\big)|y_i|^2 + \eta_i\big(|\alpha_i y_i|\big)|\alpha_i y_i|^2 \\ &+ \theta_i u_i + \sum_{j=1}^{l} F_{ij}(x_i, x_j).\end{aligned} \tag{20}$$

From (12), (20), it implies that

$$\dot{S}_i(x_i) \leq -\frac{1}{2}\sigma_i'\big(x_i^{\mathrm{T}}Q_ix_i\big)|x_i|^2 + y_i^{\mathrm{T}}\alpha_i(y_i) + \theta_i|u_i| + \sum_{j=1}^{l} F_{ij}(x_i, x_j). \tag{21}$$

So, noting from (10) that $\sigma_i'(z) = \varepsilon/z$ for sufficiently large z, we can find a $\mathcal{K}_\infty$-function $\gamma_i(\cdot)$ such that

$$\frac{1}{2}\sigma_i'\big(x_i^{\mathrm{T}}Q_ix_i\big)|x_i|^2 \geq \gamma_i\big(|x_i|\big)$$

for all $x_i \in \mathbb{R}^{n_i}$. Thus it follows from (17) and (21) that, for all values of x_i and u_i,

$$\dot{S}_i(x_i) \leq -\gamma_i\big(|x_i|\big) + y_i^{\mathrm{T}}\alpha_i(y_i) + \theta_i|u_i| + \sum_{j=1}^{l} F_{ij}(x_i, x_j). \tag{22}$$

The proof is complete. □

Theorem 3 *If CLS* (7) *satisfies assumptions* A1, A2 *and* A3, *then it is ISS.*

Proof Let $L_i(x_i) = x_i^{\mathrm{T}}P_ix_i + \sigma_i(x_i^{\mathrm{T}}Q_ix_i)$, and let $R_i(x_i) = x_i^{\mathrm{T}}P_ix_i$, $K_i = B_i^{\mathrm{T}}P_i$, then $\nabla R_i = 2x_i^{\mathrm{T}}P_i$. It follows that

$$\begin{aligned}\dot{R}_i(x_i) &= 2x_i^{\mathrm{T}}P_i\left(A_ix_i + B_i\big(-\alpha_i(y_i) + u_i\big) + \sum_{j=1}^{l} D_jx_j\right) \\ &= 2x_i^{\mathrm{T}}P_iA_ix_i + 2\sum_{j=1}^{l} x_i^{\mathrm{T}}P_iD_jx_j + 2x_i^{\mathrm{T}}P_iB_i\big(-\alpha_i(y_i) + u_i\big),\end{aligned}$$

in which

$$2x_i^{\mathrm{T}}P_iB_i\big(-\alpha_i(y_i) + u_i\big) \leq -2y_i^{\mathrm{T}}\alpha_i(y_i) + 2y_i^{\mathrm{T}}u_i$$

and

$$\begin{aligned}&2\sum_{j=1}^{l} x_i^{\mathrm{T}}P_iD_jx_j \\ &\quad= lP_i^{\mathrm{T}}P_i|x_i|^2 + \sum_{j=1}^{l} D_j^{\mathrm{T}}D_j|x_j|^2\end{aligned}$$

$$= l\big(P_i^{\mathrm{T}} P_i + D_j^{\mathrm{T}} D_j\big)|x_i|^2 + \sum_{j=1}^{l} \big(D_j^{\mathrm{T}} D_j |x_j|^2 - D_i^{\mathrm{T}} D_i |x_i|^2\big)$$

$$= l\big(x_i^{\mathrm{T}} P_i^{\mathrm{T}} P_i x_i + x_i^{\mathrm{T}} D_i^{\mathrm{T}} D_i x_i\big) + \sum_{j=1}^{l} F_{ij}(x_i, x_j).$$

Noting that $A_i^{\mathrm{T}} P_i + P_i A_i + l(P_i^{\mathrm{T}} P_i + D_i^{\mathrm{T}} D_i) \leq 0$, we get

$$\dot{R}_i(x_i) \leq -2y_i^{\mathrm{T}} \alpha_i\big(y_i + 2y_i^{\mathrm{T}} u_i\big) + \sum_{j=1}^{l} F_{ij}(x_i, x_j).$$

Considering the two cases $|u_i| \leq \varphi_i(|y_i|)/2$ and $|u_i| \geq \varphi_i(|y_i|)/2$ and using (9), we obtain the inequality

$$2y_i^{\mathrm{T}} u_i \leq 2|y_i||u_i| \leq |y_i|\big|\varphi_i\big(|y_i|\big)\big| + 2(\varphi_i)^{-1}\big(2|u_i|\big)|u_i| \leq y_i^{\mathrm{T}} \alpha_i(y_i) + 2(\varphi_i)^{-1}\big(2|u_i|\big)|u_i|,$$

which results in

$$\dot{R}_i(x_i) \leq -y_i^{\mathrm{T}} \alpha_i(y_i) + 2\varphi_i^{-1}\big(2|u_i|\big)|u_i| + \sum_{j=1}^{l} F_{ij}(x_i, x_j).$$

From Lemma 2, we know $\dot{S}_i(x_i) \leq -\gamma_i(|x_i|) + y_i^{\mathrm{T}} \alpha_i(y_i) + \theta_i |u_i| + \sum_{j=1}^{l} F_{ij}(x_i, x_j)$. Then we have

$$L_i(x_i) \leq -\gamma_i\big(|x_i|\big) + \beta_i\big(|u_i|\big) + \sum_{j=1}^{l} F_{ij}(x_i, x_j)$$

with $\beta_i(|u_i|) = \delta_i |u_i| + 2\phi_i^{-1}(2|u_i|)|u_i|$. Then we can find $\xi_i > 0$ satisfying

$$\dot{L}_i(x_i) \leq -\xi_i |x_i|^p + \sum_{j=1}^{l} F_{ij}(x_i, x_j).$$

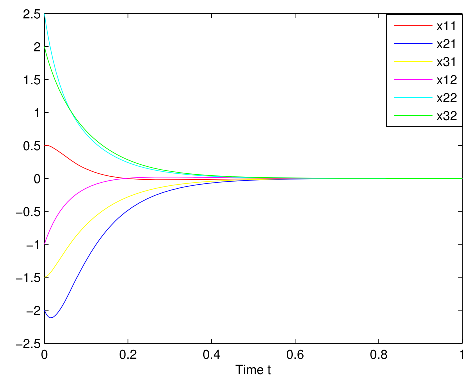
So we conclude that $L_i(x_i)$ is a vertex ISS-Lyapunov function, then from Theorem 1, CLS (7) is ISS. □

Remark 1 In recent years, Lurie systems have been studied by many researchers [4, 25]. Particularly, compared with [25], the main differences are as follows.

(i) This paper considers a coupled Lurie system, which is more complicated.
(ii) This paper uses graph theory combining with the Lyapunov method to derive the ISS of the considered system. This technique does not need us solving any linear matrix inequality. Literature [25] proposed Lyapunov-Krasovskii functionals which contain an exponential multiplier to solve the stabilization of an indirect control system.
(iii) In [25], time delay was considered, which is our further work.

Finally, two examples with their numerical simulations are provided to illustrate our results.

Figure 2 Solution to coupled Lurie system (7) with initial values $(x_{11}, x_{12}, x_{21}, x_{22}, x_{31}, x_{32})^{\mathrm{T}} = (0.5, -1, -2, 2.5, -1.5, 2)^{\mathrm{T}}$.



Example 1 Assume that there are three vertices and $x_i = (x_{i1}, x_{i2})^{\mathrm{T}} \in \mathbb{R}^2$. We now take the following coefficients for (7), and then take some numerical simulation. Here,

$$A_i = \begin{pmatrix} -10 & 1 \\ 1 & -10 \end{pmatrix}, \qquad B_i = \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$
$$D_1 = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}, \qquad D_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad D_3 = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}.$$

And let the input function $u_i = 0$, $\alpha_i(y_i) = iy_i$, $i = 1, 2, 3$. Moreover, we take $\varphi_i(y_i) = \frac{i}{2} y_i$ and $\mu_i = 1$ for $i = 1, 2, 3$. It is clear that conditions A2 and A3 are satisfied. If we let

$$P_1 = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}, \qquad P_2 = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}, \qquad P_3 = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix},$$

by calculation, we get that condition A1 holds. Therefore, by Theorem 3, we derive that (7) is ISS. The respective simulation results are shown in Figure 2, which conforms the effectiveness of the developed results.

Example 2 We consider the ISS of a system described as follows:

$$\begin{aligned} \dot{x}_i &= A_i x_i + B_i y_i + \sum_{j=1}^{3} D_j x_j, \\ \dot{y}_i &= -K_i x_i - 6y_i - \phi_i(y_i) - u_i + \sum_{j=1}^{3} y_j, \end{aligned} \qquad i = 1, 2, 3, \tag{23}$$

with the input function $u_i = C_i(x_i^{\mathrm{T}}, y_i)^{\mathrm{T}}$ in which C_i is a matrix and $x_i = (x_{i1}, x_{i2})^{\mathrm{T}}$, function $\phi_i(y_i) = y_i^3$, and there exist matrices $P_i^{\mathrm{T}} = P_i \geq 0$ such that

$$\begin{aligned} &A_i^T P_i + P_i A_i + 3\left(P_i^T P_i + D_i^T D_i\right) \leq 0, \\ &K_i^T = P_i B_i, \end{aligned} \qquad i = 1, 2, 3. \tag{24}$$

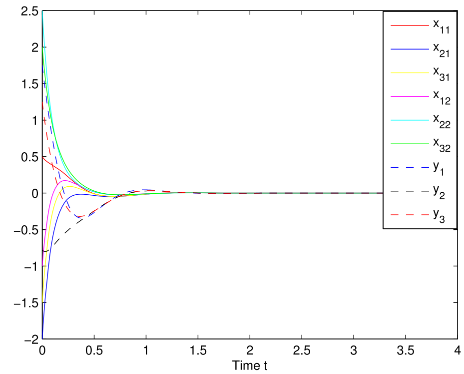


Figure 3 Solution to coupled Lurie system (23) with initial values $(x_{11}, x_{12}, y_1, x_{21}, x_{22}, y_2, x_{31}, x_{32}, y_3)^{\mathrm{T}} = (0.5, -1, 1.75, 2.5, -2, -0.78, 2, -1.5, 1.25)^{\mathrm{T}}$.

To apply Theorem 3, we rewrite (23) as in (7), with

$$A_{0i} = \begin{pmatrix} A_i & B_i \\ -K_i & -6 \end{pmatrix}, \qquad B_{0i} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad P_{0i} = \begin{pmatrix} P_i & 0 \\ 0 & 1 \end{pmatrix}, \qquad D_{0i} = \begin{pmatrix} D_i & 0 \\ 0 & 1 \end{pmatrix}$$

and $\alpha_i(y_i) = 3y_i + \phi_i(y_i)$. Then $\alpha_i(y_i)$ satisfies conditions A2 and A3. And then, we let the values of A_i, B_i, D_i, P_i, $i = 1, 2, 3$, be the same as those in Example 1 and

$$C_1 = C_2 = C_3 = (1 \quad 1 \quad 1).$$

We can see that A_{0i}, B_{0i}, P_{0i}, D_{0i} satisfy condition A1 because of (24). So, we conclude that system (23) is ISS. Figure 3 shows that the solution of system (23) is ISS.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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