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Existence of nonoscillatory solutions for system of higher-order neutral differential equations with distributed coefficients and delays

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Abstract

In this paper we consider the existence of nonoscillatory solutions for a system of higher-order neutral differential equations with distributed coefficients and delays. We use the *Banach* contraction principle to obtain new sufficient conditions for the existence of nonoscillatory solutions.

Keywords: system; higher-order; distributed coefficients and delays; nonoscillatory solutions; Banach contraction principle

1 Introduction and preliminary

In this paper, we consider the system of higher-order neutral differential equations with distributed coefficients and delays

$$\begin{aligned}&\left[r(t)\mathbf{x}(t)+\int_a^b p(t,\theta)\mathbf{x}(t-\theta)\,d\theta\right]^{(n)}\\&\quad+(-1)^{n+1}\left[\int_c^d Q_1(t,\tau)\mathbf{x}(t-\tau)\,d\tau-\int_e^f Q_2(t,\sigma)\mathbf{x}(t-\sigma)\,d\sigma-\mathbf{h}(t)\right]=\mathbf{0},\end{aligned}\tag{1}$$

(1) where n is a positive integer, $n \geq 1, 0 < a < b, 0 < c < d, 0 < e < f$;
(2) $r \in C([t_0,\infty), R^+), r(t) > 0, p \in C([t_0,\infty) \times [a,b], R), \mathbf{h} \in C([t_0,\infty), \mathbf{R})$,
(3) $\mathbf{x} \in \mathbf{R}^n, Q_i$ is continuous $n \times n$ matrix on $[t_0,\infty), i = 1,2$.

Recently there have been a lot of activities concerning the existence of nonoscillatory solutions for neutral differential equations with positive and negative coefficients. In 2005, the existence of nonoscillatory solutions of the first-order linear neutral delay differential equations

$$\frac{d}{dt}\big[x(t)+P(t)x(t-\tau)\big]+Q_1(t)x(t-\sigma_1)-Q_2(t)x(t-\sigma_2)=0$$

was investigated by Zhang *et al.* [1]. In 2012, Candan [2] studied the higher-order nonlinear differential equation

$$\bigl[r(t)\bigl[x(t)+P(t)x(t-\tau)\bigr]^{(n-1)}\bigr]'+(-1)^{n+1}\bigl[Q_1(t)g_1\bigl(x(t-\sigma_1)\bigr)-Q_2(t)g_2\bigl(x(t-\mu)\bigr)-f(t)\bigr]=0.$$

In 2013, Candan [3] has investigated the existence of nonoscillatory solutions for the system of higher-order nonlinear neutral differential equations

$$\bigl[\mathbf{x}(t)+P(t)\mathbf{x}(t-\theta)\bigr]^{(n)}+(-1)^{n+1}\bigl[Q_1(t)\mathbf{x}(t-\sigma_1)-Q_2(t)\mathbf{x}(t-\sigma_2)\bigr]=\mathbf{0}.$$

In the same year, Liu *et al.* [4] has obtained the existence of nonoscillatory solutions for the system of higher-order neutral differential equations

$$\begin{aligned}&\bigl[r(t)\bigl[\mathbf{x}(t)+P(t)\mathbf{x}(t-\theta)\bigr]^{(n-1)}\bigr]'\\&\quad+(-1)^{n}\left(\int_c^d Q_1(t,\tau)\mathbf{x}(t-\tau)\,d\tau-\int_e^f Q_2(t,\sigma)\mathbf{x}(t-\sigma)\,d\sigma\right)=\mathbf{0}.\end{aligned}$$

As can be seen from the development process of the above equations, the delay of neutral part in the discussed differential equations were all constant delays. However, the case for distributed deviating arguments is rather rare; see [5, 6]. In 2015, Candan and Gecgel [6] studied the systems of higher-order neutral differential equations with distributed delay

$$\begin{aligned}&\left[\left[x(t)+\int_{a_3}^{b_3}\tilde{P}(t,\xi)x(t-\xi)\,d\xi\right]\right]'\\&\quad+(-1)^{n+1}\left[\int_{a_1}^{b_1}Q_1(t,\xi)\mathbf{x}(t-\xi)\,d\xi-\int_{a_2}^{b_2}Q_2(t,\xi)\mathbf{x}(t-\xi)\,d\xi\right]=\mathbf{0},\end{aligned}\tag{2}$$

the discussion only covered the condition for the coefficient being $0<\int_{a_3}^{b_3}\tilde{P}(t,\xi)x(t-\xi)\,d\xi<\frac{1}{2}$ and $-\frac{1}{2}<\int_{a_3}^{b_3}\tilde{P}(t,\xi)x(t-\xi)\,d\xi<0$. However, in this paper, the difficulty in establishing a feasible operator was settled by skillful use of $r(t)$, and the coefficients $\int_a^b p_2(t,\xi)\,d\xi$ in the neutral part were all discussed in four cases, that is, $(-\infty,-1),(-1,0),(0,1),(1,+\infty)$. Thus, in view of the above, this paper may have theoretical value as well as practical application value. For related work, we refer the reader to [7–10].

A solution of the system of equations (1) is a continuous vector function $\mathbf{x}(t)$ defined on $([t_1-\mu,\infty),\mathbf{R}^n)$, for some $t_1>t_0$, such that $r(t)\mathbf{x}(t)-\int_a^b p(t,\theta)\mathbf{x}(t-\theta)\,d\theta$ is n times continuously differentiable and the system of equations (1) holds for all $n\geq 1$. Here, $\mu=\max\{b,\tau,\sigma\}$.

2 The main results

Theorem 1 *Assume that* $0\leq\int_a^b p(t,\theta)\,d\theta\leq p_1<1$ *and*

$$\begin{aligned}&\int_{t_0}^{\infty}s^{n-1}\left\|\int_c^d Q_1(t,\tau)\,d\tau\right\|ds<\infty,\qquad\int_{t_0}^{\infty}s^{n-1}\left\|\int_e^f Q_2(t,\sigma)\,d\sigma\right\|ds<\infty,\\&\int_{t_0}^{\infty}s^{n-1}\bigl\|\mathbf{h}(s)\bigr\|\,ds<\infty.\end{aligned}\tag{3}$$

Then equation (1) *has a bounded nonoscillatory solution.*

Proof Let Λ be the set of all continuous and bounded vector functions on $[t_0,\infty)$ with the sup norm. Set $A = \{x \in \Lambda, M_1 \le \|\mathbf{x}(t)\| \le M_2, t \ge t_0\}$, where M_1, M_2 are two positive constants and $\mathbf{c}$ is a constant vector, such that $p_1M_2 + \frac{M_1}{p_1} < \|\mathbf{c}\| < M_2, 1 \le r(t) \le \frac{1}{p_1}$. From (3), one can choose a $t_1 \ge t_0$, $t_1 \ge t_0 + \mu$, sufficiently large, $t \ge t_1$, such that

$$\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\left[M_2\left\|\int_c^d Q_1(t,\tau)\,d\tau\right\| + \left\|\mathbf{h}(s)\right\|\right]ds \le M_2 - \|\mathbf{c}\|, \tag{4}$$

$$\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\left[M_2\left\|\int_e^f Q_2(s,\sigma)\,d\sigma\right\| + \left\|\mathbf{h}(s)\right\|\right]ds \le \|\mathbf{c}\| - p_1M_2 + \frac{M_1}{p_1}, \tag{5}$$

$$\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\left[\left\|\int_c^d Q_1(t,\tau)\,d\tau\right\| + \left\|\int_e^f Q_2(s,\sigma)\,d\sigma\right\|\right]ds \le 1 - p_1, \tag{6}$$

and one defines an operator T on A as follows:

$$(T\mathbf{x})(t) = \begin{cases} \frac{1}{r(t)}\{\mathbf{c} - \int_a^b p(t,\theta)\mathbf{x}(t-\theta)\,d\theta + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}[\int_c^d Q_1(s,\tau)\mathbf{x}(s-\tau)\,d\tau \\ \quad - \int_e^f Q_2(s,\sigma)\mathbf{x}(s-\sigma)\,d\sigma - \mathbf{h}(s)]\,ds\} \quad t \ge t_1, \\ (T\mathbf{x})(t_1) \quad t_0 \le t \le t_1. \end{cases}$$

It is easy to see that $T\mathbf{x}$ is continuous, for $t \ge t_1, \mathbf{x} \in A$, by using (4), we have

$$\begin{aligned} \left\|(T\mathbf{x})(t)\right\| &\le \frac{1}{r(t)}\left\{\|\mathbf{c}\| + \left\|\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\left[\int_c^d Q_1(s,\tau)\mathbf{x}(s-\tau)\,d\tau - \mathbf{h}(s)\right]ds\right\|\right\} \\ &\le \|\mathbf{c}\| + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\left[M_2\left\|\int_c^d Q_1(t,\tau)\,d\tau\right\| + \left\|\mathbf{h}(s)\right\|\right]ds \\ &\le M_2, \end{aligned}$$

and taking (5) into account, we have

$$\begin{aligned} \left\|(T\mathbf{x})(t)\right\| &\ge \frac{1}{r(t)}\left\{\|\mathbf{c}\| - \int_a^b p(t,\theta)\left\|\mathbf{x}(t-\theta)\right\|d\theta \right. \\ &\quad \left. - \left\|\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\left[\int_e^f Q_2(s,\sigma)\mathbf{x}(s-\sigma)\,d\sigma - \mathbf{h}(s)\right]ds\right\|\right\} \\ &\ge p_1\left\{\|\mathbf{c}\| - p_1M_2 - \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\left(M_1\left\|\int_e^f Q_2(s,\sigma)\,d\sigma\right\| + \left\|\mathbf{h}(s)\right\|\right)ds\right\} \\ &\ge M_1. \end{aligned}$$

These show that $TA \subset A$, since A is a bounded, closed, and convex subset of Λ, in order to apply the contraction principle we have to show that T is a contraction mapping on A. For $\forall \mathbf{x}_1, \mathbf{x}_2 \in A$, and $t \ge t_1$,

$$\begin{aligned} &\left\|(T\mathbf{x}_1)(t) - (T\mathbf{x}_2)(t)\right\| \\ &\quad \le \frac{1}{r(t)}\left\{\int_a^b p(t,\theta)\left\|\mathbf{x}_1(t-\theta) - \mathbf{x}_2(t-\theta)\right\|d\theta \right. \\ &\qquad + \left\|\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\left[\int_c^d Q_1(s,\tau)\mathbf{x}_1(s-\tau)\,d\tau - \int_e^f Q_2(s,\sigma)\mathbf{x}_1(s-\sigma)\,d\sigma - \mathbf{h}(s)\right]ds\right\| \end{aligned}$$

$$
\begin{aligned}
&- \left\| \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[\int_c^d Q_1(s,\tau)\mathbf{x}_2(s-\tau)\,d\tau - \int_e^f Q_2(s,\sigma)\mathbf{x}_2(s-\sigma)\,d\sigma - \mathbf{h}(s) \right] ds \right\| \Bigg\} \\
&\le \frac{1}{r(t)} \Bigg\{ p_1 \|\mathbf{x}_1 - \mathbf{x}_2\| + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \Bigg[\left\| \int_c^d Q_1(t,\tau)\,d\tau \right\| \big\| \mathbf{x}_1(s-\tau) - \mathbf{x}_2(s-\tau) \big\| \\
&\quad + \left\| \int_e^f Q_2(s,\sigma)\,d\sigma \right\| \big\| \mathbf{x}_1(s-\sigma) - \mathbf{x}_2(s-\sigma) \big\| \Bigg] ds \Bigg\}.
\end{aligned}
$$

Using (6),

$$
\begin{aligned}
&\big\| (T\mathbf{x}_1)(t) - (T\mathbf{x}_2)(t) \big\| \\
&\le \|\mathbf{x}_1 - \mathbf{x}_2\| \left(p_1 + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[\left\| \int_c^d Q_1(t,\tau)\,d\tau \right\| + \left\| \int_e^f Q_2(s,\sigma)\,d\sigma \right\| \right] ds \right) \\
&< \|\mathbf{x}_1 - \mathbf{x}_2\|.
\end{aligned}
$$

which shows that T is a contraction mapping on A and therefore there exists a unique solution, obviously a bounded positive solution of (1) $\mathbf{x} \in A$, such that $T\mathbf{x} = \mathbf{x}$. The proof is complete. □

Theorem 2 *Assume that* $1 < p_3 \le \int_a^b p(t,\theta)\,d\theta \le p_2 < 2p_3 < +\infty$, *and that* (3) *holds. Then equation* (1) *has a bounded nonoscillatory solution.*

Proof Let Λ be the set of all continuous and bounded vector functions on $[t_0,\infty)$ with the sup norm. Set $A = \{x \in \Lambda, M_3 \le \|\mathbf{x}(t)\| \le M_4, t \ge t_0\}$, where M_3, M_4 are two positive constants such that $p_2M_4 + 2p_2M_3 < \|\mathbf{c}\| < 2p_3M_4, 2p_3 \le r(t) \le 2p_2$. From (3), one can choose a $t_1 \ge t_0 + b$, sufficiently large $t \ge t_1$, such that

$$
\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[M_4 \left\| \int_c^d Q_1(s,\tau)\,d\tau \right\| + \big\| \mathbf{h}(s) \big\| \right] ds \le 2p_3M_4 - \|\mathbf{c}\|, \tag{7}
$$

$$
\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[M_4 \left\| \int_e^f Q_2(s,\sigma)\,d\sigma \right\| + \big\| \mathbf{h}(s) \big\| \right] ds \le \|\mathbf{c}\| - p_2M_4 - 2p_2M_3, \tag{8}
$$

$$
\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[\left\| \int_c^d Q_1(s,\tau)\,d\tau \right\| + \left\| \int_e^f Q_2(s,\sigma)\,d\sigma \right\| \right] ds \le 2p_3 - p_2, \tag{9}
$$

and one defines an operator T on A as follows:

$$
(Tx)(t) = \begin{cases} \frac{1}{r(t)} \{ \mathbf{c} - \int_a^b p(t,\theta)\mathbf{x}(t-\theta)\,d\theta + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} [\int_c^d Q_1(s,\tau)\mathbf{x}(s-\tau)\,d\tau \\ \quad - \int_e^f Q_2(s,\sigma)\mathbf{x}(s-\sigma)\,d\sigma - \mathbf{h}(s)]\,ds \} & t \ge t_1, \\ (T\mathbf{x})(t_1) & t_0 \le t \le t_1. \end{cases}
$$

It is easy to see that T is continuous, for $t \ge t_1, \mathbf{x} \in A$. By using (7), we have

$$
\begin{aligned}
\big\| (T\mathbf{x})(t) \big\| &\le \frac{1}{r(t)} \left\{ \|\mathbf{c}\| + \left\| \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[\int_c^d Q_1(s,\tau)\mathbf{x}(s-\tau)\,d\tau - \mathbf{h}(s) \right] ds \right\| \right\} \\
&\le \frac{1}{2p_3} \left\{ \|\mathbf{c}\| + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[M_4 \left\| \int_c^d Q_1(s,\tau)\,d\tau \right\| + \big\| \mathbf{h}(s) \big\| \right] ds \right\} \le M_4,
\end{aligned}
$$

and taking (8) into account, we have

$$
\begin{aligned}
\left\|(T\mathbf{x})(t)\right\| &\geq \frac{1}{r(t)}\left\{\|\mathbf{c}\| - \int_a^b p(t,\theta)\left\|\mathbf{x}(t-\theta)\right\|d\theta\right.\\
&\quad\left. + \left\|\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\left[\int_e^f Q_2(s,\sigma)\mathbf{x}(s-\sigma)\,d\sigma - \mathbf{h}(s)\right]ds\right\|\right\}\\
&\geq \frac{1}{2p_2}\left\{\|\mathbf{c}\| - p_2M_4 - \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\left(M_4\left\|\int_e^f Q_2(s,\sigma)\,d\sigma\right\| + \left\|\mathbf{h}(s)\right\|\right)ds\right\}\\
&\geq M_3.
\end{aligned}
$$

These show that $TA \subset A$, since A is a bounded, closed, and convex subset of Λ, in order to apply the contraction principle, we have to show that T is a contraction mapping on A. For $\forall \mathbf{x}_1, \mathbf{x}_2 \in A$, and $t \geq t_1$,

$$
\begin{aligned}
&\left\|(T\mathbf{x}_1)(t) - (T\mathbf{x}_2)(t)\right\|\\
&\quad\leq \frac{1}{r(t)}\left\{\int_a^b p(t,\theta)\left\|\mathbf{x}_1(t-\theta) - \mathbf{x}_2(t-\theta)\right\|d\theta\right.\\
&\qquad + \left\|\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\left[\int_c^d Q_1(s,\tau)\mathbf{x}_1(s-\tau)\,d\tau - \int_e^f Q_2(s,\sigma)\mathbf{x}_1(s-\sigma)\,d\sigma - \mathbf{h}(s)\right]ds\right\|\\
&\qquad\left. - \left\|\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\left[\int_c^d Q_1(s,\tau)\mathbf{x}_2(s-\tau)\,d\tau - \int_e^f Q_2(s,\sigma)\mathbf{x}_2(s-\sigma)\,d\sigma - \mathbf{h}(s)\right]ds\right\|\right\}\\
&\quad\leq \frac{1}{r(t)}\left\{p_2\|\mathbf{x}_1 - \mathbf{x}_2\| + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\left[\left\|\int_c^d Q_1(s,\tau)\,d\tau\right\|\left\|\mathbf{x}_1(s-\tau) - \mathbf{x}_2(s-\tau)\right\|\right.\right.\\
&\qquad\left.\left. + \left\|\int_e^f Q_2(s,\sigma)\,d\sigma\right\|\left\|\mathbf{x}_1(s-\sigma) - \mathbf{x}_2(s-\sigma)\right\|\right]ds\right\},
\end{aligned}
$$

using (9),

$$
\begin{aligned}
&\left\|(T\mathbf{x}_1)(t) - (T\mathbf{x}_2)(t)\right\|\\
&\quad\leq \frac{1}{2p_3}\|\mathbf{x}_1 - \mathbf{x}_2\|\left\{p_2 + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\left[\left\|\int_c^d Q_1(s,\tau)\,d\tau\right\| + \left\|\int_e^f Q_2(s,\sigma)\,d\sigma\right\|\right]ds\right\}\\
&\quad< \|\mathbf{x}_1 - \mathbf{x}_2\|,
\end{aligned}
$$

which shows that T is a contraction mapping on A and therefore there exists a unique solution, obviously a bounded positive solution of (1) $\mathbf{x} \in A$, such that $T\mathbf{x} = \mathbf{x}$. The proof is complete. □

Theorem 3 *Assume that* $-1 < p_4 \leq \int_a^b p(t,\theta)\,d\theta \leq 0$ *and that* (2) *holds.*
Then equation (1) *has a bounded nonoscillatory solution.*

Proof Let Λ be the set of all continuous and bounded functions on $[t_0,\infty)$ with the sup norm. Set $A = \{x \in \Lambda, M_5 \leq \|\mathbf{x}(t)\| \leq M_6, t \geq t_0\}$, where M_5, M_6 are two positive constants such that $\frac{M_5}{-p_4} < \|\mathbf{c}\| < (1+p_4)M_6, 1 \leq r(t) \leq \frac{1}{-p_4}$. From (3), one can choose a $t_1 \geq t_0 + b$, sufficiently large $t \geq t_1$ such that

$$\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\left[M_6\left\|\int_c^d Q_1(s,\tau)\,d\tau\right\| + \left\|\mathbf{h}(s)\right\|\right]ds \leq (1+p_4)M_6 - \|\mathbf{c}\|, \tag{10}$$

$$\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\left[M_6\left\|\int_e^f Q_2(s,\sigma)\,d\sigma\right\| + \left\|\mathbf{h}(s)\right\|\right]ds \leq \|\mathbf{c}\| + \frac{M_5}{p_4}, \tag{11}$$

$$\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\left[\left\|\int_c^d Q_1(s,\tau)\,d\tau\right\| + \left\|\int_e^f Q_2(s,\sigma)\,d\sigma\right\|\right]ds \leq 1 + p_4, \tag{12}$$

and one defines an operator T on A as follows:

$$(T\mathbf{x})(t) = \begin{cases} \frac{1}{r(t)}\{\mathbf{c} - \int_a^b p(t,\theta)\mathbf{x}(t-\theta)\,d\theta + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}[\int_c^d Q_1(s,\tau)\mathbf{x}(s-\tau)\,d\tau \\ \quad - \int_e^f Q_2(s,\sigma)\mathbf{x}(s-\sigma)\,d\sigma - \mathbf{h}(s)]\,ds\} \quad t \geq t_1, \\ (T\mathbf{x})(t_1) \quad t_0 \leq t \leq t_1. \end{cases}$$

It is easy to see that T is continuous, for $t \geq t_1, \mathbf{x} \in A$. By using (10), we have

$$\begin{aligned} \left\|(T\mathbf{x})(t)\right\| &\leq \frac{1}{r(t)}\left\{\|\mathbf{c}\| - \int_a^b p(t,\theta)\left\|\mathbf{x}(t-\theta)\right\|d\theta \right. \\ &\quad \left. + \left\|\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\left[\int_c^d Q_1(s,\tau)\mathbf{x}(s-\tau)\,d\tau - \mathbf{h}(s)\right]ds\right\|\right\} \\ &\leq \|\mathbf{c}\| - p_4M_6 + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\left[M_6\left\|\int_c^d Q_1(s,\tau)\,d\tau\right\| + \left\|\mathbf{h}(s)\right\|\right]ds \\ &\leq M_6, \end{aligned}$$

and taking (11) into account, we have

$$\begin{aligned} \left\|(T\mathbf{x})(t)\right\| &\geq \frac{1}{r(t)}\left\{\|\mathbf{c}\| - \left\|\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\left[\int_e^f Q_2(s,\sigma)\mathbf{x}(s-\sigma)\,d\sigma - \mathbf{h}(s)\right]ds\right\|\right\} \\ &\geq -p_4\left\{\|\mathbf{c}\| - \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\left[M_6\left\|\int_e^f Q_2(s,\sigma)\,d\sigma\right\| + \left\|\mathbf{h}(s)\right\|\right]ds\right\} \\ &\geq M_5. \end{aligned}$$

These show that $TA \subset A$, since A is a bounded, closed, and convex subset of Λ, in order to apply the contraction principle, we have to show that T is a contraction mapping on A. For $\forall \mathbf{x}_1, \mathbf{x}_2 \in A$, and $t \geq t_1$,

$$\begin{aligned} &\left\|(T\mathbf{x}_1)(t) - (T\mathbf{x}_2)(t)\right\| \\ &\quad \leq \frac{1}{r(t)}\left\{\int_a^b p(t,\theta)\left\|\mathbf{x}_1(t-\theta) - \mathbf{x}_2(t-\theta)\right\|d\theta \right. \\ &\qquad + \left\|\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\left[\int_c^d Q_1(s,\tau)\mathbf{x}_1(s-\tau)\,d\tau - \int_e^f Q_2(s,\sigma)\mathbf{x}_1(s-\sigma)\,d\sigma - \mathbf{h}(s)\right]ds\right\| \\ &\qquad \left. - \left\|\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\left[\int_c^d Q_1(s,\tau)\mathbf{x}_2(s-\tau)\,d\tau - \int_e^f Q_2(s,\sigma)\mathbf{x}_2(s-\sigma)\,d\sigma - \mathbf{h}(s)\right]ds\right\|\right\} \end{aligned}$$

$$\le \frac{1}{r(t)}\left\{-p_4\|\mathbf{x}_1-\mathbf{x}_2\| + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\left[\left\|\int_c^d Q_1(s,\tau)\,d\tau\right\|\,\|\mathbf{x}_1(s-\tau)-\mathbf{x}_2(s-\tau)\| \right.\right.$$
$$\left.\left. + \left\|\int_e^f Q_2(s,\sigma)\,d\sigma\right\|\,\|\mathbf{x}_1(s-\sigma)-\mathbf{x}_2(s-\sigma)\|\right]ds\right\}.$$

Or using (12),

$$\begin{aligned}
&\|(T\mathbf{x}_1)(t)-(T\mathbf{x}_2)(t)\| \\
&\quad\le \|\mathbf{x}_1-\mathbf{x}_2\|\left(-p_4 + KL\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\left[\left\|\int_c^d Q_1(s,\tau)\,d\tau\right\| + \left\|\int_e^f Q_2(s,\sigma)\,d\sigma\right\|\right]ds\right) \\
&\quad< \|\mathbf{x}_1-\mathbf{x}_2\|,
\end{aligned}$$

which shows that T is a contraction mapping on A and therefore there exists a unique solution, obviously a bounded positive solution of (1) $\mathbf{x}\in A$, such that $T\mathbf{x}=\mathbf{x}$. The proof is complete. □

Theorem 4 *Assume that* $-\infty < 2p_5 < p_6 \le \int_a^b p(t,\theta)\,d\theta \le p_5 < -1$ *and that* (3) *holds. Then equation* (1) *has a bounded nonoscillatory solution.*

Proof Let Λ be the set of all continuous and bounded functions on $[t_0,\infty)$ with the sup norm. Set $A=\{\mathbf{x}\in\Lambda, M_7\le\|\mathbf{x}(t)\|\le M_8, t\ge t_0\}$, where M_7, M_8 are two positive constants such that $-2p_6M_7 < \|\mathbf{c}\| < (-2p_5+p_6)M_8$, $-2p_5 < r(t) < -2p_6$. From (3), one can choose a $t_1\ge t_0+b$, sufficiently large $t\ge t_1$, such that

$$\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\left[M_8\left\|\int_c^d Q_1(s,\tau)\,d\tau\right\| + \|\mathbf{h}(s)\|\right]ds \le -2p_5M_8 + p_6M_8 - \|\mathbf{c}\|, \tag{13}$$
$$\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\left[M_8\left\|\int_e^f Q_2(s,\sigma)\,d\sigma\right\| + \|\mathbf{h}(s)\|\right]ds \le \|\mathbf{c}\| + 2p_6M_7, \tag{14}$$
$$\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\left[\left\|\int_c^d Q_1(s,\tau)\,d\tau\right\| + \left\|\int_e^f Q_2(s,\sigma)\,d\sigma\right\|\right]ds \le p_6 - 2p_5, \tag{15}$$

and one defines an operator T on A as follows:

$$(T\mathbf{x})(t) = \begin{cases} \frac{1}{r(t)}\{\mathbf{c} - \int_a^b p(t,\theta)\mathbf{x}(t-\theta)\,d\theta + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}[\int_c^d Q_1(s,\tau)\mathbf{x}(s-\tau)\,d\tau \\ \quad - \int_e^f Q_2(s,\sigma)\mathbf{x}(s-\sigma)\,d\sigma - \mathbf{h}(s)]\,ds]\} \quad t\ge t_1, \\ (T\mathbf{x})(t_1) \quad t_0\le t\le t_1. \end{cases}$$

It is easy to see that T is continuous, for $t\ge t_1, \mathbf{x}\in A$. By using (13), we have

$$\|(T\mathbf{x})(t)\| \le \frac{1}{r(t)}\left\{\|\mathbf{c}\| - \int_a^b p(t,\theta)\|\mathbf{x}(t-\theta)\|\,d\theta\right.$$
$$\left. + \left\|\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\left[\int_c^d Q_1(s,\tau)\mathbf{x}(s-\tau)\,d\tau - \mathbf{h}(s)\right]ds\right\|\right\}$$

$$
\begin{aligned}
&\leq \frac{1}{-2p_5}\left\{\|\mathbf{c}\| - p_6M_8 + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\left[M_8\left\|\int_c^d Q_1(s,\tau)\,d\tau\right\| + \left\|\mathbf{h}(s)\right\|\right]ds\right\}\\
&\leq M_8,
\end{aligned}
$$

and taking (14) into account, we have

$$
\begin{aligned}
\left\|(T\mathbf{x})(t)\right\| &\geq \frac{1}{r(t)}\left\{\|\mathbf{c}\| - \left\|\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\left[\int_e^f Q_2(s,\sigma)\mathbf{x}(s-\sigma)\,d\sigma - \mathbf{h}(s)\right]ds\right\|\right\}\\
&\geq \frac{1}{-2p_6}\left\{\|\mathbf{c}\| - \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\left[M_8\left\|\int_e^f Q_2(s,\sigma)\,d\sigma\right\| + \left\|\mathbf{h}(s)\right\|\right]ds\right\}\\
&\geq M_7.
\end{aligned}
$$

These show that $TA \subset A$, since A is a bounded, closed, and convex subset of Λ, in order to apply the contraction principle, we have to show that T is a contraction mapping on A. For $\forall \mathbf{x}_1, \mathbf{x}_2 \in A$, and $t \geq t_1$,

$$
\begin{aligned}
&\left\|(T\mathbf{x}_1)(t) - (T\mathbf{x}_2)(t)\right\|\\
&\quad\leq \frac{1}{r(t)}\left\{\int_a^b p(t,\theta)\left\|\mathbf{x}_1(t-\theta) - \mathbf{x}_2(t-\theta)\right\|\,d\theta\right.\\
&\quad\quad + \left\|\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\left[\int_c^d Q_1(s,\tau)\mathbf{x}_1(s-\tau)\,d\tau - \int_e^f Q_2(s,\sigma)\mathbf{x}_1(s-\sigma)\,d\sigma - \mathbf{h}(s)\right]\right\| ds\\
&\quad\quad \left. - \left\|\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\left[\int_c^d Q_1(s,\tau)\mathbf{x}_2(s-\tau)\,d\tau - \int_e^f Q_2(s,\sigma)\mathbf{x}_2(s-\sigma)\,d\sigma - \mathbf{h}(s)\right]ds\right\|\right\}\\
&\quad\leq \frac{1}{r(t)}\left\{-p_6\|\mathbf{x}_1 - \mathbf{x}_2\| + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\left[\left\|\int_c^d Q_1(s,\tau)\,d\tau\right\|\left\|\mathbf{x}_1(s-\tau) - \mathbf{x}_2(s-\tau)\right\|\right.\right.\\
&\quad\quad \left.\left. + \left\|\int_e^f Q_2(s,\sigma)\,d\sigma\right\|\left\|\mathbf{x}_1(s-\sigma) - \mathbf{x}_2(s-\sigma)\right\|\right]ds\right\}.
\end{aligned}
$$

Or using (15),

$$
\begin{aligned}
&\left\|(T\mathbf{x}_1)(t) - (T\mathbf{x}_2)(t)\right\|\\
&\quad\leq \frac{1}{-2p_5}\|\mathbf{x}_1 - \mathbf{x}_2\|\left\{-p_6 + \frac{1}{(n-1)!}\int_t^\infty \frac{(s-t)^{n-1}}{r(s)}\left[\left\|\int_c^d Q_1(s,\tau)\,d\tau\right\|\right.\right.\\
&\quad\quad \left.\left. + \left\|\int_e^f Q_2(s,\sigma)\,d\sigma\right\|\right]ds\right\}\\
&\quad< \|\mathbf{x}_1 - \mathbf{x}_2\|,
\end{aligned}
$$

which shows that T is a contraction mapping on A and therefore there exists a unique solution, obviously a bounded positive solution of (1) $\mathbf{x} \in A$, such that $T\mathbf{x} = \mathbf{x}$. The proof is complete. □

3 Example

Consider the higher-order neutral differential equation with distributed coefficients and delays

$$\left(\frac{2}{2+\sin t}\mathbf{x}(t)+\int_{\frac{\pi}{2}}^{\pi} e^{-t}\mathbf{x}(t-\theta)\,d\theta\right)^{(3)}+\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \tau e^{-t}\begin{pmatrix}\frac{5}{2} & \frac{7}{2}\\ 5 & 1\end{pmatrix}\mathbf{x}(t-\tau)\,d\tau$$
$$-\int_{\pi}^{2\pi} e^{-t}\begin{pmatrix}2 & 4\\ \frac{7}{3} & \frac{11}{3}\end{pmatrix}\mathbf{x}(t-\sigma)\,d\sigma$$
$$=e^{-t}\left(-4\cos t-12\pi\sin t-7\pi+\pi^2\right)\begin{pmatrix}1\\1\end{pmatrix}. \tag{16}$$

Here, $n=3$,

$$r(t)=\frac{2}{2+\sin t},\qquad a=c=\frac{\pi}{2},\qquad b=e=\pi,\qquad d=\frac{3\pi}{2},$$
$$f=2\pi,\qquad \mathbf{h}(t)=e^{-t}\left(-4\cos t-12\pi\sin t-7\pi+\pi^2\right)\begin{pmatrix}1\\1\end{pmatrix},$$
$$Q_1(t)=\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \tau e^{-t}\begin{pmatrix}\frac{5}{2} & \frac{7}{2}\\ 5 & 1\end{pmatrix}d\tau,\qquad Q_2(t)=\int_{\pi}^{2\pi} e^{-t}\begin{pmatrix}2 & 4\\ \frac{7}{3} & \frac{11}{3}\end{pmatrix}d\sigma.$$

It is easy to see that $1\le r(t)\le 2$,

$$\int_{\frac{\pi}{2}}^{\pi} e^{-t}\,d\theta=\frac{\pi}{2}e^{-t}>1(t>0),\qquad \int_t^{\infty} s^2\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \tau e^{-s}\left\|\begin{pmatrix}\frac{5}{2} & \frac{7}{2}\\ 5 & 1\end{pmatrix}\right\|d\tau\,ds<\infty,$$
$$\int_t^{\infty} s^2\int_{\pi}^{2\pi} e^{-s}\left\|\begin{pmatrix}2 & 4\\ \frac{7}{3} & \frac{11}{3}\end{pmatrix}\right\|d\sigma\,ds<\infty,$$
$$\int_t^{\infty} s^2e^{-s}\left(-4\cos s-12\pi\sin s-7\pi+\pi^2\right)\left\|\begin{pmatrix}1\\1\end{pmatrix}\right\|ds<\infty,$$

thus Theorem 2 holds.

In fact, $\mathbf{x}(t)=\binom{2+\sin t}{2+\sin t}$ is a nonoscillatory solution of equation (16).

4 Remark

When $r(t)\equiv 1, \mathbf{h}(t)=\mathbf{0}$, equation (1) becomes equation (2), thus this paper improves results of Candan and Gecgel [6].

Competing interests
The authors declare that there is no conflict of interests regarding the publication of this article.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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