Li et al. *Advances in Difference Equations* (2015) 2015:325
DOI 10.1186/s13662-015-0659-4

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Asymptotic stability analysis of fractional-order neutral systems with time delay

Hong Li, Shou-ming Zhong and Hou-biao Li[*]

[*]Correspondence:
lihoubiao0189@163.com
School of Mathematics Sciences, University of Electronic Science and Technology of China, Chengdu, 611731, China

Abstract
In this paper, the asymptotic stability of linear and interval linear fractional-order neutral systems with time delay is discussed with true initial conditions. By applying the relation between integer system's characteristic equation and fractional system's characteristic equation, some brief sufficient stability conditions are deserved. The proposed method here is different from other ones in literature. In addition, some simple examples also demonstrate that this method is feasible.

Keywords: fractional-order neutral systems; time delay; characteristic equation

1 Introduction

Recently, fractional-order systems have gained increasing interests from various areas and have become one of the central subjects [1–7]. Note that a fractional-order derivative has a nonlocal property and weakly singular kernels, fractional-order systems provide an excellent tool for the description of memory and hereditary properties of dynamical processes. Therefore, it has many applied fields such as signal processing, physics, electrode-electrolyte polarization; see [1, 3].

The stability is a very essential and crucial issue for fractional-order control systems; see [6, 7]. At present, some powerful criteria have been proposed. For example, for commensurate fractional-order systems, the Matignon stability theorem is the most well-known one [4]. It permits one to determine the stability of the system through the location in the complex plane of the dynamic matrix eigenvalues of the state space. In addition, the Lambert function approach [5, 8], Lyapunov's second approach [9], the matrix measure approach [10, 11], Bellman-Gronwall's approach [6], and the LMI approach [7] are also used to investigate the stability of fractional-order linear systems. Reference [12] discussed Lyapunov theory for fractional-order system considering not only the pseudo state space representation but also a frequency-based representation involving the true state of the fractional-order system: $Z_C(\omega, t)$.

Of course, time delay has also considerable impacts on the stability of the system. Generally speaking, the analyses of time-delay systems can be usually classified into two types: those concerned with time dependent criteria and those concerned with delay independent stability. As there is no upper limit to the time delay, delay independent results can be regarded as conservative in practice. In addition, if the system contains delay both in its states and in the derivatives of its states, then the system is usually called a neutral type delay system.

Recently, a finite-time stability analysis of fractional time-delay systems has first been presented and reported in [13]. But until now, only a few papers studied the stability of fractional neutral systems. Though the Lyapunov approach of nonlinear fractional neutral system was extended in [14], it is difficult to use the Lyapunov method to study the stability of fractional neutral systems due to the complicated fractional derivatives. However, based on the algebraic approach, Zhang *et al.* [14] obtained some sufficient conditions for fractional neutral dynamical systems.

In this paper, by using the characteristic equation of the system, some brief sufficient asymptotic stability conditions are obtained with true initial conditions. These stability conditions are simpler and can be tested more easily than the conditions in [14]. In addition, the stability of an interval linear fractional neutral system with time delay is also obtained.

2 Problem formulation and preliminaries

As is well known, the differ-integral operator, denoted by ${}_aD_t^\alpha$, is a combined differentiation and integration operator commonly used in fractional calculus, which is defined by

$$ {}_aD_t^\alpha = \begin{cases} \frac{d^\alpha}{dt^\alpha}, & \alpha > 0, \\ 1, & \alpha = 0, \\ \int_a^t (d\tau)^{-\alpha}, & \alpha < 0. \end{cases} $$

However, for fractional derivatives, there exist different definitions. The most commonly used definitions are the Grünwald-Letnikov, the Riemann-Liouville, and the Caputo definitions. The Caputo definition is sometimes called a smooth fractional derivative in the literature because it is suitable to treat by the Laplace transform technique. One can also use a Laplace transform technique for the Riemann-Liouville operator, but potentially a problem is the physical meaning of the initial conditions.

For convenience, in the rest of the paper, D^α is used to denote the Caputo fractional derivative of order α,

$$ D^\alpha f(t) = \frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(\alpha - m)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}}\, d\tau, \tag{1} $$

where m is an integer satisfying $(m-1) < \alpha \leq m$.

In addition, this paper mainly focuses on the case that the fractional order is $0 < \alpha < 2$, since there exists an equivalent relation of fractional-order systems with order $0 < \alpha \leq 1$ and with order $1 \leq \beta < 2$; see [15].

Next, let us consider the linear fractional neutral system with time delay described by the following form:

$$ \frac{d^\alpha}{dt^\alpha}\bigl(x(t) - Cx(t-\tau)\bigr) = Ax(t) + Bx(t-\tau) \tag{2} $$

with the initial condition $x(t_0 + t) = \psi(t) \in C([-\tau, 0], \mathbb{R}^n)$. Here $1 \leq \alpha < 2$ is the fractional commensurate order, $x(t) \in \mathbb{R}^n$ denotes the pseudo state vector (the true initial function is usually denoted by $Z_C(\omega, t)$, see [16, 17]), C, A, and $B \in \mathbb{R}^{n\times n}$ are constant matrices, $\tau > 0$ is a pure time delay.

If the matrices A and B are uncertain, then the fractional-order neutral interval system can be described by the state space equation of the form

$$\frac{d^\alpha}{dt^\alpha}\big(x(t) - Cx(t-\tau)\big) = Ax(t) - Bx(t-\tau), \tag{3}$$

where

$$A \in \left[A^1, A^2\right] = \left\{[a_{ij}] : a_{ij}^1 \le a_{ij} \le a_{ij}^2\right\},$$
$$B \in \left[B^1, B^2\right] = \left\{[b_{ij}] : b_{ij}^1 \le b_{ij} \le b_{ij}^2, 1 \le i,j \le n\right\}.$$

Throughout this article, the following conventions are used:

$\mu(A)$: matrix measure of the matrix A, *i.e.*, $\mu(A) = \frac{1}{2}\lambda_{\max}(A + A^*)$.
$\rho(A)$: spectral radius of the matrix A.
$|A|$: modulus matrix of the matrix A.
$A \otimes B$: Kronecker product of A and B.
$\|A\|$: spectral norm of matrix A; $\|A\| = \sqrt{\lambda_{\max}(A^*A)}$.
$\lambda_{\max}(A)$: the maximum eigenvalue of the matrix A.
A^*: the conjugate transpose of the matrix A.
$A \le B$: the element of A and B satisfy the inequality $a_{ij} \le b_{ij}$.

In addition, to prove the main results in the next section, we need also the following lemma.

Lemma 2.1 [18] *Let R, T, and $V \in \mathbb{C}^{\kappa\times\kappa}$. If $|R| \le V$, then*

$$\rho(R+T) \le \rho\big(|R+T|\big) \le \rho\big(|R| + |T|\big) \le \rho\big(V + |T|\big);$$
$$\rho(RT) \le \rho\big(|R||T|\big) \le \rho\big(V|T|\big); \qquad \rho(R) \le \rho\big(|R|\big) \le \rho(V);$$
$$\operatorname{Re}\big(\lambda_j(A)\big) \le \mu(A); \qquad \mu(A+B) \le \mu(A) + \mu(B); \qquad \mu(A) \le \|A\|.$$

3 Main results

Throughout this paper, we define $\Theta = \left(\begin{smallmatrix} \sin\frac{\alpha\pi}{2} & \cos\frac{\alpha\pi}{2} \\ -\cos\frac{\alpha\pi}{2} & \sin\frac{\alpha\pi}{2}\end{smallmatrix}\right)$, so $\|\Theta\| = 1$.

3.1 Stability of linear fractional neutral systems with delay

Lemma 3.1 [18] *Let $A, B \in \mathbb{R}^{n\times n}$ be Hermitian matrices, then*

$$\lambda_{\max}(A+B) \le \lambda_{\max}(A) + \lambda_{\max}(B).$$

Lemma 3.2 [18] *Let $C \in \mathbb{R}^{n\times n}$. If $\rho(C) < 1$, then* $\det(I \pm C) \ne 0$ *and*

$$(I - C)^{-1} = I + C + C^2 + \cdots.$$

Specially, if $\|C\| < 1$, then $(I-C)^{-1}$ exists, and $\|(I-C)^{-1}\| \le 1/(1 - \|C\|)$.

Lemma 3.3 *Let $B \in \mathbb{R}^{n\times n}$, define $B_u = \Theta \otimes B$, $B_l = \Theta \otimes (-iB)$, where $i^2 = -1$. If $\mathrm{Re}(s) \geq 0$, then the following inequality holds*:

$$\mu\left(\Theta \otimes Be^{-s\tau}\right) \leq \sqrt{\mu^2(B_u) + \mu^2(B_l)}.$$

Proof Let $s = \alpha + i\beta$, then $Be^{-s\tau} = Be^{-\alpha\tau}(\cos(\beta\tau) - i\sin(\beta\tau))$. Noticing that $\mathrm{Re}(s) = \alpha \geq 0$, so $0 < e^{-\alpha\tau} < 1$. Using Lemma 2.1 and Lemma 3.1, we have

$$\begin{aligned}
&\mu\left(\Theta \otimes Be^{-s\tau}\right)\\
&\quad = \frac{1}{2}\lambda_{\max}\left(\Theta \otimes Be^{-s\tau} + \Theta^* \otimes \left(Be^{-s\tau}\right)^*\right)\\
&\quad = \frac{1}{2}\lambda_{\max}\left(\Theta \otimes Be^{-\alpha\tau}\left(\cos(\beta\tau) - i\sin(\beta\tau)\right) + \Theta^* \otimes B^* e^{-\alpha\tau}\left(\cos(\beta\tau) + i\sin(\beta\tau)\right)\right)\\
&\quad = e^{-\alpha\tau}\frac{1}{2}\lambda_{\max}\left(\left(\Theta \otimes B + \Theta^* \otimes B^*\right)\cos(\beta\tau) + \left(\Theta^* \otimes iB^* + \Theta \otimes (-iB)\right)\sin(\beta\tau)\right)\\
&\quad \leq e^{-\alpha\tau}\left(\frac{1}{2}\lambda_{\max}\left(\Theta \otimes B + \Theta^* \otimes B^*\right)\cos(\beta\tau)\right.\\
&\qquad \left. + \frac{1}{2}\lambda_{\max}\left(\Theta^* \otimes iB^* - \Theta \otimes (-iB)\right)\sin(\beta\tau)\right)\\
&\quad = e^{-\alpha\tau}\left(\cos(\beta\tau)\mu(\Theta \otimes B) + \sin(\beta\tau)\mu\left(\Theta \otimes (-iB)\right)\right)\\
&\quad = e^{-\alpha\tau}\left(\cos(\beta\tau)\mu(B_u) + \sin(\beta\tau)\mu(B_l)\right)\\
&\quad = e^{-\alpha\tau}\sqrt{\mu^2(B_u) + \mu^2(B_l)}\left(\cos(\beta\tau)\frac{\mu(B_u)}{\sqrt{\mu^2(B_u) + \mu^2(B_l)}} + \sin(\beta\tau)\frac{\mu(B_l)}{\sqrt{\mu^2(B_u) + \mu^2(B_l)}}\right)\\
&\quad \leq e^{-\alpha\tau}\sqrt{\mu^2(B_u) + \mu^2(B_l)}\\
&\quad \leq \sqrt{\mu^2(B_u) + \mu^2(B_l)}.
\end{aligned}\tag{4}$$

Thus, the proof is completed. □

Lemma 3.4 [18] *Let $A \in \mathbb{R}^{m\times m}$, $B \in \mathbb{R}^{n\times n}$; λ_r $(r = 1, \ldots, m)$ are eigenvalues of the matrix A and u_k $(k = 1, \ldots, n)$ are eigenvalues of the matrix B, then $\lambda_r u_k$ $(r = 1, \ldots, m, k = 1, \ldots, n)$ are eigenvalues of the Kronecker product $A \otimes B$.*

Lemma 3.5 *Let $A \in \mathbb{R}^{m\times m}$, $B \in \mathbb{R}^{n\times n}$, we have*

$$\|A \otimes B\| = \|A\| \cdot \|B\|.$$

Proof Using Lemma 3.4, we can obtain

$$\begin{aligned}
\|A \otimes B\| &= \sqrt{\lambda_{\max}\left((A \otimes B)^*(A \otimes B)\right)} = \sqrt{\lambda_{\max}\left(\left(A^* \otimes B^*\right)(A \otimes B)\right)}\\
&= \sqrt{\lambda_{\max}\left(A^*A \otimes B^*B\right)} = \sqrt{\lambda_{\max}\left(A^*A\right) \cdot \lambda_{\max}\left(B^*B\right)}\\
&= \sqrt{\lambda_{\max}\left(A^*A\right)} \cdot \sqrt{\lambda_{\max}\left(B^*B\right)}\\
&= \|A\| \cdot \|B\|.
\end{aligned}\tag{5}$$

The proof is completed. □

Lemma 3.6 *Let* $A \in \mathbb{R}^{n\times n}$, $\det(s^\alpha I - A) \neq 0$ *for* $\forall \operatorname{Re}(s) \geq 0$ *if and only if* $\det(sI - \Theta \otimes A) \neq 0$ *for* $\forall \operatorname{Re}(s) \geq 0$, *where* $1 \leq \alpha < 2$.

Proof According to Theorem 3 in [19], we can easily prove this lemma. □

Next, we consider the global stability of system (2) in the case of the true initial state space, according to the true initial function $Z_C(\omega, t)$; see [16, 17].

Definition 3.1 The system (2) is said to be asymptotically stable if for any pseudo initial function $\psi(t), \psi'(t) \in C([-\tau, 0], \mathbb{R}^n)$, any true initial function $Z_C(\omega, t) \in C([0, +\infty; -\tau, 0], \mathbb{R}^n)$ and for any time delay $\tau > 0$, the analytic solution $x(t)$ of the system (2) satisfies $\lim_{t\to+\infty} x(t) = 0$.

Theorem 3.1 *If all the roots of the characteristic equation*

$$D(s) = \det\bigl(s^\alpha\bigl(I - Ce^{-s\tau}\bigr) - \bigl(A + Be^{-s\tau}\bigr)\bigr) = 0$$

have negative real parts, then the system (2) *with* $1 \leq \alpha < 2$ *is asymptotically stable.*

Proof Taking the Laplace transform of the system (2), we have

$$\begin{aligned}
&\bigl(s^\alpha I - A - Cs^\alpha e^{-s\tau} - Be^{-s\tau}\bigr)X(s) \\
&\quad = s^{\alpha-1}\psi(0) + s^{\alpha-2}\psi'(0) - Ce^{-s\tau}s^{\alpha-1}\psi(0) - Ce^{-s\tau}s^{\alpha-2}\psi'(0) \\
&\qquad + Be^{-s\tau}\int_{-\tau}^{0} e^{-s\tau}Z_C(\omega, t)\,dt + Ce^{-s\tau}\int_{-\tau}^{0} e^{-s\tau}Z_C(\omega, t)\,dt \\
&\qquad + Ce^{-s\tau}\int_0^{+\infty}\frac{u_{2-n}(\omega)Z_C(\omega, 0)}{s+\omega}\,d\omega - \int_0^{+\infty}\frac{u_{2-n}(\omega)Z_C(\omega, 0)}{s+\omega}\,d\omega.
\end{aligned} \tag{6}$$

Let $D(s,\tau) = s^\alpha I - A - Cs^\alpha e^{-s\tau} - Be^{-s\tau}$. Multiplying s on both sides of (6) gives

$$\begin{aligned}
&D(s,\tau)sX(s) \\
&\quad = s^\alpha\psi(0) + s^{\alpha-1}\psi'(0) - Ce^{-s\tau}s^\alpha\psi(0) - Ce^{-s\tau}s^{\alpha-1}\psi'(0) \\
&\qquad + Bse^{-s\tau}\int_{-\tau}^{0} e^{-s\tau}Z_C(\omega, t)\,dt + Cse^{-s\tau}\int_{-\tau}^{0} e^{-s\tau}Z_C(\omega, t)\,dt \\
&\qquad + Cse^{-s\tau}\int_0^{+\infty}\frac{u_{2-n}(\omega)Z_C(\omega, 0)}{s+\omega}\,d\omega - s\int_0^{+\infty}\frac{u_{2-n}(\omega)Z_C(\omega, 0)}{s+\omega}\,d\omega.
\end{aligned} \tag{7}$$

Similar to [20], one can easily prove the theorem. □

Theorem 3.2 *The system* (2) *with* $1 \leq \alpha < 2$ *is asymptotically stable, if the following inequalities are satisfied*:

$$\begin{aligned}
&(1)\quad \|C\| < 1; \\
&(2)\quad \mu(\Theta \otimes A) + \sqrt{\mu^2(B_u) + \mu^2(B_l)} + \frac{\|CA\| + \|CB\|}{1 - \|C\|} < 0.
\end{aligned} \tag{8}$$

Proof The system (2) is asymptotically stable if and only if the characteristic equation $D(s) \neq 0$ for $\forall \operatorname{Re}(s) \geq 0$. Since $0 < e^{-\operatorname{Re}(s)\tau} \leq 1$ for any $\operatorname{Re}(s) \geq 0$, we have from Lemma 2.1

$$\rho(Ce^{-s\tau}) \leq \rho(|Ce^{-s\tau}|) \leq \rho(|C|) \leq \|C\| < 1. \tag{9}$$

By Lemma 3.2, we further know that $(I - Ce^{-s\tau})^{-1}$ exists for $\forall \operatorname{Re}(s) \geq 0$. Thus $D(s) \neq 0$ for $\forall \operatorname{Re}(s) \geq 0$, *i.e.*,

$$\det(s^\alpha I - (I - Ce^{-s\tau})^{-1}(A + Be^{-s\tau})) \neq 0, \quad \forall \operatorname{Re}(s) \geq 0. \tag{10}$$

In addition, from Lemma 3.6, (10) is equivalent to the following inequality:

$$\det(sI - \Theta \otimes (I - Ce^{-s\tau})^{-1}(A + Be^{-s\tau})) \neq 0, \quad \forall \operatorname{Re}(s) \geq 0, \tag{11}$$

which is equivalent to

$$s \neq \lambda_j(\Theta \otimes (I - Ce^{-s\tau})^{-1}(A + Be^{-s\tau})), \quad \forall \operatorname{Re}(s) \geq 0, j \in 1, 2, \ldots, n. \tag{12}$$

Thus, if we have $\operatorname{Re}_{\lambda_j}(\Theta \otimes (I - Ce^{-s\tau})^{-1}(A + Be^{-s\tau})) < 0$, then we can prove the stability of the system (2).

In fact, employing the well-known relation

$$(I - Ce^{-s\tau})^{-1} = I + (I - Ce^{-s\tau})^{-1}Ce^{-s\tau}$$

and using Lemmas 2.1, 3.2, and 3.3, we can obtain

$$\begin{aligned}
&\operatorname{Re}_{\lambda_j}(\Theta \otimes (I - Ce^{-s\tau})^{-1}(A + Be^{-s\tau})) \\
&\quad = \operatorname{Re}_{\lambda_j}(\Theta \otimes (I + (I - Ce^{-s\tau})^{-1}Ce^{-s\tau})(A + Be^{-s\tau})) \\
&\quad = \operatorname{Re}_{\lambda_j}(\Theta \otimes (A + Be^{-s\tau} + (I - Ce^{-s\tau})^{-1}(CAe^{-s\tau} + CBe^{-2s\tau}))) \\
&\quad = \operatorname{Re}_{\lambda_j}(\Theta \otimes A + \Theta \otimes Be^{-s\tau} + \Theta \otimes (I - Ce^{-s\tau})^{-1}(CAe^{-s\tau} + CBe^{-2s\tau})) \\
&\quad \leq \mu(\Theta \otimes A + \Theta \otimes Be^{-s\tau} + \Theta \otimes (I - Ce^{-s\tau})^{-1}(CAe^{-s\tau} + CBe^{-2s\tau})) \\
&\quad \leq \mu(\Theta \otimes A) + \mu(\Theta \otimes Be^{-s\tau}) + \mu(\Theta \otimes (I - Ce^{-s\tau})^{-1}(CAe^{-s\tau} + CBe^{-2s\tau})) \\
&\quad \leq \mu(\Theta \otimes A) + \sqrt{\mu^2(B_u) + \mu^2(B_l)} + \|\Theta \otimes (I - Ce^{-s\tau})^{-1}(CAe^{-s\tau} + CBe^{-2s\tau})\| \\
&\quad \leq \mu(\Theta \otimes A) + \sqrt{\mu^2(B_u) + \mu^2(B_l)} + \|\Theta\| \cdot \|(I - Ce^{-s\tau})^{-1}(CAe^{-s\tau} + CBe^{-2s\tau})\| \\
&\quad \leq \mu(\Theta \otimes A) + \sqrt{\mu^2(B_u) + \mu^2(B_l)} + \frac{\|CA\| + \|CB\|}{1 - \|C\|} \\
&\quad < 0.
\end{aligned} \tag{13}$$

This completes the proof. □

Corollary 3.3 *The system* (2) *with* $1 \leq \alpha < 2$ *is asymptotically stable, if there exists an invertible matrix* $P \in \mathbb{R}^{n\times n}$, *such that the following inequalities are satisfied*:

(1) $\|C\| < 1$;

$$(2)\quad \mu\bigl(P^{-1}(\Theta\otimes A)P\bigr)+\sqrt{\mu^2\bigl(P^{-1}B_uP\bigr)+\mu^2\bigl(P^{-1}B_lP\bigr)} + \frac{\|P^{-1}\|\|CA\|\|P\|+\|P^{-1}\|\|CB\|\|P\|}{1-\|C\|}<0. \tag{14}$$

Proof According to Theorem 3.2, we see that the system (2) is stable if and only if $\mathrm{Re}_{\lambda_j}(\Theta\otimes(I-Ce^{-s\tau})^{-1}(A+Be^{-s\tau}))<0$, $\forall\,\mathrm{Re}(s)\geq 0$, which is equivalent to

$$\mathrm{Re}_{\lambda_j}\bigl(P^{-1}\bigl(\Theta\otimes\bigl(I-Ce^{-s\tau}\bigr)^{-1}\bigl(A+Be^{-s\tau}\bigr)\bigr)P\bigr)<0,\quad \forall\,\mathrm{Re}(s)\geq 0.$$

So, we can prove this corollary easily. □

3.2 Stability of interval linear fractional neutral systems with delay

Now, we consider the interval neutral fractional linear system (3).

Let

$$\begin{aligned}
&\bar{A}=\frac{1}{2}\bigl(A^1+A^2\bigr),\qquad \Delta A=A-\bar{A},\qquad A_M=A_2-\bar{A};\\
&\bar{B}=\frac{1}{2}\bigl(B^1+B^2\bigr),\qquad \Delta B=B-\bar{B},\qquad B_M=B_2-\bar{B},
\end{aligned}\tag{15}$$

then $|\Delta A|\leq A_M$, $|\Delta B|\leq B_M$.

We have the following lemmas and theorems as regards the interval system (3).

Lemma 3.7 *Let $B\in\mathbb{R}^{n\times n}$, if $\mathrm{Re}(s)\geq 0$, then the following inequality holds*:

$$\mu\bigl(\Theta\otimes Be^{-s\tau}\bigr)\leq\sqrt{2}\|B\|.$$

Proof Using Lemma 3.3, we have

$$\mu\bigl(\Theta\otimes Be^{-s\tau}\bigr)=e^{-\alpha\tau}\bigl(\cos(\beta\tau)\mu(\Theta\otimes B)+\sin(\beta\tau)\mu\bigl(\Theta\otimes(-iB)\bigr)\bigr).\tag{16}$$

Noticing that $\mathrm{Re}(s)=\alpha\geq 0$, we have $0<e^{-\alpha\tau}<1$. According to Lemma 3.1, we can obtain

$$\begin{aligned}
&e^{-\alpha\tau}\bigl(\cos(\beta\tau)\mu(\Theta\otimes B)+\sin(\beta\tau)\mu\bigl(\Theta\otimes(-iB)\bigr)\bigr)\\
&\quad\leq e^{-\alpha\tau}\bigl(\bigl|\cos(\beta\tau)\bigr|\cdot\|\Theta\otimes B\|+\bigl|\sin(\beta\tau)\bigr|\cdot\bigl\|\Theta\otimes(-iB)\bigr\|\bigr)\\
&\quad= e^{-\alpha\tau}\bigl(\bigl|\cos(\beta\tau)\bigr|\cdot\|\Theta\|\|B\|+\bigl|\sin(\beta\tau)\bigr|\cdot\|\Theta\|\|-iB\|\bigr)\\
&\quad= e^{-\alpha\tau}\|\Theta\|\|B\|\bigl(\bigl|\cos(\beta\tau)\bigr|+\bigl|\sin(\beta\tau)\bigr|\bigr)\\
&\quad= e^{-\alpha\tau}\|\Theta\|\|B\|\sqrt{2}\biggl(\frac{1}{\sqrt{2}}\bigl|\cos(\beta\tau)\bigr|+\frac{1}{\sqrt{2}}\bigl|\sin(\beta\tau)\bigr|\biggr)\\
&\quad\leq\sqrt{2}\|\Theta\|\|B\|\\
&\quad=\sqrt{2}\|B\|.
\end{aligned}\tag{17}$$

Thus, the proof is completed. □

Theorem 3.4 *The system* (3) *with $1\leq\alpha<2$ is asymptotically stable, if the following inequalities are satisfied*:

(1) $\|C\|<1$;

$$\text{(2)}\quad \mu(\Theta\otimes\bar{A})+\sqrt{\mu^2(\bar{B}_u)+\mu^2(\bar{B}_l)}+\|A_M\|+\sqrt{2}\|B_M\| + \frac{\|C\bar{A}\|+\|C\bar{B}\|+\|C\|(\|A_M\|+\|B_M\|)}{1-\|C\|}<0. \tag{18}$$

Proof According to Theorem 3.2, we see that the interval system (3) is asymptotically stable if and only if

$$\mathrm{Re}_{\lambda_j}\bigl(\Theta\otimes\bigl(I+\bigl(I-Ce^{-s\tau}\bigr)^{-1}Ce^{-s\tau}\bigr)\bigl(A+Be^{-s\tau}\bigr)\bigr)<0,\quad \forall\,\mathrm{Re}(s)\geq 0.$$

Using Lemma 2.1, we can obtain

$$\begin{aligned}
&\mathrm{Re}_{\lambda_j}\bigl(\Theta\otimes\bigl(I+\bigl(I-Ce^{-s\tau}\bigr)^{-1}Ce^{-s\tau}\bigr)\bigl(A+Be^{-s\tau}\bigr)\bigr)\\
&\quad=\mathrm{Re}_{\lambda_j}\bigl(\Theta\otimes\bigl(I+\bigl(I-Ce^{-s\tau}\bigr)^{-1}Ce^{-s\tau}\bigr)\bigl(\bar{A}+\Delta A+(\bar{B}+\Delta B)e^{-s\tau}\bigr)\bigr)\\
&\quad=\mathrm{Re}_{\lambda_j}\bigl(\Theta\otimes\bigl(\bar{A}+\bar{B}e^{-s\tau}+\Delta A+\Delta B\\
&\qquad+\bigl(I-Ce^{-s\tau}\bigr)^{-1}\bigl(C\bar{A}e^{-s\tau}+C\bar{B}e^{-2s\tau}+C\Delta Ae^{-s\tau}+C\Delta Be^{-2s\tau}\bigr)\bigr)\bigr)\\
&\quad=\mathrm{Re}_{\lambda_j}\bigl(\Theta\otimes\bar{A}+\Theta\otimes\bar{B}e^{-s\tau}+\Theta\otimes\Delta A+\Theta\otimes\Delta Be^{-s\tau}\\
&\qquad+\Theta\otimes\bigl(I-Ce^{-s\tau}\bigr)^{-1}\bigl(C\bar{A}e^{-s\tau}+C\bar{B}e^{-2s\tau}+C\Delta Ae^{-s\tau}+C\Delta Be^{-2s\tau}\bigr)\bigr)\\
&\quad\leq\mu\bigl(\Theta\otimes\bar{A}+\Theta\otimes\bar{B}e^{-s\tau}+\Theta\otimes\Delta A+\Theta\otimes\Delta Be^{-s\tau}\\
&\qquad+\Theta\otimes\bigl(I-Ce^{-s\tau}\bigr)^{-1}\bigl(C\bar{A}e^{-s\tau}+C\bar{B}e^{-2s\tau}+C\Delta Ae^{-s\tau}+C\Delta Be^{-2s\tau}\bigr)\bigr)\\
&\quad\leq\mu(\Theta\otimes\bar{A})+\mu\bigl(\Theta\otimes\bar{B}e^{-s\tau}\bigr)+\mu(\Theta\otimes\Delta A)+\mu\bigl(\Theta\otimes\Delta Be^{-s\tau}\bigr)\\
&\qquad+\mu\bigl(\Theta\otimes\bigl(I-Ce^{-s\tau}\bigr)^{-1}\bigl(C\bar{A}e^{-s\tau}+C\bar{B}e^{-2s\tau}+C\Delta Ae^{-s\tau}+C\Delta Be^{-2s\tau}\bigr)\bigr).
\end{aligned}\tag{19}$$

In addition, according to Lemma 3.3 and Lemma 3.7, we can further obtain

$$\begin{aligned}
&\mu(\Theta\otimes\bar{A})+\mu\bigl(\Theta\otimes\bar{B}e^{-s\tau}\bigr)+\mu(\Theta\otimes\Delta A)+\mu\bigl(\Theta\otimes\Delta Be^{-s\tau}\bigr)\\
&\qquad+\mu\bigl(\Theta\otimes\bigl(I-Ce^{-s\tau}\bigr)^{-1}\bigl(C\bar{A}e^{-s\tau}+C\bar{B}e^{-2s\tau}+C\Delta Ae^{-s\tau}+C\Delta Be^{-2s\tau}\bigr)\bigr)\\
&\quad\leq\mu(\Theta\otimes\bar{A})+\sqrt{\mu^2(\bar{B}_u)+\mu^2(\bar{B}_l)}+\|\Theta\otimes\Delta A\|+\bigl\|\Theta\otimes\Delta Be^{-s\tau}\bigr\|\\
&\qquad+\bigl\|\Theta\otimes\bigl(I-Ce^{-s\tau}\bigr)^{-1}\bigl(C\bar{A}e^{-s\tau}+C\bar{B}e^{-2s\tau}+C\Delta Ae^{-s\tau}+C\Delta Be^{-2s\tau}\bigr)\bigr\|\\
&\quad\leq\mu(\Theta\otimes A)+\sqrt{\mu^2(\bar{B}_u)+\mu^2(\bar{B}_l)}+\|\Theta\|\|\Delta A\|+\sqrt{2}\|\Delta B\|\\
&\qquad+\|\Theta\|\cdot\bigl\|\bigl(I-Ce^{-s\tau}\bigr)^{-1}\bigl(C\bar{A}e^{-s\tau}+C\bar{B}e^{-2s\tau}+C\Delta Ae^{-s\tau}+C\Delta Be^{-2s\tau}\bigr)\bigr\|\\
&\quad\leq\mu(\Theta\otimes\bar{A})+\sqrt{\mu^2(\bar{B}_u)+\mu^2(\bar{B}_l)}+\|A_M\|+\sqrt{2}\|B_M\|\\
&\qquad+\frac{\|C\bar{A}\|+\|C\bar{B}\|+\|C\|(\|A_M\|+\|B_M\|)}{1-\|C\|}\\
&\quad<0.
\end{aligned}\tag{20}$$

This completes the proof. □

Remark 3.1 If we use $\Theta_1 = \left(\begin{smallmatrix} -\sin\frac{\alpha\pi}{2} & +\cos\frac{\alpha\pi}{2} \\ -\cos\frac{\alpha\pi}{2} & -\sin\frac{\alpha\pi}{2} \end{smallmatrix}\right)$ to replace Θ in Theorems 3.2, 3.4 and Corollary 3.3, we obtain the following conclusions:

1. The system (2) with $0 < \alpha < 1$ is unstable, if the following inequalities are satisfied:

$$\begin{aligned} &(1) \quad \|C\| < 1; \\ &(2) \quad \mu(\Theta_1 \otimes A) + \sqrt{\mu^2(B'_u) + \mu^2(B'_l)} + \frac{\|CA\| + \|CB\|}{1 - \|C\|} < 0. \end{aligned} \tag{21}$$

2. The system (2) with $0 < \alpha < 1$ is unstable, if there exists an invertible matrix $P \in \mathbb{R}^{n\times n}$, such that the following inequalities are satisfied:

$$\begin{aligned} &(1) \quad \|C\| < 1; \\ &(2) \quad \mu\bigl(P^{-1}(\Theta_1 \otimes A)P\bigr) + \sqrt{\mu^2\bigl(P^{-1}B'_uP\bigr) + \mu^2\bigl(P^{-1}B'_lP\bigr)} \\ &\qquad + \frac{\|P^{-1}CAP\| + \|P^{-1}CBP\|}{1 - \|C\|} < 0, \end{aligned} \tag{22}$$

   where $B'_u = \Theta_1 \otimes B$, $B'_l = \Theta_1 \otimes (-iB)$.

3. The system (3) with $0 < \alpha < 1$ is unstable, if the following inequalities are satisfied:

$$\begin{aligned} &(1) \quad \|C\| < 1; \\ &(2) \quad \mu(\Theta_1 \otimes \bar{A}) + \sqrt{\mu^2(\bar{B}'_u) + \mu^2(\bar{B}'_l)} + \|A_M\| + \sqrt{2}\|B_M\| \\ &\qquad + \frac{\|C\bar{A}\| + \|C\bar{B}\| + \|C\|(\|A_M\| + \|B_M\|)}{1 - \|C\|} < 0, \end{aligned} \tag{23}$$

   where $\bar{B}'_u = \Theta_1 \otimes \bar{B}$, $\bar{B}'_l = \Theta_1 \otimes (-i\bar{B})$.

Remark 3.2 As stated in [16, 17], fractional differentiation usually needs an initialization to avoid long range memory phenomenon, so one should use $Z_C(\omega, 0)$ to initialize Caputo derivative. In Theorem 3.1, we give a simple proof for this case. In fact, for the case of the pseudo state model, one can also prove this theorem according to [20] by taking the Laplace transform.

4 Numerical examples

Next, we apply Matlab to help us demonstrate our results.

Example 4.1 Consider the stability of the following fractional-order neutral system with delay:

$$\frac{d^\alpha}{dt^\alpha}\bigl(x(t) - Cx(t-\tau)\bigr) = Ax(t) - Bx(t-\tau), \tag{24}$$

where $\alpha = 3/2$, and

$$A = \begin{pmatrix} -2 & -\frac{1}{3} \\ \frac{1}{3} & -3 \end{pmatrix}, \qquad B = \begin{pmatrix} -\frac{1}{2} & \frac{1.1}{3} \\ \frac{1.1}{3} & -\frac{1}{2} \end{pmatrix}, \qquad C = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}.$$

Since $\alpha = 3/2$, we can obtain $\theta = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$. By computation, we have

$$\mu(\theta \otimes A) = -1.34, \qquad \sqrt{\mu^2(B_u) + \mu^2(B_l)} = 0.6673, \qquad \frac{\|CA\| + \|CB\|}{1 - \|C\|} = 0.4288.$$

So $\|C\| = 0.1 < 1$ and

$$\begin{aligned} &\mu(\Theta \otimes A) + \sqrt{\mu^2(B_u) + \mu^2(B_l)} + \frac{\|CA\| + \|CB\|}{1 - \|C\|} \\ &\quad = -1.34 + 0.6673 + 0.4288 = -0.2475 < 0. \end{aligned}$$

Therefore, from Theorem 3.2, we know that the fractional system (24) is asymptotically stable.

Example 4.2 Consider the stability of the following fractional-order neutral system with delay:

$$\frac{d^\alpha}{dt^\alpha}\bigl(x(t) - Cx(t - \tau)\bigr) = Ax(t) - Bx(t - \tau), \tag{25}$$

where $\alpha = 3/2$, and

$$\begin{aligned} &A^1 = \begin{pmatrix} -2.2 & 0.1 \\ 0.2 & -3.1 \end{pmatrix}, \qquad A^2 = \begin{pmatrix} -1.8 & \frac{3}{10} \\ 0.2 & -2.9 \end{pmatrix}, \\ &B^1 = \begin{pmatrix} 0.54 & \frac{1.1}{5} \\ \frac{1.1}{5} & -0.52 \end{pmatrix}, \qquad B^2 = \begin{pmatrix} 0.46 & \frac{1.1}{5} \\ \frac{1.1}{5} & -0.48 \end{pmatrix}, \qquad C = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}. \end{aligned}$$

First, note that

$$\begin{aligned} &\theta = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}, \qquad \bar{A} = \begin{pmatrix} -2 & \frac{1}{5} \\ \frac{1}{5} & -3 \end{pmatrix}, \qquad A_M = \begin{pmatrix} 0.2 & 0.1 \\ 0 & 0.1 \end{pmatrix}, \\ &\bar{B} = \begin{pmatrix} -0.5 & \frac{1.1}{5} \\ \frac{1.1}{5} & -0.5 \end{pmatrix}, \qquad B_M = \begin{pmatrix} 0.01 & 0 \\ 0 & 0.02 \end{pmatrix}. \end{aligned}$$

By computation, we have

$$\begin{aligned} &\mu(\theta \otimes A) = -1.3877, \qquad \sqrt{\mu^2(B_u) + \mu^2(B_l)} = 0.6673, \qquad \|A_M\| = 0.26, \\ &\|B_M\| = 0.02, \qquad \frac{\|CA\| + \|CB\|}{1 - \|C\|} = 0.4288, \qquad \frac{\|C\|(\|A_M\| + \|B_M\|)}{1 - \|C\|} = 0.00292. \end{aligned}$$

So $\|C\| = 0.1 < 1$ and

$$\begin{aligned} &\mu(\Theta \otimes \bar{A}) + \sqrt{\mu^2(\bar{B}_u) + \mu^2(\bar{B}_l)} + \|A_M\| + \sqrt{2}\|B_M\| \\ &\qquad + \frac{\|C\bar{A}\| + \|C\bar{B}\| + \|C\|(\|A_M\| + \|B_M\|)}{1 - \|C\|} \\ &\quad = -1.3877 + 0.6673 + 0.26 + 0.028 + 0.4288 + 0.00292 = -0.00428 < 0. \end{aligned}$$

Therefore, from Theorem 3.4, we know that the fractional system (25) is asymptotically stable.

5 Conclusions

In summary, this paper mainly presents some brief sufficient conditions for the stability of a class of fractional-order neutral systems with uncertain parameters. The proposed method here is quite different from other ones in the literature. Two simple examples also demonstrate that this method is feasible.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors completed the paper together. HL and H-bL have equal contributions to this work. All authors read and approved the final manuscript.

Acknowledgements
The authors are grateful to the anonymous referees for their constructive comments and helpful suggestions, improving this paper greatly. In addition, this work is also supported by National Natural Science Foundation of China (61472462, 11101071, 11271001, 51175443) and the Fundamental Research Funds for China Scholarship Council.

Received: 7 January 2015 Accepted: 6 October 2015 Published online: 19 October 2015

References

1. Podlubny, I: Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications. Mathematics in Science and Engineering, vol. 198, pp. 261-300. Academic press, London (1998)
2. Ita, J, Stixrude, L: Petrology, elasticity, and composition of the mantle transition zone. J. Geophys. Res. **97**(B5), 6849-6866 (1992)
3. Oustaloup, A, Moreau, X, Nouillant, M: The crone suspension. Control Eng. Pract. **4**(8), 1101-1108 (1996)
4. Matignon, D: Stability results for fractional differential equations with applications to control processing. In: Computational Engineering in Systems Applications, Lille, France, vol. 2, pp. 963-968 (1996).
5. Chen, Y, Moore, KL: Analytical stability bound for a class of delayed fractional-order dynamic systems. Nonlinear Dyn. **29**(1), 191-200 (2002)
6. Lazarević, M: Stability and stabilization of fractional order time delay systems. Sci. Tech. Rev. **61**(1), 31-45 (2011)
7. Sabatier, J, Moze, M, Farges, C: LMI stability conditions for fractional order systems. Comput. Math. Appl. **59**(5), 1594-1609 (2010)
8. Hwang, C, Cheng, Y-C: Use of Lambert *W* function to stability analysis of time-delay systems. In: Proceedings of the American Control Conference, pp. 4283-4288. IEEE Press, New York (2005)
9. Liu, K, Jiang, W: Stability of fractional neutral systems. Adv. Differ. Equ. **2014**, 78 (2014)
10. Chen, J, Xu, D, Shafai, B: On sufficient conditions for stability independent of delay. IEEE Trans. Autom. Control **40**(9), 1675-1680 (1995)
11. Mori, T: Criteria for asymptotic stability of linear time-delay systems. IEEE Trans. Autom. Control **30**(2), 158-161 (1985)
12. Trigeassou, J-C, Maamri, N, Sabatier, J, Oustaloup, A: A Lyapunov approach to the stability of fractional differential equations. Signal Process. **91**(3), 437-445 (2011)
13. Han, Q-L: Stability of linear neutral systems with linear fractional norm-bounded uncertainty. In: Proceedings of the American Control Conference, pp. 2827-2832. IEEE Press, New York (2005)
14. Zhang, H, Wu, D, Cao, J: Asymptotic stability of Caputo type fractional neutral dynamical systems with multiple discrete delays. Abstr. Appl. Anal. **2014**, Article ID 138124 (2014)
15. Li, H, Zhong, S-M, Li, H-B: Stability of interval fractional-order systems with order $0 < \alpha < 1$. Int. J. Comput. Math. Sci. **6**, 89-93 (2012)
16. Trigeassou, J-C, Maamri, N, Sabatier, J, Oustaloup, A: Transients of fractional-order integrator and derivatives. Signal Image Video Process. **6**(3), 359-372 (2012)
17. Sabatier, J, Farges, C, Trigeassou, J-C: Fractional systems state space description: some wrong ideas and proposed solutions. J. Vib. Control **20**(7), 1076-1084 (2014)
18. Horn, RA, Johnson, CR: Matrix Analysis, 2nd edn. Cambridge University Press, New York (2013)
19. Tavazoei, MS, Haeri, M: A note on the stability of fractional order systems. Math. Comput. Simul. **79**(5), 1566-1576 (2009)
20. Li, H, Zhong, S-M, Li, H-B: Stability analysis of fractional order systems with time delay. Int. J. Math. Comput. Sci. Eng. **8**(4), 14-17 (2014)