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Periodicity in a neutral predator-prey system with monotone functional responses

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Abstract

By utilizing the coincidence degree theory and the related continuation theorem, as well as some prior estimates, we investigate the existence of positive periodic solutions of a neutral predator-prey system with monotone functional responses. New sufficient criteria are established for the existence of periodic solutions. Some well-known results in the literature are generalized.

MSC: 34C25; 92D25

Keywords: coincidence degree; periodic solution; neutral; monotone functional responses; delay

1 Introduction

Recently, many contributions [1–5] focused on the existence of periodic solution of neutral predator-prey systems. In these references, by using the effective Mawhin coincidence degree theory, a set of new and interesting sufficient conditions are obtained for the existence of periodic solution of neutral predator-prey systems. Attracted by the above work, in this paper, we are interested in the following neutral predator-prey system with monotone functional responses [6–10]:

$$\begin{cases} x'(t) = x(t)[a(t) - b(t)x(t-\sigma(t)) - c(t)x'(t-\sigma(t))] - r(t)f(\frac{x(t)}{y(t)})y(t), \\ y'(t) = y(t)[-d(t) + k(t)f(\frac{x(t-\tau(t))}{y(t-\tau(t))})], \end{cases} \tag{1.1}$$

where $x(t), y(t)$ stand for the population densities of prey and predator at time t, respectively. $a(t), b(t), r(t), d(t), k(t)$ are the intrinsic growth rate of the prey, the density-dependent coefficient of the prey, capture rates, the death rates of predator and maximal predator growth rates at time t, respectively. $\sigma(t), \tau(t)$ are delays. The function $f(\nu)$ is called the functional response. In many predator-prey interactions, especially in microbial dynamics or chemical kinetics, the functional response has the monotone property. Throughout this paper, we impose the following hypotheses on the function $f(\nu)$ due to its monotonicity.

(1) $f \in \mathcal{C}^1[0,+\infty), f(0) = 0$,
(2) $f'(\nu) > 0$ for $\nu \in [0,+\infty)$,
(3) $\lim_{\nu\to+\infty} f(\nu) = \mu > 0$, μ is a constant.

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The most popular monotone functional response which has been studied extensively by the authors is $f(\nu) = \nu/(\alpha + \nu)$, that proposed by Michaelis and Menten in the context of studying enzymatic reactions [11–13]. Another type, known as the Holling type-III functional response [14], takes the form $f(\nu) = \nu^2/(\alpha + \nu^2)$ and, in the general case, $f(\nu) = \nu^n/(\alpha + \nu^n), n > 2$, it is known as the sigmoidal response function. A similar type of celebrated response function is $f(\nu) = \nu^2/((\alpha + \nu)(\beta + \nu))$ found in [15].

In this paper, using a prior estimation technique and a continuation theorem, we show that system (1.1) with all of the monotone functional responses always has at least one positive ω-periodic solution. Our result extends previous work. In Section 2, for the reader's convenience, we will present some basic results from Gaines and Mawhin, then we study the existence of periodic solutions in Section 3. In Section 4, we will apply the main result of the paper to some more concrete predator-prey models.

2 Preliminaries

Before exploring the existence of periodic solution, we shall make some preparations. We now recall Mawhin's coincidence degree theory [16], which our study is based upon.

Let $\mathbb{X}, \mathbb{Z}$ be normed vector spaces, $L : \operatorname{Dom} L \subset \mathbb{X} \to \mathbb{Z}$ be a linear mapping, $N : \mathbb{X} \to \mathbb{Z}$ be a continuous mapping.

The mapping L will be called a Fredholm mapping of index zero if $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in $\mathbb{Z}$.

If L is a Fredholm mapping of index zero there exist continuous projectors $P : \mathbb{X} \to \mathbb{X}$ and $Q : \mathbb{Z} \to \mathbb{Z}$ such that $\operatorname{Im} P = \operatorname{Ker} L, \operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q)$. It follows that $L| \operatorname{Dom} L \cap \operatorname{Ker} P : (I - P)\mathbb{X} \to \operatorname{Im} L$ is invertible. We denote the inverse of that map by K_P.

If Ω is an open bounded subset of $\mathbb{X}$, the mapping N will be called L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I - Q)N : \bar{\Omega} \to \mathbb{X}$ is compact.

Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$.

Lemma 2.1 (Continuation theorem) *Let L be a Fredholm mapping of index zero and let N be L-compact on $\bar{\Omega}$. Suppose the following.*

(i) *For each $\lambda \in (0, 1)$, every solution x of $Lx = \lambda Nx$ is such that $x \notin \partial\Omega$;*
(ii) *$QNx \neq 0$ for each $x \in \partial\Omega \cap \operatorname{Ker} L$;*
(iii) $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$.

Then the equation $Lx = Nx$ has at least one solution lying in $\operatorname{Dom} L \cap \bar{\Omega}$.

For convenience of our formulation, we introduce the following notations:

$$|\phi|_0 = \max_{t\in[0,\omega]} \{|\phi(t)|\}, \qquad \bar{\phi} = \frac{1}{\omega}\int_0^\omega \phi(t)\,\mathrm{d}t, \qquad \hat{\phi} = \frac{1}{\omega}\int_0^\omega |\phi(t)|\,\mathrm{d}t,$$

where $\phi(t)$ is a continuous ω-periodic function.

In addition, let

$$\psi(t) = \frac{\mathrm{e}^{u(t)}}{\mathrm{e}^{\nu(t)}}, \qquad g(\nu) = \frac{f(\nu)}{\nu}, \qquad m = \sup_{\nu\in[0,+\infty)} g(\nu).$$

It is easy to see that $0 < m < +\infty$.

3 The existence of positive periodic solution

In this section, we shall study the existence of at least one positive periodic solution of system (1.1). By exploiting the continuation theorem of the coincidence degree, we derive a set of easily verifiable sufficient conditions for the existence of positive periodic solution of system (1.1).

Now, we state our main theorem.

Theorem 3.1 *Suppose that the following conditions hold.*

(H1) $a, \sigma, \tau \in \mathcal{C}(\mathbb{R}, \mathbb{R})$, $b, c, r, d, k \in \mathcal{C}(\mathbb{R}, [0, +\infty))$ *are* ω*-periodic functions*, $\bar{a}, \bar{b}, \bar{d} > 0$.

(H2) $c \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$, $\sigma \in \mathcal{C}^2(\mathbb{R}, \mathbb{R})$, $\sigma'(t) < 1$, $b(t) > h'(t)$ *for any* $t \in [0, \omega]$, *where* $h(t) = \frac{c(t)}{1-\sigma'(t)}$.

(H3) $|c|_0 \mathrm{e}^{A_1} < 1$, *where* $A_1 = \ln[2\bar{a}\max_{t\in[0,\omega]}\{\frac{1-\sigma'(t)}{b(t)-h'(t)}\}] + |h|_0 \max_{t\in[0,\omega]}\{\frac{2\bar{a}}{b(t)-h'(t)}\} + (\hat{a} + \bar{a})\omega$.

(H4) $\bar{a} > m\bar{r}$, $\mu\bar{k} > \bar{d}$.

Then system (1.1) *has at least one positive* ω*-periodic solution*.

Proof It is easy to see the solution of system (1.1) remains positive for $t \geq 0$. So we can make change of variables as follows:

$$x(t) = \mathrm{e}^{u(t)}, \qquad y(t) = \mathrm{e}^{v(t)}$$

and derive that

$$\begin{cases} u'(t) = a(t) - b(t)\mathrm{e}^{u(t-\sigma(t))} - c(t)\mathrm{e}^{u(t-\sigma(t))}u'(t-\sigma(t)) - r(t)g(\psi(t)), \\ v'(t) = -d(t) + k(t)f(\psi(t-\tau(t))). \end{cases} \tag{3.1}$$

It is easy to see that if system (3.1) has an ω-periodic solution $(u^*, v^*)^T$, then $(x^*, y^*)^T = (e^{u^*}, e^{v^*})^T$ is a positive ω-periodic solution of system (1.1). To this end, it suffices to prove that system (3.1) has at least one ω-periodic solution.

In order to apply Lemma 2.1, we set

$$\mathbb{X} = \mathbb{Z} = \{z(t) = (u(t), v(t))^T \in \mathcal{C}(\mathbb{R}, \mathbb{R}^2), z(t+\omega) = z(t)\}$$

and define

$$|z|_\infty = \max_{t\in[0,\omega]}\{|u(t)| + |v(t)|\}, \qquad \|z\| = |z|_\infty + |z'|_\infty.$$

Then both $\mathbb{X}$ and $\mathbb{Z}$ are Banach spaces when they are endowed with the norms $\|\cdot\|$ and $|\cdot|_\infty$, respectively.

Define operators L, P, and Q as follows:

$$L : \operatorname{Dom} L \cap \mathbb{X} \to \mathbb{Z}, \qquad Lz = z', \qquad Pz = Qz = \frac{1}{\omega}\int_0^\omega z(t)\,\mathrm{d}t,$$

where $\operatorname{Dom} L = \{z | z \in \mathbb{X} : z(t) \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}^2)\}$. Obviously, $\operatorname{Ker} L = \mathbb{R}^2, \operatorname{Im} L = \{z \in \mathbb{Z} : \int_0^\omega z(t)\,\mathrm{d}t = 0\}$ is closed in $\mathbb{Z}$ and

$$\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L = 2.$$

Therefore, L is a Fredholm mapping of index zero. This indicates that L has a unique inverse. We define by $K_p : \operatorname{Im} L \to \operatorname{Ker} P \cap \operatorname{Dom} L$ the inverse of L. By a direct calculation, we obtain

$$K_p(z) = \int_0^t z(s)\,\mathrm{d}s - \frac{1}{\omega}\int_0^\omega \int_0^t z(s)\,\mathrm{d}s\,\mathrm{d}t.$$

Moreover, by the above definitions of P and Q, it can be seen that P, Q are both continuous projectors satisfying

$$\operatorname{Im} P = \operatorname{Ker} L, \qquad \operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q).$$

Define $N : \mathbb{X} \to \mathbb{Z}$ by the form

$$Nz = \begin{pmatrix} a(t) - b(t)\mathrm{e}^{u(t-\sigma(t))} - c(t)\mathrm{e}^{u(t-\sigma(t))}u'(t-\sigma(t)) - r(t)g(\psi(t)) \\ -d(t) + k(t)f(\psi(t-\tau(t))) \end{pmatrix}.$$

Then it follows that $QN : \mathbb{X} \to \mathbb{Z}$ reads

$$\begin{aligned} QNz &= \begin{pmatrix} \frac{1}{\omega}\int_0^\omega [a(t) - b(t)\mathrm{e}^{u(t-\sigma(t))} - c(t)\mathrm{e}^{u(t-\sigma(t))}u'(t-\sigma(t)) - r(t)g(\psi(t))]\,\mathrm{d}t \\ \frac{1}{\omega}\int_0^\omega [-d(t) + k(t)f(\psi(t-\tau(t)))]\,\mathrm{d}t \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\omega}\int_0^\omega [a(t) - (b(t) - h'(t))\mathrm{e}^{u(t-\sigma(t))} - r(t)g(\psi(t))]\,\mathrm{d}t \\ \frac{1}{\omega}\int_0^\omega [-d(t) + k(t)f(\psi(t-\tau(t)))]\,\mathrm{d}t \end{pmatrix}. \end{aligned}$$

Here, at the first row of the matrix, we use the deformation

$$\begin{aligned} \frac{1}{\omega}\int_0^\omega c(t)\mathrm{e}^{u(t-\sigma(t))}u'\bigl(t-\sigma(t)\bigr) &= \frac{1}{\omega}\int_0^\omega \frac{c(t)}{1-\sigma'(t)}\bigl(\mathrm{e}^{u(t-\sigma(t))}\bigr)'\,\mathrm{d}t \\ &= \frac{h(t)}{\omega}\mathrm{e}^{u(t-\sigma(t))}\Big|_0^\omega - \frac{1}{\omega}\int_0^\omega h'(t)\mathrm{e}^{u(t-\sigma(t))}\,\mathrm{d}t \\ &= -\frac{1}{\omega}\int_0^\omega h'(t)\mathrm{e}^{u(t-\sigma(t))}\,\mathrm{d}t. \end{aligned}$$

From this, in turn, we obtain $K_p(I-Q)N : \mathbb{X} \to \mathbb{X}$,

$$\begin{aligned} K_p(I-Q)Nz = {}& \begin{pmatrix} \int_0^t [a(s) - (b(s) - h'(s))\mathrm{e}^{u(s-\sigma(s))} - r(s)g(\psi(s))]\,\mathrm{d}s \\ -h(t)\mathrm{e}^{u(t-\sigma(t))} + h(0)\mathrm{e}^{u(-\sigma(0))} \\ \int_0^t [-d(s) + k(s)f(\psi(s-\tau(s)))]\,\mathrm{d}s \end{pmatrix} \\ &- \begin{pmatrix} \frac{1}{\omega}\int_0^\omega\int_0^t [a(s) - (b(s) - h'(s))\mathrm{e}^{u(s-\sigma(s))} - r(s)g(\psi(s))]\,\mathrm{d}s\,\mathrm{d}t \\ -\frac{1}{\omega}\int_0^\omega [h(t)\mathrm{e}^{u(t-\sigma(t))} - h(0)\mathrm{e}^{u(-\sigma(0))}]\,\mathrm{d}t \\ \frac{1}{\omega}\int_0^\omega\int_0^t [-d(s) + k(s)f(\psi(s-\tau(s)))]\,\mathrm{d}s \end{pmatrix} \\ &- \begin{pmatrix} (\frac{t}{\omega} - \frac{1}{2})\int_0^\omega [a(s) - (b(s) - h'(s))\mathrm{e}^{u(s-\sigma(s))} - r(s)g(\psi(s))]\,\mathrm{d}s \\ (\frac{t}{\omega} - \frac{1}{2})\int_0^\omega [-d(s) + k(s)f(\psi(s-\tau(s)))]\,\mathrm{d}s \end{pmatrix}. \end{aligned}$$

It is not difficult to check by the Lebesgue convergence theorem that QN and $K_p(I-Q)N$ are both continuous. By virtue of the Arzela-Ascoli theorem, we know that $K_p(I-Q)N(\overline{\Omega})$

is relatively compact and $QN(\overline{\Omega})$ is bounded for any open set $\Omega \subset \mathbb{X}$. Hence N is L-compact on $\overline{\Omega}$.

Now we need to find an appropriate open and bounded subset $\Omega \subset \mathbb{X}$. Corresponding to the operator equation $Lz = \lambda Nz$ for $\lambda \in (0,1)$, we have

$$\begin{cases} u'(t) = \lambda[a(t) - b(t)e^{u(t-\sigma(t))} - c(t)e^{u(t-\sigma(t))}u'(t-\sigma(t)) - r(t)g(\psi(t))], \\ v'(t) = \lambda[-d(t) + k(t)f(\psi(t-\tau(t)))]. \end{cases} \tag{3.2}$$

Suppose that $(u(t), v(t))^T$ is an arbitrary ω-periodic solution of system (3.2) for a certain $\lambda \in (0,1)$. Integrating (3.2) over the interval $[0,\omega]$, we obtain

$$\int_0^\omega \left[\left(b(t) - h'(t)\right)e^{u(t-\sigma(t))} + r(t)g\left(\psi(t)\right)\right] dt = \bar{a}\omega \tag{3.3}$$

and

$$\int_0^\omega k(t)f\left(\psi\left(t - \tau(t)\right)\right) = \bar{d}\omega. \tag{3.4}$$

From the first equation of (3.2) and (3.3), one can see

$$\begin{aligned} &\int_0^\omega \left|\frac{d}{dt}\left[u(t) + \lambda h(t)e^{u(t-\sigma(t))}\right]\right| dt \\ &\quad= \lambda \int_0^\omega \left|a(t) - \left(b(t) - h'(t)\right)e^{u(t-\sigma(t))} - r(t)g\left(\psi(t)\right)\right| dt \\ &\quad\le \int_0^\omega \left|a(t)\right| dt + \int_0^\omega \left[\left(b(t) - h'(t)\right)e^{u(t-\sigma(t))} + r(t)g\left(\psi(t)\right)\right] dt \\ &\quad= (\hat{a} + \bar{a})\omega. \end{aligned} \tag{3.5}$$

Let $t = \varphi(\varsigma)$ be the inverse function of $\varsigma = t - \sigma(t)$. It is easy to verify that $b(\varphi(\varsigma))$, $h'(\varphi(\varsigma)), \sigma'(\varphi(\varsigma))$ are all ω-periodic functions.

From (3.3), it follows that

$$\begin{aligned} \int_0^\omega \left(b(t) - h'(t)\right)e^{u(t-\sigma(t))} dt &= \int_{-\sigma(0)}^{\omega-\sigma(\omega)} \frac{b(\varphi(\varsigma)) - h'(\varphi(\varsigma))}{1 - \sigma'(\varphi(\varsigma))} e^{u(\varsigma)} d\varsigma \\ &= \int_0^\omega \frac{b(\varphi(t)) - h'(\varphi(t))}{1 - \sigma'(\varphi(t))} e^{u(t)} dt < \bar{a}\omega. \end{aligned}$$

Then

$$\int_0^\omega \left[\frac{b(\varphi(t)) - h'(\varphi(t))}{1 - \sigma'(\varphi(t))} e^{u(t)} + \left(b(t) - h'(t)\right)e^{u(t-\sigma(t))}\right] dt < 2\bar{a}\omega.$$

By the mean value theorem of differential calculus, we see that there exists $\varrho \in [0,\omega]$ such that

$$\frac{b(\varphi(\varrho)) - h'(\varphi(\varrho))}{1 - \sigma'(\varphi(\varrho))} e^{u(\varrho)} + \left(b(\varrho) - h'(\varrho)\right)e^{u(\varrho-\sigma(\varrho))} < 2\bar{a}.$$

Noting that $b(t) > g'(t), t \in [0,\omega]$, we derive

$$\frac{b(\varphi(\varrho)) - h'(\varphi(\varrho))}{1 - \sigma'(\varphi(\varrho))} \mathrm{e}^{u(\varrho)} < 2\bar{a}, \qquad \big(b(\varrho) - h'(\varrho)\big)\mathrm{e}^{u(\varrho - \sigma(\varrho))} < 2\bar{a},$$

which imply that

$$u(\varrho) \le \ln\left[2\bar{a} \max_{t\in[0,\omega]}\left\{\frac{1-\sigma'(t)}{b(t) - h'(t)}\right\}\right], \qquad \mathrm{e}^{u(\varrho-\sigma(\varrho))} \le \max_{t\in[0,\omega]}\left\{\frac{2\bar{a}}{b(t)-h'(t)}\right\}.$$

Combining with (3.5), we obtain

$$\begin{aligned} u(t) + \lambda h(t)\mathrm{e}^{u(t-\sigma(t))} &\le u(\varrho) + \lambda h(\varrho)\mathrm{e}^{u(\varrho-\sigma(\varrho))} + \int_0^\omega \left|\frac{\mathrm{d}}{\mathrm{d}t}\big[u(t) + \lambda h(t)\mathrm{e}^{u(t-\sigma(t))}\big]\right| \mathrm{d}t \\ &\le \ln\left[2\bar{a} \max_{t\in[0,\omega]}\left\{\frac{1-\sigma'(t)}{b(t)-h'(t)}\right\}\right] + |h|_0 \max_{t\in[0,\omega]}\left\{\frac{2\bar{a}}{b(t)-h'(t)}\right\} \\ &\quad + (\hat{a} + \bar{a})\omega \\ &:= A_1. \end{aligned}$$

As $\lambda h(t)\mathrm{e}^{u(t-\sigma(t))} \ge 0$, we get

$$u(t) \le A_1, \quad t \in [0,\omega]. \tag{3.6}$$

From the first equation of (3.2) and (3.6), for any $t \in [0,\omega]$, we have

$$\begin{aligned} \big|u'(t)\big| &= \lambda\big|a(t) - b(t)\mathrm{e}^{u(t-\sigma(t))} - c(t)\mathrm{e}^{u(t-\sigma(t))}u'\big(t-\sigma(t)\big) - r(t)g\big(\psi(t)\big)\big| \\ &\le |a|_0 + |b|_0\mathrm{e}^{A_1} + |c|_0\mathrm{e}^{A_1}\big|u'\big|_0 + m|r|_0, \end{aligned}$$

which implies that

$$\big|u'\big|_0 \le \frac{1}{1 - |c|_0\mathrm{e}^{A_1}}\big[|a|_0 + |b|_0\mathrm{e}^{A_1} + m|r|_0\big] := A'. \tag{3.7}$$

Note that $(u(t), v(t))^T \in \mathbb{X}$, there exist $\xi_i, \eta_i \in [0,\omega], i = 1,2$, such that

$$\begin{aligned} &u(\xi_1) = \min_{t\in[0,\omega]}\big\{u(t)\big\}, \qquad u(\eta_1) = \max_{t\in[0,\omega]}\big\{u(t)\big\}, \\ &v(\xi_2) = \min_{t\in[0,\omega]}\big\{v(t)\big\}, \qquad v(\eta_2) = \max_{t\in[0,\omega]}\big\{v(t)\big\}. \end{aligned} \tag{3.8}$$

Let $q(t) = b(t) - h'(t)$. It follows from (3.3) that

$$\begin{aligned} \bar{a}\omega - \bar{q}\mathrm{e}^{u(\eta_1)} &\le \bar{a}\omega - \int_0^\omega \big(b(t) - h'(t)\big)\mathrm{e}^{u(t-\sigma(t))}\,\mathrm{d}t \\ &= \int_0^\omega r(t)g\big(\psi(t)\big)\,\mathrm{d}t \le m\bar{r}\omega, \end{aligned}$$

which indicates

$$u(\eta_1) \ge \ln\left(\frac{\bar{a} - m\bar{r}}{\bar{q}}\right).$$

Therefore, it follows from (3.7) that, for any $t \in [0, \omega]$,

$$u(t) \geq u(\eta_1) - \int_0^\omega |u'(t)|\, \mathrm{d}t \geq \ln\left(\frac{\bar{a} - m\bar{r}}{\bar{q}}\right) - A'\omega := A_2,$$

which together with (3.6) implies that

$$|u|_0 \leq \max\{|A_1|, |A_2|\} := A. \tag{3.9}$$

By virtue of (3.4), we know that there exists a $\theta \in [0, \omega]$ satisfying

$$f(\psi(\theta - \tau(\theta)))\bar{k}\omega = \bar{d}\omega.$$

Noting that $\psi(t)$ is an ω-periodic function in terms of the periodicity of $(u(t), v(t))$ and $\tau(t)$, we choose $\zeta \in [0, \omega]$ such that $\psi(\zeta) = \psi(\theta - \tau(\theta))$. Then, from the above expression, we have

$$f(\psi(\zeta)) = \frac{\bar{d}}{\bar{k}}. \tag{3.10}$$

From the properties of f, it is easy to see that f has an inverse function on the interval $[0, +\infty)$, namely f^{-1}. Then we have

$$\psi(\zeta) = \frac{\mathrm{e}^{u(\zeta)}}{\mathrm{e}^{v(\zeta)}} = f^{-1}\left(\frac{\bar{d}}{\bar{k}}\right) > 0.$$

Combining with (3.9), we get

$$\mathrm{e}^{v(\xi_2)} \leq \mathrm{e}^{A}/f^{-1}\left(\frac{\bar{d}}{\bar{k}}\right), \qquad \mathrm{e}^{v(\eta_2)} \geq \mathrm{e}^{-A}/f^{-1}\left(\frac{\bar{d}}{\bar{k}}\right).$$

From the second equation of (3.2) and (3.4), we obtain

$$\int_0^\omega |v'(t)|\, \mathrm{d}t = \lambda \int_0^\omega \left|-d(t) + k(t)f(\psi(t - \tau(t)))\right| \mathrm{d}t \leq (\hat{d} + \bar{d})\omega.$$

Consequently,

$$v(t) \leq v(\xi_2) + \int_0^\omega |v'(t)|\, \mathrm{d}t \leq A - \ln\left\{f^{-1}\left(\frac{\bar{d}}{\bar{k}}\right)\right\} + (\bar{d} + \hat{d})\omega := B_1$$

and

$$v(t) \geq v(\eta_2) - \int_0^\omega |v'(t)|\, \mathrm{d}t \leq -A - \ln\left\{f^{-1}\left(\frac{\bar{d}}{\bar{k}}\right)\right\} + (\bar{d} + \hat{d})\omega := B_2,$$

which yield

$$|v|_0 \leq \max\{|B_1|, |B_2|\} := B. \tag{3.11}$$

By using the property of f and (3.2), we have

$$\left|v'\right|_0 = \max_{t\in[0,\omega]}\left\{\left|v'(t)\right|\right\} \le |d|_0 + \mu|k|_0 := B'. \tag{3.12}$$

From (3.7), (3.9), (3.11), and (3.12), we obtain

$$\|z\| = \left\|(u,v)^T\right\| = |z|_\infty + \left|z'\right|_\infty \le A + A' + B + B'.$$

In addition, it follows from the given conditions that the system of algebraic equations

$$\begin{cases} \bar{a} - \bar{b}\mathrm{e}^u - \bar{r}g(\frac{\mathrm{e}^u}{\mathrm{e}^v}) = 0, \\ \bar{d} - \bar{k}f(\frac{\mathrm{e}^u}{\mathrm{e}^v}) = 0, \end{cases} \tag{3.13}$$

has a unique solution $(u^*, v^*)^T \in \mathbb{R}^2$.

Set $D = A + A' + B + B' + D_0$, where D_0 is taken sufficiently large so that the unique solution $(u^*, v^*)^T$ of (3.13) satisfies

$$\left\|\left(u^*, v^*\right)^T\right\| = \left|u^*\right| + \left|v^*\right| < D_0.$$

Clearly, D is independent of λ. We now take

$$\Omega = \left\{z = \left(u(t), v(t)\right)^T \middle| z \in \mathbb{X}, \|z\| < D\right\}.$$

It is clear that Ω satisfies the condition (i) in Lemma 2.1. When $z = (u,v)^T \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap \mathbb{R}^2, z = (u,v)^T$ is a constant vector in $\mathbb{R}^2$ with $\|z\| = \|(u,v)^T\| = |u| + |v| = D$.

Thus, we have

$$QNz = \begin{pmatrix} \bar{a} - \bar{b}\mathrm{e}^u - \bar{r}g(\frac{\mathrm{e}^u}{\mathrm{e}^v}) \\ \bar{d} - \bar{k}f(\frac{\mathrm{e}^u}{\mathrm{e}^v}) \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This implies that condition (ii) in Lemma 2.1 is satisfied.

Finally, we show that condition (iii) in Lemma 2.1 holds too.

Taking $J = I : \operatorname{Im} Q \to \operatorname{Ker} L, (u,v)^T \to (u,v)^T$, a direct calculation gives

$$\begin{aligned} \deg(JQN, \Omega \cap \operatorname{Ker} L, 0) &= \operatorname{sgn}\det\begin{pmatrix} -\bar{b}\mathrm{e}^u - \bar{r}g'(\psi)\psi'_u & -\bar{r}g'(\psi)\psi'_v \\ \bar{k}f'(\psi)\psi'_u & -\bar{k}f'(\psi)\psi'_v \end{pmatrix} \\ &= \operatorname{sgn}\left\{\bar{k}\bar{b}f'(\psi)\mathrm{e}^{2u-v}\right\} \neq 0. \end{aligned}$$

We have verified all the requirements of the Mawhin coincidence theorem in Ω. So system (3.1) has at least one ω-periodic solution. By the medium of (3.2), we derive that system (1.1) has at least one positive ω-periodic solution. This completes the proof. □

4 Applications

In order to illustrate some features of our maim theorem, we explore the existence of positive periodic solutions of some more concrete models, which have been extensively studied in the literature. In the sequel, the parameters $\alpha, \beta > 0$ are constants.

Example 4.1 Taking the monotone functional response $f(\nu) = \nu/(\alpha + \nu)$, we obtain the following neutral predator-prey system:

$$\begin{cases} x'(t) = x(t)[a(t) - b(t)x(t - \sigma(t)) - c(t)x'(t - \sigma(t))] - \frac{r(t)x(t)y(t)}{\alpha y(t)+x(t)}, \\ y'(t) = y(t)[-d(t) + \frac{k(t)x(t-\tau(t))}{\alpha y(t-\tau(t))+x(t-\tau(t))}]. \end{cases} \tag{4.1}$$

Theorem 4.1 *Assume that* (H1), (H2), (H3) *are satisfied*. *Moreover*,

(H5) $\alpha\bar{a} > \bar{r}, \qquad \bar{k} > \bar{d}$.

Then system (4.1) *has at least one positive ω-periodic solution*.

Example 4.2 Choose the monotone functional response $f(\nu) = \nu^n/(\alpha + \nu^n), n \geq 2$, we have the following neutral predator-prey system:

$$\begin{cases} x'(t) = x(t)[a(t) - b(t)x(t - \sigma(t)) - c(t)x'(t - \sigma(t))] - \frac{r(t)x^n(t)y(t)}{\alpha y^n(t)+x^n(t)}, \\ y'(t) = y(t)[-d(t) + \frac{k(t)x^n(t-\tau(t))}{\alpha y^n(t-\tau(t))+x^n(t-\tau(t))}]. \end{cases} \tag{4.2}$$

Theorem 4.2 *Assume that* (H1), (H2), (H3) *are satisfied*. *Moreover*,

(H6) $n\sqrt[n]{\alpha}\bar{a} > \bar{r}, \qquad \bar{k} > \bar{d}$.

Then system (4.2) *has at least one positive ω-periodic solution*.

Example 4.3 Select the monotone functional responses $f(\nu) = \nu^2/((\alpha + \nu)(\beta + \nu))$, we get the following neutral predator-prey system:

$$\begin{cases} x'(t) = x(t)[a(t) - b(t)x(t - \sigma(t)) - c(t)x'(t - \sigma(t))] - \frac{r(t)x^2(t)y(t)}{(\alpha y(t)+x(t))(\beta y(t)+x(t))}, \\ y'(t) = y(t)[-d(t) + \frac{k(t)x^2(t-\tau(t))}{(\alpha y(t-\tau(t))+x(t-\tau(t)))(\beta y(t-\tau(t))+x(t-\tau(t)))}]. \end{cases} \tag{4.3}$$

Theorem 4.3 *Assume that* (H1), (H2), (H3) *are satisfied*. *Moreover*,

(H7) $(2\sqrt{\alpha\beta} + \alpha + \beta)\bar{a} > \bar{r}, \qquad \bar{k} > \bar{d}$.

Then system (4.3) *has at least one positive ω-periodic solution*.

Competing interests

The author declares to have no competing interests.

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