Su et al. *Advances in Difference Equations* (2017) 2017:47
DOI 10.1186/s13662-017-1098-1

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Homoclinic orbits for a class of second order dynamic equations on time scales via variational methods

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Abstract

In this paper, we study the existence of nontrivial homoclinic orbits of a dynamic equation on time scales $\mathbb{T}$ of the form

$$\begin{cases} (p(t)u^{\Delta}(t))^{\Delta} + q^{\sigma}(t)u^{\sigma}(t) = f(\sigma(t), u^{\sigma}(t)), & \Delta\text{-a.e. } t \in \mathbb{T}, \\ u(\pm\infty) = u^{\Delta}(\pm\infty) = 0. \end{cases}$$

We construct a variational framework of the above-mentioned problem, and some new results on the existence of a homoclinic orbit or an unbounded sequence of homoclinic orbits are obtained by using the mountain pass lemma and the symmetric mountain pass lemma, respectively. The interesting thing is that the variational method and the critical point theory are used in this paper. It is notable that in our study any periodicity assumptions on $p(t)$, $q(t)$ and $f(t,u)$ are not required.

MSC: 34B15; 34C25; 34N05

Keywords: time scales; variational structure; homoclinic orbits; critical point theorem

1 Introduction

In the past decades, there has been an increasing interest in the study of dynamic equations on time scales, employing and developing a variety of methods (such as the variational method, the fixed point theory, the method of upper and lower solutions, the coincidence degree theory, and the topological degree arguments [1–13]) motivated, at least in part, by the fact that the existence of homoclinic and heteroclinic solutions is of utmost importance in the study of ordinary differential equations.

Although considerable attention has been dedicated to the existence of homoclinic and heteroclinic solutions for continuous or discrete ordinary differential equations, see [14–19] and the references therein, to the best of our knowledge, there is little work on homoclinic orbits for differential equations on time scales [20]. One of interesting and open problems on dynamic equations on time scales is to investigate discrete or continuous differential equations on time scales with one goal being the unified treatment of differential equations (the continuous case) and difference equations (the discrete case). In particular, not much work has been seen on the existence of solutions or homoclinic orbits to dynamic equations on time scales through the variational method and the critical point theory [20–23].

In this paper, we consider the existence of nontrivial homoclinic orbits to zero of equation on time scales $\mathbb{T}$ of the form

$$\begin{cases} (p(t)u^{\Delta}(t))^{\Delta} + q^{\sigma}(t)u^{\sigma}(t) = f(\sigma(t), u^{\sigma}(t)), & \Delta\text{-a.e. } t \in \mathbb{T}, \\ u(\pm\infty) = u^{\Delta}(\pm\infty) = 0, \end{cases} \tag{1}$$

where $p(t) : \mathbb{T} \to \mathbb{R}$ is nonzero and is Δ-differential, $q : \mathbb{T} \to \mathbb{R}$ is Lebesgue integrable and $f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is Lebesgue integrable with respect to t for Δ-a.e. $t \in \mathbb{T}$. Providing that $f(t,x)$ grows superlinearly both at origin and at infinity or is an odd function with respect to $x \in \mathbb{R}$, we explore the existence of a nontrivial homoclinic orbit of the dynamic equation (1) by means of the mountain pass lemma and the existence of an unbounded sequence of nontrivial homoclinic orbits by using the symmetric mountain pass lemma. The interesting thing is that the variational method and the critical point theory are used in this paper. It is notable that in our study any periodicity assumptions on $p(t)$, $q(t)$ and $f(t,u)$ are not required.

We say that a property holds for Δ-a.e. $t \in A \subset \mathbb{T}$ or Δ-a.e. on $A \subset \mathbb{T}$ whenever there exists a set $E \subset A$ with the null Lebesgue Δ-measure such that this property holds for every $t \in A \setminus E$.

Definition 1 We say that a solution u of equation (1) is homoclinic to zero if it satisfies $u(t) \to 0$ as $t \to \pm\infty$, where $t \in \mathbb{T}$. In addition, if $u \neq 0$, then u is called a nontrivial homoclinic solution.

Throughout this paper, we make the following assumptions:

(H_0) $\lim_{x\to 0} \frac{f(t,x)}{x} = 0$ uniformly for Δ-a.e. $t \in \mathbb{T}$;
(H_1) there exists a constant $\beta > 2$ such that

$$xf(t,x) \le \beta \int_0^x f(t,s)\,ds < 0 \quad \text{for } \Delta\text{-a.e. } t \in \mathbb{T} \text{ and for all } x \in \mathbb{R} \setminus \{0\}; \tag{2}$$

(H_2) $p(t) > 0$ for Δ-a.e. $t \in \mathbb{T}$ and $\int_{(-\infty,\infty)_{\mathbb{T}}} p^2(t)\Delta t < +\infty$;
(H_3) $q^{\sigma}(t) < 0$ for Δ-a.e. $t \in \mathbb{T}$, $\lim_{|t|\to\infty} q^{\sigma}(t) = -\infty$ and $\int_{(-\infty,\infty)_{\mathbb{T}}} |q^{\sigma}(t)|^2 \Delta t < +\infty$.

Let $F(t,x) = \int_0^x f(t,s)\,ds$, it follows from (2) that

$$\frac{dF}{F} \ge \frac{\beta}{x}\,dx \quad \text{for } |x| \ge 1,$$

which implies that there is a real function $\alpha(t) > 0$ such that

$$\int_0^x f(t,s)\,ds \le -\alpha(t)|x|^{\beta} \quad \text{for } \Delta\text{-a.e. } t \in \mathbb{T} \text{ and } |x| \ge 1. \tag{3}$$

It follows from (2) and (3) that

$$\lim_{|x|\to\infty} \frac{f(t,x)}{x} = -\infty \quad \text{uniformly for } \Delta\text{-a.e. } t \in \mathbb{T}. \tag{4}$$

Hence, we have the following remark.

Remark 1

(1) $u(t) \equiv 0$ is a trivial homoclinic solution of equation (1).

(2) $f(t,x)$ grows superlinearly both at infinity and at origin.

The paper is structured as follows. In Section 2, we introduce two technical lemmas which will be used in the proofs of our main results. In Section 3, the variational structure of the dynamic equation (1) is presented. In Section 4, we summarize our main results on the existence homoclinic solution of the dynamic equation (1) on time scales and present two examples. We demonstrate the proofs in Section 5.

2 Preliminaries

In this section, we present two lemmas which can help us to better understand our main results and proofs. For the basic terminologies such as measure, absolute continuity, the Lebesgue integral and Sobolev's spaces on time scales, we refer the reader to references [23–29].

Let us recall the mountain pass theorem [30] and the symmetric mountain pass theorem [31], respectively.

Lemma 1 ([30]) *Let X be a real Banach space and $\varphi : X \to \mathbb{R}$ be a C^1-smooth functional. Suppose that φ satisfies the following conditions*:

(i) $\varphi(0) = 0$;

(ii) *every sequence $\{u_j\}_{j\in\mathbb{N}}$ in X such that $\{\varphi(u_j)\}_{j\in\mathbb{N}}$ is bounded in $\mathbb{R}$ and $\varphi'(u_j) \to 0$ in X^* as $j \to +\infty$ contains a convergent subsequence as $j \to +\infty$* (*the PS condition*);

(iii) *there exist constants ϱ and $\alpha > 0$ such that $\varphi|_{\partial B_\varrho(0)} \geq \alpha$*;

(iv) *there exists $e \in X \setminus \bar{B}_\varrho(0)$ such that $\varphi(e) \leq 0$, where $B_\varrho(0)$ is an open ball in X of radius ϱ centered at* 0.

Then φ possesses a critical value $c \geq \alpha$ given by

$$c = \inf_{g\in\Gamma} \max_{s\in[0,1]} \varphi\bigl(g(s)\bigr),$$

where

$$\Gamma = \bigl\{g \in C\bigl([0,1],E\bigr) : g(0) = 0, g(1) = e\bigr\}.$$

Lemma 2 ([31]) *Let X be a real Banach space and $\varphi : X \to \mathbb{R}$ be a C^1-smooth functional. Suppose that φ satisfies the following conditions*:

(i) $\varphi(0) = 0$;

(ii) *φ satisfies the PS condition*;

(iii) *there exist constants ϱ and $\alpha > 0$ such that $\varphi|_{\partial B_\varrho(0)} \geq \alpha$*;

(iv) *for each finite-dimensional subspace $\widetilde{E} \subset E$, there is $\gamma = \gamma(\widetilde{E})$ such that $\varphi \leq 0$ on $\widetilde{E} \setminus \beta_\gamma$.*

Then φ possesses an unbounded sequence of critical values.

3 Variational framework

In this section, we state some basic notations, some lemmas which are closely related to our main results, and construct a variational framework of our problem.

For $p \in \mathbb{R}$ and $p \geq 1$, we let the space

$$L^p_\Delta\big((-\infty,\infty)_\mathbb{T},\mathbb{R}\big) = \left\{f:(-\infty,\infty)_\mathbb{T} \to \mathbb{R} : \int_{(-\infty,\infty)_\mathbb{T}} \big|f(t)\big|^p \Delta t < +\infty\right\}$$

be equipped with the norm

$$\|f\|_{L^p_\Delta} = \left[\int_{(-\infty,\infty)_\mathbb{T}} \big|f(s)\big|^p \Delta s\right]^{\frac{1}{p}}.$$

Then $L^p_\Delta((-\infty,\infty)_\mathbb{T},\mathbb{R})$ is a Banach space together with the inner product given by

$$\langle f,g\rangle_{L^p_\Delta} = \int_{(-\infty,\infty)_\mathbb{T}} f(t)g(t)\Delta t,$$

where $(f,g) \in L^p_\Delta((-\infty,\infty)_\mathbb{T},\mathbb{R}) \times L^p_\Delta((-\infty,\infty)_\mathbb{T},\mathbb{R})$.

Let

$$\begin{aligned} H^{1,2}_\Delta &= H^{1,2}_\Delta\big((-\infty,\infty)_\mathbb{T},\mathbb{R}\big) \\ &= \left\{ u:(-\infty,\infty)_\mathbb{T} \to \mathbb{R} \;\middle|\; \begin{array}{l} u \text{ is absolutely continuous and} \\ \text{bounded measurable functional,} \\ u^\Delta \in L^2_\Delta((-\infty,\infty)_\mathbb{T},\mathbb{R}) \end{array} \right\}. \end{aligned}$$

It is a Hilbert space with the norm defined by

$$\|u\| = \|u\|_{H^{1,2}_\Delta} = \left(\int_{(-\infty,\infty)_\mathbb{T}} |u|^2 \Delta t + \int_{(-\infty,\infty)_\mathbb{T}} \big|u^\Delta\big|^2 \Delta t\right)^{\frac{1}{2}}$$

for $u \in H^{1,2}_\Delta$.

Define

$$E = \left\{ u \in H^{1,2}_\Delta \;\middle|\; \begin{array}{l} \int_{(-\infty,\infty)_\mathbb{T}} [p(t)(u^\Delta)^2 - q^\sigma(t)(u^\sigma)^2]\Delta t < +\infty, \\ \text{and there exist } 0, a \in (-\infty,\infty)_\mathbb{T} \text{ are real} \\ \text{such that } \int_{(0,a)_\mathbb{T}} u(t)\Delta t = 0 \end{array} \right\}.$$

Then E is a Hilbert space with the norm defined by

$$\|u\|^2_E = \int_{(-\infty,\infty)_\mathbb{T}} \big[p(t)\big(u^\Delta\big)^2 - q^\sigma(t)\big(u^\sigma\big)^2\big]\Delta t \quad \text{for } u \in E,$$

and the inner product is

$$\langle u,v\rangle = \int_{(-\infty,\infty)_\mathbb{T}} \big[p(t)u^\Delta v^\Delta - q^\sigma(t)\big(u^\sigma\big)^2 v\big]\Delta t \quad \text{for any } u,v \in E.$$

Let

$$L_{\Delta}^{\infty}\big((-\infty,+\infty)_{\mathbb{T}},\mathbb{R}\big)=\left\{u:(-\infty,+\infty)_{\mathbb{T}}\to\mathbb{R}\,\middle|\,\begin{array}{l}u\text{ is bounded measurable}\\ \text{function a.e. on }(-\infty,+\infty)_{\mathbb{T}}\end{array}\right\},$$

and $L_{\Delta}^{\infty}((-\infty,+\infty)_{\mathbb{T}},\mathbb{R})$ is called the essentially bounded space on time scales, which is equipped with the norm

$$\|u\|_{L_{\Delta}^{\infty}}:=\operatorname{ess\,sup}\big\{\big|u(t)\big|:t\in(-\infty,+\infty)_{\mathbb{T}}\big\}=\inf_{\mu(E_0)=0,E_0\subset E}\ \sup_{t\in(-\infty,+\infty)_{\mathbb{T}}\setminus E_0}\big|u(t)\big|,$$

where $u(t)$ is bounded on $(-\infty,+\infty)_{\mathbb{T}}\setminus E_0$, and E_0 is a set of measure zero in the space $(-\infty,+\infty)_{\mathbb{T}}$.

Now, we list three technical lemmas which will be used in the proofs of our main results in the next section.

We have the following lemma.

Lemma 3 *There exist positive constants C^* and L such that the following inequality holds*:

$$\|u\|_{L_{\Delta}^{\infty}}\leq C^*\|u\|. \tag{5}$$

Moreover, there exist $0,a\in(-\infty,\infty)_{\mathbb{T}}$ are real such that $\int_{(0,a)_{\mathbb{T}}}u(t)\Delta t=0$, then

$$\|u\|_{L_{\Delta}^{\infty}}\leq L\|u^{\Delta}\|_{L_{\Delta}^{2}}, \tag{6}$$

where $t\in(-\infty,+\infty)_{\mathbb{T}}$, holds.

Proof Going to the components of $u(t)$, we can assume that $n=1$, and there exist $0,a\in(0,+\infty)_{\mathbb{T}}$ are real. If $u(t)\in H_{\Delta}^{1,2}$, then there exists $\tau\in[0,a]_{\mathbb{T}}$ such that $u(\tau)=\inf_{t\in[0,a]_{\mathbb{T}}}u(t)$, it follows that

$$\frac{1}{a}\int_{(0,a)_{\mathbb{T}}}u(t)\Delta t\geq\frac{1}{a}\int_{(0,a)_{\mathbb{T}}}u(\tau)\Delta t=u(\tau).$$

Thus, there exists constant $c_3>0$ such that $|u(\tau)|\leq c_3|\int_{(0,a)_{\mathbb{T}}}u(t)\Delta t|$. Hence, for $t\in(-\infty,\infty)_{\mathbb{T}}$, one can get

$$\begin{aligned}\big|u(t)\big|&=\left|u(\tau)+\int_{(\tau,t)_{\mathbb{T}}}u^{\Delta}(t)\Delta t\right|\leq\big|u(\tau)\big|+\left|\int_{(\tau,t)_{\mathbb{T}}}u^{\Delta}(t)\Delta t\right|\\&\leq c_3\left|\int_{(0,a)_{\mathbb{T}}}u(t)\Delta t\right|+|t-\tau|^{\frac{1}{2}}\left(\int_{(\tau,t)_{\mathbb{T}}}\big|u^{\Delta}(t)\big|^2\Delta t\right)^{\frac{1}{2}}\\&\leq c_3a^{\frac{1}{2}}\left(\int_{(-\infty,\infty)_{\mathbb{T}}}\big|u(t)\big|^2\Delta t\right)^{\frac{1}{2}}\\&\quad+|t-\tau|^{\frac{1}{2}}\left(\int_{(-\infty,\infty)_{\mathbb{T}}}\big|u^{\Delta}(t)\big|^2\Delta t\right)^{\frac{1}{2}},\end{aligned}$$

then

$$\begin{aligned}
\|u\|_{L^\infty_\Delta} &= \inf_{\mu(E_0)=0, E_0\subset E} \sup_{t\in(-\infty,+\infty)_\mathbb{T}\setminus E_0} \bigl|u(t)\bigr| \\
&\le \max\Bigl\{c_3 a^{\frac{1}{2}}, \inf_{\mu(E_0)=0, E_0\subset E} \sup_{t\in(-\infty,+\infty)_\mathbb{T}\setminus E_0} |t-\tau|^{\frac{1}{2}}\Bigr\} \\
&\quad \times \biggl(\biggl(\int_{(-\infty,\infty)_\mathbb{T}} \bigl|u(s)\bigr|^2\Delta t\biggr)^{\frac{1}{2}} + \biggl(\int_{(-\infty,\infty)_\mathbb{T}} \bigl|u^\Delta(s)\bigr|^2\Delta t\biggr)^{\frac{1}{2}}\biggr) \\
&\le C^*\|u\|.
\end{aligned}$$

If $\int_{(0,a)_\mathbb{T}} u(t)\Delta t = 0$, then

$$\begin{aligned}
\bigl|u(t)\bigr| &= \biggl|u(\tau) + \int_{(\tau,t)_\mathbb{T}} u^\Delta(t)\Delta t\biggr| \le \bigl|u(\tau)\bigr| + \biggl|\int_{(\tau,t)_\mathbb{T}} u^\Delta(t)\Delta t\biggr| \\
&\le c_3\biggl|\int_{(0,a)_\mathbb{T}} u(t)\Delta t\biggr| + |t-\tau|^{\frac{1}{2}}\biggl(\int_{(\tau,t)_\mathbb{T}} \bigl|u^\Delta(t)\bigr|^2\Delta t\biggr)^{\frac{1}{2}},
\end{aligned}$$

which implies (6) holds. □

Lemma 4 *Assume that the sequence $\{u_n\}\subset E$ such that $u_n \rightharpoonup u$ in E, then the sequence u_n satisfies $u_n \to u$ in $L^2_\Delta((-\infty,\infty)_\mathbb{T},\mathbb{R})$.*

Proof Without loss of generality, assume that $u_n \rightharpoonup 0$ in E for any $\varepsilon > 0$. It follows from (H_3) that there exists negative $T_0 \in \mathbb{T}$ such that

$$-\frac{1}{q^\sigma(t)} \le \varepsilon \quad \text{for } \Delta\text{-a.e. } t\in(-\infty,T_0)_\mathbb{T}. \tag{7}$$

Similarly, we also have there exists positive $T_1 \in \mathbb{T}$ such that

$$-\frac{1}{q^\sigma(t)} \le \varepsilon \quad \text{for } \Delta\text{-a.e. } t\in(T_0,\infty)_\mathbb{T}. \tag{8}$$

From (H_2) and (H_3), we have $u_n \rightharpoonup u$ in E_I, where

$$E_I = \biggl\{u\in H^{1,2}_\Delta \Bigm| \int_{(T_0,T_1)_\mathbb{T}} \bigl[p(t)\bigl(u^\Delta(t)\bigr)^2 - q^\sigma(t)\bigl(u^\sigma(t)\bigr)^2\bigr]\Delta t < +\infty\biggr\}.$$

Hence, $\{u_n\}$ is bounded in E_I, which implies that $\{u_n\}$ is bounded in $L^2_\Delta((T_0,T_1)_\mathbb{T},\mathbb{R})$. Due to the uniqueness of the weak limit in $L^2_\Delta((T_0,T_1)_\mathbb{T},\mathbb{R})$, one obtains $u_n \to 0$ on $(T_0,T_1)_\mathbb{T}$, then there is n_0 such that

$$\int_{(T_0,T_1)_\mathbb{T}} \bigl|u_n(t)\bigr|^2\Delta t \le \varepsilon \quad \text{for all } n\ge n_0 \tag{9}$$

since

$$\sup_n \int_{(-\infty,\infty)_\mathbb{T}} \bigl[p(t)\bigl(u_n^\Delta(t)\bigr)^2 - q^\sigma(t)\bigl(u_n(t)\bigr)^2\bigr]\Delta t < +\infty.$$

Let

$$A_1 = \max\left\{\int_{(-\infty,T_0)_{\mathbb{T}}} q^\sigma(t)\left|u_n(t)\right|^2\Delta t, \int_{(-\infty,T_0)_{\mathbb{T}}} q^\sigma(t)\left|u_n^\sigma(t)\right|^2\Delta t\right\},$$

then $0 < A_1 < +\infty$.

According to (7), we have

$$\begin{aligned}
&\int_{(-\infty,T_0)_{\mathbb{T}}} \left|u_n(t)\right|^2\Delta t \\
&\quad\le -\varepsilon \max\left\{\int_{(-\infty,T_0)_{\mathbb{T}}} q^\sigma(t)\left|u_n(t)\right|^2\Delta t, \int_{(-\infty,T_0)_{\mathbb{T}}} q^\sigma(t)\left|u_n^\sigma(t)\right|^2\Delta t\right\} \\
&\quad\le \varepsilon A_1.
\end{aligned} \tag{10}$$

Let

$$A_2 = \max\left\{\int_{(T_1,\infty)_{\mathbb{T}}} q^\sigma(t)\left|u_n(t)\right|^2\Delta t, \int_{(T_1,\infty)_{\mathbb{T}}} q^\sigma(t)\left|u_n^\sigma(t)\right|^2\Delta t\right\},$$

then $0 < A_2 < +\infty$.

In view of (8), we have

$$\begin{aligned}
&\int_{(T_1,\infty)_{\mathbb{T}}} \left|u_n(t)\right|^2\Delta t \\
&\quad\le -\varepsilon \max\left\{\int_{(T_1,\infty)_{\mathbb{T}}} q^\sigma(t)\left|u_n(t)\right|^2\Delta t, \int_{(T_1,\infty)_{\mathbb{T}}} q^\sigma(t)\left|u_n^\sigma(t)\right|^2\Delta t\right\} \\
&\quad\le \varepsilon A_2.
\end{aligned} \tag{11}$$

Since ε is arbitrary, combining (9), (10) and (11), one has

$$u_n \to u \quad \text{in } L^2_\Delta\big((-\infty,\infty)_{\mathbb{T}},\mathbb{R}\big). \qquad \square$$

In the following, we define and prove the variational framework of the dynamic equation (1).

Define the functional $E \to \mathbb{R}$ by

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\int_{(-\infty,\infty)_{\mathbb{T}}} \left(p(t)\left(u^\Delta(t)\right)^2 - q^\sigma(t)\left(u^\sigma(t)\right)^2\right)\Delta t + \int_{(-\infty,\infty)_{\mathbb{T}}} F\left(\sigma(t),u^\sigma(t)\right)\Delta t \\
&= \frac{1}{2}\|u\|_E^2 + \int_{(-\infty,\infty)_{\mathbb{T}}} F\left(\sigma(t),u^\sigma(t)\right)\Delta t,
\end{aligned} \tag{12}$$

where $F(t,\xi) = \int_0^\xi f(t,s)\,ds$.

Lemma 5 *The functional φ is continuously differentiable on E, and*

$$\varphi'(u)v = \int_{(-\infty,\infty)_{\mathbb{T}}} \left(p(t)u^\Delta v^\Delta - q^\sigma(t)u^\sigma v^\sigma\right)\Delta t + \int_{(-\infty,\infty)_{\mathbb{T}}} f\left(\sigma(t),u^\sigma\right)v^\sigma\,\Delta t \quad \textit{for } u,v \in E.$$

Proof Let us first consider the existence of the Gâteaux derivative.

For any $v \in E$ and $\varepsilon \in \mathbb{R}$ $(0 < |\varepsilon| < 1)$, we have

$$\begin{aligned}
&\frac{1}{\varepsilon}\big[\varphi(u+\varepsilon v)-\varphi(u)\big] \\
&\quad= \int_{(-\infty,\infty)_{\mathbb{T}}} \frac{1}{2\varepsilon}\Big[2p(t)\varepsilon u^{\Delta}v^{\Delta} + p(t)\varepsilon^2\big(v^{\Delta}\big)^2 - 2\varepsilon q^{\sigma}(t)u^{\sigma}(t)v^{\sigma}(t) + \varepsilon^2 q^{\sigma}(t)\big(v^{\sigma}(t)\big)^2\Big] \\
&\qquad + \int_{(-\infty,\infty)_{\mathbb{T}}} \frac{F(\sigma(t),u^{\sigma}+\varepsilon v^{\sigma}) - F(\sigma(t),u^{\sigma})}{\varepsilon}\Delta t.
\end{aligned}$$

Given $u \in \mathbb{R}$, the mean value theorem indicates that there exists $\lambda_2 \in (0,1)$ such that

$$\begin{aligned}
&\frac{1}{|\varepsilon|}\big|F\big(\sigma(t),u^{\sigma}+\varepsilon v^{\sigma}\big) - F\big(\sigma(t),u^{\sigma}\big)\big| \\
&\quad= \frac{1}{|\varepsilon|}\left|\frac{\partial F}{\partial \xi}\bigg|_{(\sigma(t),u^{\sigma}+\lambda_2\varepsilon v^{\sigma})}\right|\big|\varepsilon v^{\sigma}\big| = \big|f\big(\sigma(t),u^{\sigma}+\lambda_2\varepsilon v^{\sigma}\big)\big|\big|v^{\sigma}\big|.
\end{aligned}$$

Note that

$$\big|f\big(\sigma(t),u^{\sigma}+\lambda_2\varepsilon v^{\sigma}\big)\big|\big|v^{\sigma}\big| \in L^1_{\Delta}\big((-\infty,\infty)_{\mathbb{T}},\mathbb{R}\big).$$

It follows from Lebesgue's dominated convergence theorem on time scales that

$$\begin{aligned}
\varphi'(u)v &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\big[\varphi(u+\varepsilon v)-\varphi(u)\big] \\
&= \int_{(-\infty,\infty)_{\mathbb{T}}}\big(p(t)u^{\Delta}v^{\Delta} - q^{\sigma}(t)u^{\sigma}v^{\sigma}\big)\Delta t + \int_{(-\infty,\infty)_{\mathbb{T}}} f\big(\sigma(t),u^{\sigma}\big)v^{\sigma}\,\Delta t.
\end{aligned}$$

Next, we show the continuity of the Gâteaux derivative.

Assume that the sequence $\{u_n\} \subset E$ satisfies $u_n \to u$ as $n \to \infty$ in E. Using Lebesgue's dominated convergence theorem on time scales and (H_0) yields

$$\int_{(-\infty,\infty)_{\mathbb{T}}}\big|f\big(\sigma(t),u_n^{\sigma}\big) - f\big(\sigma(t),u^{\sigma}\big)\big|\Delta t \to 0 \quad \text{as } n \to \infty. \tag{13}$$

It follows from Theorem 4.5 in [21] that $E \hookrightarrow L^2_{\Delta}((-\infty,\infty)_{\mathbb{T}},\mathbb{R})$ is compact, then $u_n \to u$ as $n \to \infty$ in $L^2_{\Delta}((-\infty,\infty)_{\mathbb{T}},\mathbb{R})$. For arbitrary $v \in E$, there holds

$$\begin{aligned}
&\varphi'(u_n)v - \varphi'(u)v \\
&\quad= \int_{(-\infty,\infty)_{\mathbb{T}}} p(t)\big(u_n^{\Delta}-u^{\Delta}\big)v^{\Delta}\,\Delta t \\
&\qquad - \int_{(-\infty,\infty)_{\mathbb{T}}} q^{\sigma}(t)\big(u_n^{\sigma}-u^{\sigma}\big)v^{\sigma}\,\Delta t + \int_{(-\infty,\infty)_{\mathbb{T}}}\big(f\big(\sigma(t),u_n^{\sigma}\big) - f\big(\sigma(t),u^{\sigma}\big)\big)v^{\sigma}\,\Delta t.
\end{aligned}$$

Hölder's inequality on time scales and Lemma 3 reduce to

$$\begin{aligned}
&\big|\varphi'(u_n)v - \varphi'(u)v\big| \\
&\quad\le \int_{(-\infty,\infty)_{\mathbb{T}}}\big|p(t)\big(u_n^{\Delta}-u^{\Delta}\big)\big|\big|v^{\Delta}\big|\Delta t + \int_{(-\infty,\infty)_{\mathbb{T}}}\big|q^{\sigma}(t)\big(u_n^{\sigma}-u^{\sigma}\big)\big|\big|v^{\sigma}\big|\Delta t
\end{aligned}$$

$$
\begin{aligned}
&+\int_{(-\infty,\infty)_{\mathbb{T}}}\left|\left(f\left(\sigma(t),u_n^\sigma\right)-f\left(\sigma(t)u^\sigma\right)v^\sigma\right)\right|\Delta t\\
&\le\|v\|_{L^\infty_\Delta}\left(\int_{(-\infty,\infty)_{\mathbb{T}}}\left|u_n^\Delta-u^\Delta\right|^2\Delta t\right)^{\frac{1}{2}}\left(\int_{(-\infty,\infty)_{\mathbb{T}}}\left|p(t)\right|^2\Delta t\right)^{\frac{1}{2}}\\
&\quad+\left\|v^\sigma\right\|_{L^\infty_\Delta}\left(\int_{(-\infty,\infty)_{\mathbb{T}}}\left|u_n^\sigma-u^\sigma\right|^2\Delta t\right)^{\frac{1}{2}}\left(\int_{(-\infty,\infty)_{\mathbb{T}}}\left|q^\sigma(t)\right|^2\Delta t\right)^{\frac{1}{2}}\\
&\quad+\int_{(-\infty,\infty)_{\mathbb{T}}}\left|\left(f\left(\sigma(t),u_n^\sigma\right)-f\left(\sigma(t),u^\sigma\right)v^\sigma\right)\right|\Delta t\\
&\le C^*\|v\|\left\|u_n^\Delta-u^\Delta\right\|_{L^2_\Delta}\left(\int_{(-\infty,\infty)_{\mathbb{T}}}\left|p(t)\right|^2\Delta t\right)^{\frac{1}{2}}\\
&\quad+C^*\left\|v^\sigma\right\|\left\|u_n^\sigma-v^\sigma\right\|_{L^2_\Delta}\left(\int_{(-\infty,\infty)_{\mathbb{T}}}\left|q^\sigma(t)\right|^2\Delta t\right)^{\frac{1}{2}}\\
&\quad+C^*\left\|v^\sigma\right\|\int_{(-\infty,\infty)_{\mathbb{T}}}\left|f\left(\sigma(t),u_n^\sigma\right)-f\left(\sigma(t),u^\sigma\right)\right|\Delta t.
\end{aligned}
$$

Thus, from the above discussion, (13), (H_1) and (H_2), we have

$$
\begin{aligned}
&\left\|\varphi'(u_n)-\varphi'(u)\right\|\\
&\le C^*\left\|u_n^\Delta-u^\Delta\right\|_{L^2_\Delta}\left(\int_{(-\infty,\infty)_{\mathbb{T}}}\left|p(t)\right|^2\Delta t\right)^{\frac{1}{2}}\\
&\quad+C^*\frac{\|v^\sigma\|}{\|v\|}\left\|u_n^\sigma-v^\sigma\right\|_{L^2_\Delta}\left(\int_{(-\infty,\infty)_{\mathbb{T}}}\left|q^\sigma(t)\right|^2\Delta t\right)^{\frac{1}{2}}\\
&\quad+C^*\frac{\|v^\sigma\|}{\|v\|}\int_{(-\infty,\infty)_{\mathbb{T}}}\left|f\left(\sigma(t),u_n^\sigma\right)-f\left(\sigma(t),u^\sigma\right)\right|\Delta t\to 0\quad\text{as } n\to\infty,
\end{aligned}
$$

which implies $\varphi'(u_n)\to\varphi'(u)$ as $n\to\infty$. □

For any $v^\sigma\in E$, the dynamic equation (1) gives

$$
\begin{aligned}
&\int_{(-\infty,\infty)_{\mathbb{T}}}\left(p(t)u^\Delta(t)\right)^\Delta v^\sigma\,\Delta t+\int_{(-\infty,\infty)_{\mathbb{T}}}q^\sigma(t)u^\sigma(t)v^\sigma\,\Delta t\\
&\quad-\int_{(-\infty,\infty)_{\mathbb{T}}}f\left(\sigma(t),u^\sigma(t)\right)v^\sigma\,\Delta t\\
&=\int_{(-\infty,\infty)_{\mathbb{T}}}\left(-p(t)u^\Delta(t)v^\sigma+q^\sigma(t)u^\sigma(t)v^\sigma\right)\Delta t-\int_{(-\infty,\infty)_{\mathbb{T}}}f\left(\sigma(t),u^\sigma\right)v^\sigma\,\Delta t\\
&=0.
\end{aligned}
$$

So, finding the homoclinic solutions to the zero of dynamic equation (1) is equivalent to finding the critical points of the associated functional φ defined in (12).

4 Main results

In this section, we state the results of the existence of nontrivial homoclinic orbits of the dynamic equation (1) on time scales. As an elementary illustration, two examples are given to show the usefulness of these criteria.

Theorem 1 *If conditions* (H_0), (H_1), (H_2) *and* (H_3) *are satisfied, then the dynamic equation* (1) *has one nontrivial homoclinic orbit to* 0 *such that*

$$0 < \int_{(-\infty,\infty)_{\mathbb{T}}} \left[\frac{1}{2}\left(p(t)\left(u^{\Delta}(t)\right)^2 - q^{\sigma}(t)\left(u^{\sigma}(t)\right)^2\right) + F\left(\sigma(t), u^{\sigma}\right)\right]\Delta t < +\infty.$$

Example 2 Let

$$\mathbb{T} = \{0, 5, 121, 131, 143, 150, 162, 173, 180, 190\} \cup [190.5, +\infty) \cup (-\infty, -190.5).$$

Consider the following second order boundary value problem on time scales $\mathbb{T}$ of the form

$$\begin{cases} (3t^2 u^{\Delta}(t))^{\Delta} - (t^{\sigma})^2 u^{\sigma} = -\frac{1}{2}\sigma(t)(u^{\sigma}(t))^3, & \Delta\text{-a.e. } t \in \mathbb{T}, \\ u(\pm\infty) = u^{\Delta}(\pm\infty) = 0. \end{cases} \tag{14}$$

Since $\int_0^x f(t,s)\,ds = -\frac{t}{8}x^4$, one can check that all conditions of Theorem 1 are fulfilled. It follows from Theorem 1 that the dynamic equation (1) has one nontrivial homoclinic orbit to 0.

Theorem 3 *If conditions* (H_0), (H_1), (H_2), (H_3) *and the following condition are satisfied*

(H_4) $f(t,-x) = -f(t,x)$ *for all* $x \in \mathbb{R}$ *and* Δ*-a.e.* $t \in \mathbb{T}$,

then the dynamic equation (1) *has an unbounded sequence in* E *of a homoclinic orbit to* 0.

Example 4 Let $a, b > 0$ be real numbers,

$$P_1 = \bigcup_{k=0}^{\infty}\left[k(a+b), k(a+b)+a\right],$$

and

$$P_2 = \bigcup_{k=0}^{\infty}\left[-k(a+b)-a, -k(a+b)\right].$$

Consider the following second order boundary value problem on time scales $P_1 \cup P_2$ of the form

$$\begin{cases} (3t^4 u^{\Delta}(t))^{\Delta} - |t^{\sigma}| u^{\sigma} = -\frac{1}{2}\sigma(t)(u^{\sigma}(t))^5, & \Delta\text{-a.e. } t \in P_1 \cup P_2, \\ u(\pm\infty) = u^{\Delta}(\pm\infty) = 0. \end{cases} \tag{15}$$

Since $\int_0^x f(t,s)\,ds = -\frac{t}{12}x^6$, one can check that all conditions of Theorem 3 are fulfilled. It follows from Theorem 3 that the dynamic equation (1) has an unbounded sequence in E of a homoclinic orbit to 0.

5 Proof of theorems

In this section, we show our main results on the existence of nontrivial homoclinic orbits of the dynamic equation (1) on time scales.

Proof of Theorem 1 Since we have already known that $\varphi \in C^1(E,\mathbb{R})$ and $\varphi(0) = 0$, in the following we prove that all the other conditions of Lemma 1 are fulfilled with respect to the functional φ.

Firstly, we claim that φ satisfies the PS condition.

Assume that there exist a sequence $\{u_n\} \subset E$ and a constant c such that

$$\varphi'(u_n) \to 0 \quad \text{as } n \to \infty \quad \text{and} \quad \varphi(u_n) \le c, \quad n = 1,2,\ldots, \tag{16}$$

we show that $\{u_n\}$ has a convergent subsequence in E.

It follows from (16) and (H_2) that there is a constant $d \ge 0$ such that

$$\begin{aligned} d + \|u_n\|_E &\ge \varphi(u_n) - \frac{1}{\beta}\varphi'(u_n)u_n \\ &= \left(\frac{1}{2} - \frac{1}{\beta}\right)\|u\|_E^2 + \int_{(-\infty,\infty)_{\mathbb{T}}} \bigl(F\bigl(\sigma(t),u^\sigma\bigr) - f\bigl(\sigma(t),u^\sigma\bigr)v^\sigma\bigr)\Delta t \\ &\ge \left(\frac{1}{2} - \frac{1}{\beta}\right)\|u\|_E^2, \end{aligned}$$

which implies that $\{u_n\}$ is bounded in E. Hence, there is a subsequence (still denoted by $\{u_n\}$, $u_n \rightharpoonup u_0$ in E). It follows from Lemma 4 that $u_n \to u_0$ in $L^2_\Delta((-\infty,\infty)_{\mathbb{T}},\mathbb{R})$. Now, according to (H_0), $u_n, u_0 \in E$, for any $\varepsilon > 0$, we have that there exist constants $\delta_1 > 0$, $\delta_2 > 0$ and $L \in \mathbb{T}$ such that

$$|u_n| < \delta_1, \qquad |u_0| < \delta_2 \quad \text{and} \quad \|u_n - u_0\|_{L^2_\Delta} < \varepsilon \quad \text{for } \Delta\text{-a.e. } |t| > L, \tag{17}$$

which implies that

$$\bigl|f\bigl(\sigma(t),u_n^\sigma\bigr)\bigr| \le \varepsilon\bigl|u_n^\sigma\bigr| \quad \text{and} \quad \bigl|f\bigl(\sigma(t),u_0^\sigma\bigr)\bigr| \le \varepsilon\bigl|u_0^\sigma\bigr| \quad \text{for } \Delta\text{-a.e. } |t| > L. \tag{18}$$

Since

$$\begin{aligned} &\int_{(-\infty,\infty)_{\mathbb{T}}} \bigl(f\bigl(\sigma(t),u_n^\sigma\bigr) - f\bigl(\sigma(t),u_0^\sigma\bigr)\bigr)(u_n - u_0)\Delta t \\ &\quad = \int_{[-L,L]_{\mathbb{T}}} \bigl(f\bigl(\sigma(t),u_n^\sigma\bigr) - f\bigl(\sigma(t),u_0^\sigma\bigr)\bigr)(u_n - u_0)\Delta t \\ &\qquad + \int_{(-\infty,-L)_{\mathbb{T}}} \bigl(f\bigl(\sigma(t),u_n^\sigma\bigr) - f\bigl(\sigma(t),u_0^\sigma\bigr)\bigr)(u_n - u_0)\Delta t \\ &\qquad + \int_{(L,\infty)_{\mathbb{T}}} \bigl(f\bigl(\sigma(t),u_n^\sigma\bigr) - f\bigl(\sigma(t),u_0^\sigma\bigr)\bigr)(u_n - u_0)\Delta t, \end{aligned} \tag{19}$$

let

$$L^2_{\Delta,\mathrm{loc}}(\mathbb{T},\mathbb{R}) = \bigl\{\varpi : \mathbb{T} \to \mathbb{R} \mid \text{for arbitrary compact interval } K \subset \mathbb{T}, \varpi_{I_K} \in L^2_\Delta(\mathbb{T},\mathbb{R})\bigr\},$$

where I_K is an indicator function of interval K and

$$\varpi_{I_K} = \begin{cases} \varpi(x), & x \in K, \\ 0, & x \notin K. \end{cases}$$

It follows from the uniform continuity of $f(t,x)$ in x and $u_n \to u_0$ in $L^2_{\Delta,\mathrm{loc}}(\mathbb{T},\mathbb{R}^n)$ that

$$\int_{[-L,L]_{\mathbb{T}}} \bigl(f\bigl(\sigma(t),u_n^\sigma\bigr) - f\bigl(\sigma(t),u_0^\sigma\bigr)\bigr)(u_n - u_0)\Delta t \to 0 \quad \text{as } n \to \infty.$$

Combining Hölder's inequality on time scales, (17) and (18) leads to

$$\begin{aligned} &\left|\int_{(-\infty,-L)_{\mathbb{T}}} \bigl(f\bigl(\sigma(t),u_n^\sigma\bigr) - f\bigl(\sigma(t),u_0^\sigma\bigr)\bigr)(u_n - u_0)\Delta t\right| \\ &\quad \le \left(\int_{(-\infty,-L)_{\mathbb{T}}} \bigl|f\bigl(\sigma(t),u_n^\sigma\bigr) - f\bigl(\sigma(t),u_0^\sigma\bigr)\bigr|^2\Delta t\right)^{\frac{1}{2}} \left(\int_{(-\infty,-L)_{\mathbb{T}}} (u_n - u_0)^2\Delta t\right)^{\frac{1}{2}} \\ &\quad \le \varepsilon^2 \left(\int_{(-\infty,-L)_{\mathbb{T}}} \bigl(\bigl|u_n^\sigma\bigr| + \bigl|u_0^\sigma\bigr|\bigr)^2\Delta t\right)^{\frac{1}{2}} \\ &\quad \le \varepsilon^2 M_1. \end{aligned}$$

By using the same technique, we obtain

$$\left|\int_{(L,\infty)_{\mathbb{T}}} \bigl(f\bigl(\sigma(t),u_n^\sigma\bigr) - f\bigl(\sigma(t),u_0^\sigma\bigr)\bigr)(u_n - u_0)\Delta t\right| \le \varepsilon^2 M_2,$$

where M_1, M_2 depend on the bounds for u_n and u_0 in E. Then

$$\int_{(-\infty,\infty)_{\mathbb{T}}} \bigl(f\bigl(\sigma(t),u_n^\sigma\bigr) - f\bigl(\sigma(t),u_0^\sigma\bigr)\bigr)(u_n - u_0)\Delta t \to 0 \quad \text{as } n \to \infty \tag{20}$$

since

$$\begin{aligned} &\bigl(\varphi'(u_n) - \varphi'(u_0)\bigr)(u_k - u_0) \\ &\quad = \|u_n - u_0\|_E^2 - \int_{(-\infty,\infty)_{\mathbb{T}}} \bigl(f\bigl(\sigma(t),u_n^\sigma\bigr) - f\bigl(\sigma(t),u_0^\sigma\bigr)\bigr)(u_n - u_0)\Delta t. \end{aligned} \tag{21}$$

Equations (20) and (21) imply that $u_n \to u_0$ in E. Consequently, φ satisfies the PS condition.

Secondly, we prove that there exist constants ϱ and $\alpha > 0$ such that φ satisfies the assumption (iii) of Lemma 1.

It follows from Lemma 4 that there exists $\alpha_0 > 0$ such that

$$\|u\|_{L^2_\Delta} \le \alpha_0 \|u\|_E \quad \text{for } u \in E.$$

On the other hand, according to (H_2) and (H_3), we have that there exists $\alpha_1 > 0$ such that

$$\|u\|_\infty \le \alpha_1 \|u\|_E,$$

where

$$\|u\|_\infty = \max_{t \in (-\infty,\infty)_{\mathbb{T}}} \bigl|u(t)\bigr|.$$

(H_0) implies that there is $\delta > 0$ such that

$$\left|F(t,x)\right| \leq \varepsilon|x|^2 \quad \text{for } |x| \leq \delta.$$

Let $\rho = \frac{\delta}{\alpha_1}$ and $\|u\|_E \leq \rho$, we have $\|u\|_\infty \leq \frac{\delta}{\alpha_1}\alpha_1 = \delta$, then

$$\left|F\left(t,u^\sigma\right)\right| \leq \varepsilon\left|u^\sigma\right|^2 \quad \text{for } \left|u^\sigma\right| \leq \delta \text{ and } \Delta\text{-a.e. } t \in \mathbb{T},$$

which implies that

$$\int_{(-\infty,\infty)_{\mathbb{T}}} F\left(t,u^\sigma\right)\Delta t \geq -\varepsilon\|u\|^2_{L^2_\Delta} \geq -\varepsilon\alpha_0^2\|u\|^2_E.$$

Hence, if $\|u\|_E = \rho$, we have

$$\begin{aligned} \varphi(u) &= \frac{1}{2}\|u\|^2_E + \int_{(-\infty,\infty)_{\mathbb{T}}} F\left(\sigma(t),u^\sigma\right)\Delta t \\ &\geq \frac{1}{2}\|u\|^2_E - \varepsilon\alpha_0^2\|u\|^2_E = \left(\frac{1}{2} - \varepsilon\alpha_0^2\right)\rho^2. \end{aligned}$$

Choosing $\varepsilon = \frac{1}{4}\alpha_0^2$, we have

$$\varphi(u) \geq \frac{1}{4}\rho^2 = \alpha > 0.$$

Thirdly, we claim that there exists $e \in X \setminus \bar{B}_\rho(0)$ such that φ satisfies the assumption (iv) of Lemma 1.

Let $\overline{u} \in E$ be such that $|\overline{u}(t)| \geq 1$, for any $\sigma \geq 1$, it follows from (3) that

$$\begin{aligned} \varphi(\sigma\overline{u}) &= \frac{\sigma^2}{2}\|\overline{u}\|^2_E + \int_{(-\infty,\infty)_{\mathbb{T}}} F\left(\sigma(t),\sigma\overline{u}^\sigma\right)\Delta t \\ &\leq \frac{\sigma^2}{2}\|\overline{u}\|^2_E - \int_{(-\infty,\infty)_{\mathbb{T}}} \left|\sigma\overline{u}^\sigma\right|^\beta \alpha_0(t)\Delta t \\ &= \frac{\sigma^2}{2}\|\overline{u}\|^2_E - |\sigma|^\beta \int_{(-\infty,\infty)_{\mathbb{T}}} \left|\overline{u}^\sigma\right|^\beta \alpha_0(t)\Delta t, \end{aligned}$$

which implies that there exists $\sigma \geq 1$ such that $\|\sigma\overline{u}\| > \rho$ and $\varphi(\sigma\overline{u}) \leq 0 = \varphi(0)$.

Hence, all the conditions of Lemma 1 are satisfied, the desired results follow. □

Proof of Theorem 3 It follows from (H_4) that φ is even. In addition, we have already proved that $\varphi \in C^1(E,\mathbb{T})$, $\varphi(0) = 0$ and φ satisfies the Palais-Smale condition. We prove that all the other conditions of the symmetric mountain pass theorem are satisfied with respect to the functional φ. We have already showed that φ satisfies condition (iii) of the symmetric mountain pass theorem in the proof of Theorem 3.

In the following, we claim that φ satisfies condition (iv) of the symmetric mountain pass theorem.

Let $\widetilde{E} \subset E$ be a finite-dimensional subspace. Consider $u \in \widetilde{E} \subset E$ with $u \neq 0$. It follows from (3) that

$$\int_{(1,\infty)_{\mathbb{T}}} F(t, u^{\sigma}) \Delta t \leq -\int_{(1,\infty)_{\mathbb{T}}} \alpha(t) |u(t)|^{\beta} \Delta t,$$

and

$$\int_{(-\infty,-1)_{\mathbb{T}}} F(t, u^{\sigma}) \Delta t \leq -\int_{(-\infty,-1)_{\mathbb{T}}} \alpha(t) |u(t)|^{\beta} \Delta t.$$

We also have

$$\|u\|_E^2 \leq c\|u\|_\infty^2 \quad \text{for } u \in \widetilde{E},$$

where $c = c(\widetilde{E})$.

Define $m = \inf_{\|u\|_\infty=2}(\int_{(1,\infty)_{\mathbb{T}}} \alpha(t)|u(t)|^{\beta} \Delta t + \int_{(-\infty,-1)_{\mathbb{T}}} \alpha(t)|u(t)|^{\beta} \Delta t)$, if $m = 0$, we have $\|u\| = 0$ for Δ-a.e. $t \in \{t \mid |u(t)| > 1\}$, which contradicts $\|u\|_\infty = 2$, then $m > 0$, and we have

$$\begin{aligned}
\varphi(u) &\leq \frac{1}{2}c\|u\|_\infty^2 + \int_{(-\infty,1)_{\mathbb{T}}} F(\sigma(t), u^{\sigma}) \Delta t \\
&\quad + \int_{(1,\infty)_{\mathbb{T}}} F(\sigma(t), u^{\sigma}) \Delta t + \int_{[-1,1]_{\mathbb{T}}} F(\sigma(t), u^{\sigma}) \Delta t \\
&\leq \frac{1}{2}c\|u\|_\infty^2 + \int_{(-\infty,1)_{\mathbb{T}}} F(\sigma(t), u^{\sigma}) \Delta t + \int_{(1,\infty)_{\mathbb{T}}} F(\sigma(t), u^{\sigma}) \Delta t \\
&\leq \frac{1}{2}c\|u\|_\infty^2 - \int_{(-\infty,1)_{\mathbb{T}}} \alpha(t) |u(t)|^{\beta} \Delta t - \int_{(1,\infty)_{\mathbb{T}}} \alpha(t) |u(t)|^{\beta} \Delta t \\
&= \frac{1}{2}c\|u\|_\infty^2 - \frac{1}{2^{\beta}}\|u\|_\infty^{\beta} \left(\int_{(-\infty,1)_{\mathbb{T}}} \alpha(t) \left(\frac{2|u(t)|}{\|u\|_\infty} \right)^{\beta} \Delta t \right. \\
&\quad \left. + \int_{(1,\infty)_{\mathbb{T}}} \alpha(t) \left(\frac{2|u(t)|}{\|u\|_\infty} \right)^{\beta} \Delta t \right) \\
&\leq \frac{1}{2}c\|u\|_\infty^2 - \frac{m}{2^{\beta}}\|u\|_\infty^{\beta}.
\end{aligned}$$

Since $\beta > 2$, there exists a constant C_1 such that $\varphi(u) \leq 0$ if $\|u\|_\infty \geq C$.

Consequently, it follows from Lemma 2 that the functional φ possesses an unbounded sequence of critical values $\{c_j\}$ with $c_j = \varphi(u_j)$, where u_j satisfies

$$0 = \varphi'(u_j)u_j = \|u_j\|_E^2 + \int_{(-\infty,\infty)_{\mathbb{T}}} f(\sigma(t), u_j^{\sigma}) u_j \Delta t,$$

which implies that

$$-\|u_j\|_E^2 = \int_{(-\infty,\infty)_{\mathbb{T}}} f(\sigma(t), u_j^{\sigma}) u_j \Delta t.$$

(H_1) implies that

$$\begin{aligned} c_j &= -\frac{1}{2}\int_{(-\infty,\infty)_{\mathbb{T}}} f\big(\sigma(t),u_j^{\sigma}\big)u_j\Delta t + \int_{(-\infty,\infty)_{\mathbb{T}}} F\big(\sigma(t),u_j^{\sigma}\big)\Delta t \\ &\leq -\frac{1}{2}\int_{(-\infty,\infty)_{\mathbb{T}}} f\big(\sigma(t),u_j^{\sigma}\big)u_j\Delta t = \frac{1}{2}\|u_j\|_E^2. \end{aligned}$$

Then $\{u_j\}$ is unbounded in E because of $c_j \to \infty$ as $j \to \infty$. The proof is completed. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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Acknowledgements
This work is partially supported by the Natural Science Foundation of China (Nos. 11361047, 11501560), the Natural Science Foundation of JiangSu Province (No. BK20151160), the Six Talent Peaks Project of Jiangsu Province (2013-JY-003) and 333 High-Level Talents Training Program of Jiangsu Province (BRA2016275).

Received: 4 January 2017 Accepted: 30 January 2017 Published online: 08 February 2017

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