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The Bogdanov-Takens bifurcation study of $2m$ coupled neurons system with $2m+1$ delays

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Abstract

In this paper the Bogdanov-Takens (BT) bifurcation of an $2m$ coupled neurons network model with multiple delays is studied, where one neuron is excitatory and the next is inhibitory. When the origin of the model has a double zero eigenvalue, by using center manifold reduction of delay differential equations (DDEs), the second-order and third-order universal unfoldings of the normal forms are deduced, respectively. Some bifurcation diagrams and numerical simulations are presented to verify our main results.

Keywords: neurons; delays; Bogdanov-Takens bifurcation; normal form

1 Introduction

By using the methods developed by the authors in [1–3], the codimension one and two bifurcations for some neural network models with time delays have been studied; see [4–19] for example and the references therein. But except the authors in [20, 21] who have carried out the Hopf bifurcation of some models with n neurons, there are few codimension two bifurcation results about the neural network models with n neurons and more delays. Recently, the BT bifurcation, Hopf-transcritical, and Hopf-pitchfork bifurcations of the following model:

$$\begin{aligned} \dot{u}_1(t) &= -ku_1(t) + af\bigl(u_1(t-r)\bigr) + b_1g_1\bigl(u_2(t-\tau_2)\bigr), \\ \dot{u}_2(t) &= -ku_2(t) + af\bigl(u_2(t-r)\bigr) + b_2g_2\bigl(u_1(t-\tau_1)\bigr), \end{aligned} \tag{1.1}$$

have been studied by Yuan and Wei in [10]. Guo *et al.* in [11] analyzed the fold and Hopf bifurcations, fold-Hopf bifurcations and Hopf-Hopf bifurcations of system (1.1) with $k=1$, $g_1 = g_2 = f$.

In the follow-up the authors in [12] studied the stability and bifurcation of the following four coupled model:

$$\begin{aligned} \dot{u}_1(t) &= -ku_1(t) + f\bigl(u_1(t-r)\bigr) + g_1\bigl(u_4(t-\tau_4)\bigr), \\ \dot{u}_2(t) &= -ku_2(t) + f\bigl(u_2(t-r)\bigr) + g_2\bigl(u_1(t-\tau_1)\bigr), \\ \dot{u}_3(t) &= -ku_3(t) + f\bigl(u_3(t-r)\bigr) + g_3\bigl(u_2(t-\tau_2)\bigr), \\ \dot{u}_4(t) &= -ku_4(t) + f\bigl(u_4(t-r)\bigr) + g_4\bigl(u_3(t-\tau_3)\bigr). \end{aligned} \tag{1.2}$$

Fan *et al.* in [17] considered the following coupled model of two neurons:

$$\begin{aligned} \dot{u}_1(t) &= -u_1(t) + a\tanh\bigl(u_1(t-\tau_v)\bigr) - a_{12}\tanh\bigl(u_2(t-\tau_2)\bigr), \\ \dot{u}_2(t) &= -u_2(t) + a_{21}\tanh\bigl(u_1(t-\tau_1)\bigr) - a\tanh\bigl(u_2(t-\tau_v)\bigr), \end{aligned} \tag{1.3}$$

the coupling strengths will change their signs. The authors developed the universal unfolding of BT bifurcation with Z_2 symmetry at the origin of the system (1.3) in the special case of $\tau_v = 0$, $\tau_1 = \tau_2 = \tau > 0$, and $a_{12} = a_{21} = b$.

The relation of systems (1.1) and (1.2) motivates us to extend the system (1.3) involving n neurons, *i.e.*, the following system:

$$\begin{aligned} &\dot{u}_1(t) = -u_1(t) + af_1\bigl(u_1(t-\tau_s)\bigr) - a_{2m,1}g_{2m}\bigl(u_{2m}(t-\tau_{2m})\bigr), \\ &\ldots, \\ &\dot{u}_i(t) = -u_i(t) + (-1)^{i+1}af_i\bigl(u_i(t-\tau_s)\bigr) + (-1)^i a_{i-1,i}g_{i-1}\bigl(u_{i-1}(t-\tau_{i-1})\bigr), \\ &\ldots, \\ &\dot{u}_{2m}(t) = -u_{2m}(t) - af_{2m}\bigl(u_{2m}(t-\tau_s)\bigr) + a_{2m-1,2m}g_{2m-1}\bigl(u_{2m-1}(t-\tau_{2m-1})\bigr), \end{aligned} \tag{1.4}$$

where m is an integer, $a > 0$ is the feedback strength, τ_s is the feedback delay; $\tau_1, \tau_2, \ldots, \tau_{2m}$ denote the connection delays, $a_{12}, a_{23}, \ldots, a_{2m-1,2m}, a_{2m,1}$ represent the connection strengths. Each neuron comes with a delayed self-feedback and a delayed connection from the other neuron, and one neuron is excitatory and the other inhibitory. As regards the relations of each neuron one can see Figure 1.

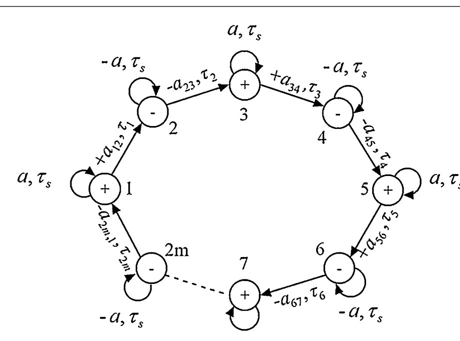
For simplicity, we assume

(H_1) $f_i(0) = g_i(0) = 0, f_i'(0) = g_i'(0) = 1$, $i = 1, 2, \ldots, 2m$.

The universal unfoldings about the BT bifurcation at the origin of system (1.4) will be given. Therefore, our study is not trivial and our results are general.

The rest of this paper is organized as follows: in Section 2, the conditions under which the origin of system (1.4) is a BT singularity are demonstrated; in Section 3, the second- and third-order normal forms at the BT singularity of the coupled system are presented; in Section 4, some bifurcation diagrams and numerical simulations are shown.

Figure 1 Architecture of the model with 2*m* neurons, *m* is an integer.



2 The existence of BT singularity

Since the origin is always the equilibrium of system (1.4), linearizing system (1.4) at the origin yields

$$\begin{aligned}
&\dot{u}_1(t) = -u_1(t) + au_1(t-\tau_s) - a_{2m,1}u_n(t-\tau_{2m}),\\
&\ldots,\\
&\dot{u}_i(t) = -u_i(t) + (-1)^{i+1}au_i(t-\tau_s) + (-1)^i a_{i-1,i}u_{i-1}(t-\tau_{i-1}),\\
&\ldots,\\
&\dot{u}_{2m}(t) = -u_{2m}(t) - au_{2m}(t-\tau_s) + a_{2m-1,2m}u_{2m-1}(t-\tau_{2m-1}).
\end{aligned} \tag{2.1}$$

Then the corresponding characteristic equation is

$$F(\lambda) = \left((\lambda+1)^2 - \delta \mathrm{e}^{-2\lambda\tau_s}\right)^m + (-1)^{m+1}\beta^m \mathrm{e}^{-2m\lambda\tau_0} = 0, \tag{2.2}$$

where $\delta = a^2$, $\beta^m = a_{12}a_{23}a_{34}\cdots a_{2m-1,2m}a_{2m,1}$, $\tau_0 = \frac{\tau_1+\tau_2+\cdots+\tau_{2m}}{2m}$.

By (2.2) we can obtain

$$\begin{aligned}
&F(0) = (1-\delta)^m + (-1)^{m+1}\beta^m,\\
&F'(0) = 2m(1-\delta)^{m-1}(1+\delta\tau_s) - 2(-1)^{m+1}m\tau_0\beta^m,\\
&F''(0) = 4m(m-1)(1-\delta)^{m-2}(1+\delta\tau_s)^2 + 2m(1-\delta)^{m-1}\left(1-2\delta\tau_s^2\right)\\
&\qquad + (-1)^{m+1}2m^2\tau_0^2\beta^m.
\end{aligned} \tag{2.3}$$

Solving $F(0) = 0$, we have $\delta = \beta + 1$, then by $F'(0) = 0$ we have $\beta = \frac{1+\tau_s}{\tau_0-\tau_s} > 0$, which implies $\delta = \frac{1+\tau_0}{\tau_0-\tau_s}$ and then

$$F''(0) = (-1)^{m-1}\frac{2m(1+\tau_s)^{m-1}(2\tau_0+2\tau_s\tau_0+1+2\tau_s)}{(\tau_0-\tau_s)^{m-1}} \neq 0.$$

To show that the origin of system (1.4) is a BT singularity, we should investigate the conditions under which all the roots of (2.2), except $\lambda = 0$, have negative real parts.

Let $\delta = \beta + 1$. First, when $\tau_s = \tau_0 = 0$, solving (2.2) one can obtain $\lambda_1 = 0$ and $\lambda_2 = -2$.

Second, when $\tau_s \neq \tau_0$, we assume $\tau_s = 0$, then (2.2) can be written as

$$F(\lambda) = \left[(\lambda+1)^2 - (\beta+1)\right]^m + (-1)^{m+1}\beta^m \mathrm{e}^{-2m\lambda\tau_0} = 0, \tag{2.4}$$

if $\lambda = iq$ ($q > 0$) is a root of (2.4), then it needs $(iq+1)^2 - (\beta+1) = -\beta \mathrm{e}^{-2iq\tau_0}$, *i.e.*,

$$-q^2 - \beta + \beta\cos(2q\tau_0) + i\left(2q - \beta\sin(2q\tau_0)\right) = 0,$$

by separating the real and negative parts of the above equation and a simple computation we have

$$\begin{cases}\cos(2q\tau_0) = \frac{\beta+q^2}{\beta},\\ \sin(2q\tau_0) = \frac{2q}{\beta}.\end{cases} \tag{2.5}$$

Hence, q should satisfy the equation $q^2 + 2\beta + 4 = 0$, due to $\beta > 0$, a positive q does not exist.

When $\tau_s \neq 0$, then $\lambda = iw$ ($w > 0$) is a root of (2.2) if and only if w satisfies the following equation:

$$(iw+1)^2 - (\beta+1)\mathrm{e}^{-2iw\tau_s} = -\beta \mathrm{e}^{-2iw\tau_0}, \tag{2.6}$$

then one can obtain

$$\begin{cases} 1 - w^2 = (\beta+1)\cos(2w\tau_s) - \beta\cos(2w\tau_0), \\ 2w = \beta\sin(2w\tau_0) - (\beta+1)\sin(2w\tau_s), \end{cases}$$

which yields

$$\left(1-w^2\right)^2 + 4w^2 = (\beta+1)^2 + \beta^2 - 2\beta(\beta+1)\cos\bigl(2w(\tau_0-\tau_s)\bigr). \tag{2.7}$$

We rewrite (2.7) as

$$\cos\bigl(2w(\tau_0-\tau_s)\bigr) = \frac{2\beta(\beta+1) - 2w^2 - w^4}{2\beta(\beta+1)}. \tag{2.8}$$

To investigate the existence of positive root of (2.8), we first consider the following equations:

$$\cos\bigl(2w(\tau_0-\tau_s)\bigr) = 0, \qquad \frac{2\beta(\beta+1) - 2w^2 - w^4}{2\beta(\beta+1)} = 0,$$

which, respectively, have the positive roots

$$w_0 = \frac{\pi}{4(\tau_0-\tau_s)}, \qquad w_* = \sqrt{-1 + \sqrt{\beta^2 + (\beta+1)^2}}. \tag{2.9}$$

It is easy to verify that (2.8) does not have a positive root w if $w_* < w_0$ and has a positive root $\bar{w}$ if $w_* \geq w_0$. To make this clear one can see Figure 2.

Together with the above discussion, we have the following lemma.

Lemma 2.1 *Let* (H_2) $\delta = \frac{1+\tau_0}{\tau_0-\tau_s}$, $\beta = \frac{1+\tau_s}{\tau_0-\tau_s}$, $\tau_0 > \tau_s$, $w_* < w_0$. *Then all the roots of system* (2.2), *except* $\lambda = 0$, *have negative real parts, i.e., the origin of system* (1.4) *is a BT singularity, where* $w_0 = \frac{\pi}{4(\tau_0-\tau_s)}$, $w_* = \sqrt{-1 + \sqrt{\beta^2 + (\beta+1)^2}}$.

3 Normal forms of BT bifurcation

From Lemma 2.1 we know that when (H_1) and (H_2) hold, then system (1.4) at the origin will undergo a BT bifurcation. In the following, we will generalize the methods introduced in [1, 3] to compute the second- and third-order normal forms of the BT bifurcation. For simplicity, as the authors have done in [17], we take $a_{12} = a_{2m,1} = b$ and choose a and a_{21} as the bifurcation parameters, *i.e.*, we consider $a = a_0 + \alpha_1$ and $a_{12} = a_{2m,1} = b_0 + \alpha_2$, where α_1 and α_2 are all near $(0,0)$ and $\delta = a_0^2 = \frac{1+\tau_0}{\tau_0-\tau_s}$, $\beta = \frac{1+\tau_s}{\tau_0-\tau_s}$, $\beta^m = a_{12}^0 a_{23} a_{34} \cdots a_{2m-1,2m} a_{2m,1}^0 =$

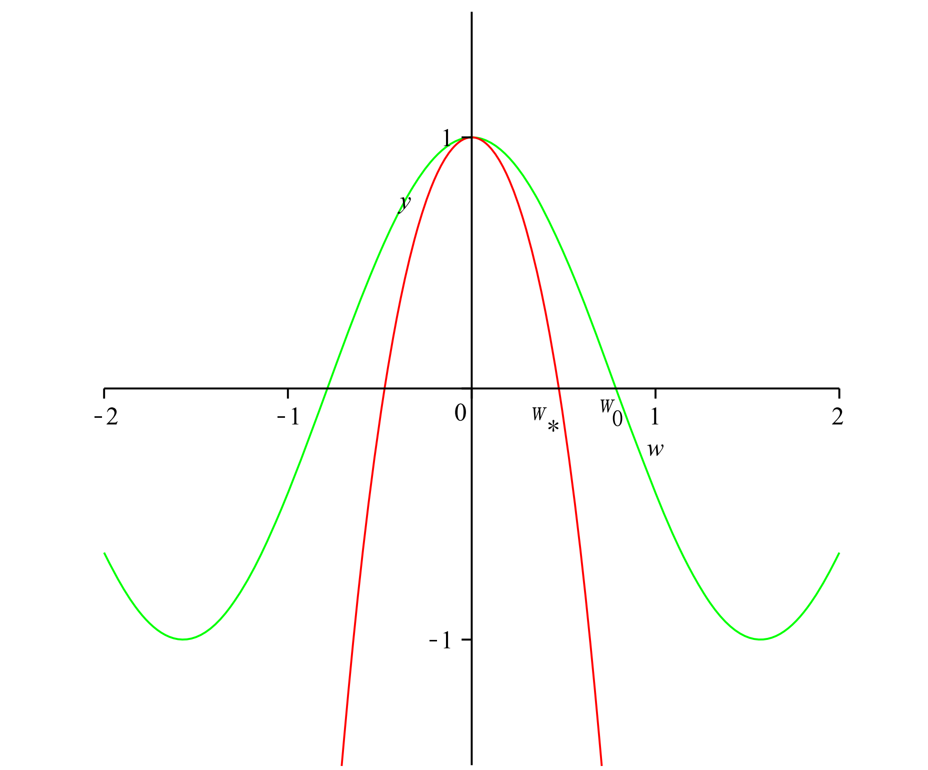


Figure 2 The distribution of the roots of (2.8).

$b_0^2 a_{23}a_{34}\cdots a_{2m-1,2m}$. Then system (1.4) becomes

$$\begin{aligned}
&\dot{u}_1(t) = -u_1(t) + (a_0+\alpha_1)f_1\big(u_1(t-\tau_s)\big) - (b_0+\alpha_2)g_{2m}\big(u_{2m}(t-\tau_{2m})\big),\\
&\dot{u}_2(t) = -u_2(t) - (a_0+\alpha_1)f_2\big(u_2(t-\tau_s)\big) + (b_0+\alpha_2)g_1\big(u_1(t-\tau_1)\big),\\
&\ldots,\\
&\dot{u}_i(t) = -u_i(t) + (-1)^{i+1}(a_0+\alpha_1)f_i\big(u_i(t-\tau_s)\big) + (-1)^i a_{i-1,i}g_{i-1}\big(u_{i-1}(t-\tau_{i-1})\big),\\
&\ldots,\\
&\dot{u}_{2m}(t) = -u_{2m}(t) - (a_0+\alpha_1)f_{2m}\big(u_{2m}(t-\tau_s)\big) + a_{2m-1,2m}g_{2m-1}\big(u_{2m-1}(t-\tau_{2m-1})\big).
\end{aligned} \tag{3.1}$$

For simplicity, we rewrite system (3.1) as the following retarded functional differential equation (FDE) with parameters in the phase space $C = C([-\tau_1, 0]; R^n)$ [1]:

$$\dot{U}(t) = L(\alpha)U_t + G(U_t, \alpha), \tag{3.2}$$

with $\varphi = (\varphi_1, \varphi_2 \cdots, \varphi_n)^T \in C$.

The operator $L_0 = L(0)$ has the form

$$L_0(\varphi) = \int_{-\tau_1}^{0} d\eta(\theta)\varphi(\theta) = Au(t) + \sum_{l=1}^{2m} B_l u(t-\tau_l) + B_{2m+1}u(t-\tau_s).$$

Define Λ to be the set of eigenvalues with zero real part, for a BT bifurcation, we have $\Lambda_0 = \{0, 0\}$, using the formal adjoint theory to decompose the phase space of an FDE. P denotes the generalized eigenspace associated with the eigenvalues in Λ, and P^* is the dual space of P. Then the phase space C can be decomposed as $C = P \oplus Q$ by Λ, where

$$Q = \{\phi \in C : \langle \psi, \phi \rangle = 0\}.$$

Denote the dual bases of P and P^* by Φ and Ψ, respectively, satisfying $\langle \Psi(s), \Phi(\theta) \rangle = I_2$, $\dot{\Phi} = \Phi B$ and $-\dot{\Psi} = B\Psi$, with $B = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Following similar methods to Lemma 3.1 in [3], we can obtain

$$\Phi(\theta) = \begin{pmatrix} 1 & \theta \\ \phi_{21} & \phi_{22} + \theta\phi_{21} \\ \phi_{31} & \phi_{32} + \theta\phi_{31} \\ \cdots & \cdots \\ \phi_{2m,1} & \phi_{n2} + \theta\phi_{2m,1} \end{pmatrix}, \qquad \Psi(0) = \begin{pmatrix} \psi_{11} & \psi_{12} & \cdots & \psi_{1,2m} \\ \psi_{21} & \psi_{22} & \cdots & \psi_{2,2m} \end{pmatrix},$$

where

$$\phi_{21} = \frac{b_0}{1 + a_0}, \qquad \phi_{31} = \frac{b_0 a_{23}}{(1 + a_0)(-1 + a_0)}, \qquad \ldots,$$

$$\phi_{i1} = \frac{b_0 a_{23} a_{34} \cdots a_{i-1,i}}{(1 + a_0)^{\frac{i}{2}}(-1 + a_0)^{\frac{i}{2}-1}} \quad (i \text{ is even}),$$

$$\phi_{i1} = \frac{b_0 a_{23} a_{34} \cdots a_{i-1,i}}{(1 + a_0)^{\frac{i-1}{2}}(-1 + a_0)^{\frac{i-1}{2}}} \quad (i \text{ is odd}), \qquad \ldots, \qquad \phi_{2m,1} = \frac{1 + a_0}{b_0};$$

$$\phi_{22} = \frac{b_0[a_0(\tau_s - \tau_1) - 1 - \tau_1]}{(1 + a_0)^2}, \qquad \phi_{32} = \frac{b_0 a_{23}[a_0^2(2\tau_s - \tau_1 - \tau_2) + 2 + \tau_1 + \tau_2]}{(1 + a_0)^2(-1 + a_0)^2},$$

$$\ldots,$$

$$\phi_{i2} = \frac{b_0 a_{23} a_{34} \cdots a_{i-1,i}[a_0^2((i-1)\tau_s - \sum_{l=1}^{i-1}\tau_l) - a_0(\tau_s + 1) + i - 1 + \sum_{l=1}^{i-1}\tau_l]}{(1 + a_0)^{\frac{i}{2}+1}(-1 + a_0)^{\frac{i}{2}}}$$

$$(i \text{ is even}),$$

$$\phi_{i2} = \frac{b_0 a_{23} a_{34} \cdots a_{i-1,i}[a_0^2((i-1)\tau_s - \sum_{l=1}^{i-1}\tau_l) + i - 1 + \sum_{l=1}^{i-1}\tau_l]}{(1 + a_0)^{\frac{i+1}{2}}(-1 + a_0)^{\frac{i+1}{2}}} \quad (i \text{ is odd}),$$

$$\ldots,$$

$$\phi_{2m,2} = \frac{a_0(\tau_{2m} - \tau_s) - (1 + \tau_{2m})}{b_0}; \qquad \psi_{21} = \frac{1 + a_0}{\xi}, \qquad \xi = 2m\tau_s + (1 + \tau_s)\sum_{l=1}^{2m}\tau_l + \frac{2m}{2},$$

$$\psi_{22} = -\frac{(-1 + a_0)\psi_{21}}{b_0}, \qquad \psi_{23} = \frac{(1 + a_0)(-1 + a_0)\psi_{21}}{b_0 a_{23}}, \qquad \ldots,$$

$$\psi_{2i} = -\frac{(1 + a_0)^{\frac{i}{2}-1}(-1 + a_0)^{\frac{i}{2}}\psi_{21}}{b_0 a_{23} a_{34} \cdots, a_{i-1,i}} \quad (i \text{ is even}),$$

$$\psi_{2i} = \frac{(1 + a_0)^{\frac{i-1}{2}}(-1 + a_0)^{\frac{i-1}{2}}\psi_{21}}{b_0 a_{23} a_{34} \cdots a_{i-1,i}} \quad (i \text{ is odd}), \qquad \ldots,$$

$$\begin{aligned}
\psi_{2,2m} &= \frac{b_0\psi_{21}}{-(1+a_0)}, \\
\psi_{11} &= \frac{1+a_0}{6m\xi^2}\left[3a_0\xi\left(\sum_{l=1}^{2m}\tau_l - 2m\tau_s\right) - \sum_{l=1}^{2m}\tau_l\left(\sum_{l=1}^{2m}\tau_l + 3m\right)\right. \\
&\quad \left. - \tau_s\left(\sum_{l=1}^{2m}\tau_l + 2m\right)\left(\sum_{l=1}^{2m}\tau_l - 4m\tau_s\right)\right], \\
\psi_{12} &= -\frac{\psi_{11}(-1+a_0) + \psi_{21}[a_0(\tau_s - \tau_1) - 1 - \tau_1]}{b_0}, \\
\psi_{13} &= \frac{\psi_{11}(-1+a_0)(1+a_0) + \psi_{21}[a_0^2(\tau_1 + \tau_2 - 2\tau_s) - 2 - \tau_1 - \tau_2]}{b_0 a_{23}}, \\
&\ldots, \\
\psi_{1i} &= -\frac{\psi_{11}(1+a_0)^{\frac{i-2}{2}}(-1+a_0)^{\frac{i}{2}}}{b_0 a_{23}a_{34}\cdots a_{i-1,i}} - \frac{\psi_{21}(1+a_0)^{\frac{i-4}{2}}(-1+a_0)^{\frac{i-2}{2}}}{b_0 a_{23}a_{34}\cdots a_{i-1,i}} \\
&\quad \times\left[a_0^2\left(\sum_{l=1}^{i-1}\tau_l - (i-1)\tau_s\right) - a_0(\tau_s+1) - (i-1) - \sum_{l=1}^{i-1}\tau_l\right] \quad (i \text{ is even}), \\
\psi_{1i} &= \frac{\psi_{11}(1+a_0)^{\frac{i-1}{2}}(-1+a_0)^{\frac{i-1}{2}}}{b_0 a_{23}a_{34}\cdots a_{i-1,i}} + \frac{\psi_{21}(1+a_0)^{\frac{i-3}{2}}(-1+a_0)^{\frac{i-3}{2}}}{b_0 a_{23}a_{34}\cdots a_{i-1,i}} \\
&\quad \times\left[a_0^2\left(\sum_{l=1}^{i-1}\tau_l - (i-1)\tau_s\right) - (i-1) - \sum_{l=1}^{i-1}\tau_l\right] \quad (i \text{ is odd}), \\
&\ldots, \\
\psi_{1,2m} &= -\frac{\psi_{11}(1+a_0)^{m-1}(-1+a_0)^{m}}{b_0 a_{23}a_{34}\cdots a_{2m-1,2m}} - \frac{\psi_{21}(1+a_0)^{m-2}(-1+a_0)^{m-1}}{b_0 a_{23}a_{34}\cdots a_{2m-1,2m}} \\
&\quad \times\left[a_0^2\left(\sum_{l=1}^{2m-1}\tau_l - (2m-1)\tau_s\right) - a_0(\tau_s+1) - (2m-1) - \sum_{l=1}^{2m-1}\tau_l\right].
\end{aligned}$$

The representation of $\Psi(s)$ is given in the Appendix. Denote the Taylor expansion of $\widehat{F}(u_t,\alpha)$ with respect to u_t and α in system (3.1) as $\widehat{F}(u_t,\alpha) = \sum_{j\geq 2}\frac{1}{j!}\widehat{F}_j(u_t,\alpha)$, we have

$$\begin{aligned}
\frac{1}{2}\widehat{F}_2(u_t,\alpha) &= A_1u(t)\alpha_1 + A_2u(t)\alpha_2 + \sum_{l=1}^{2m+1}\left[B_{l1}u(t-\tau_l)\alpha_1 + B_{l2}u(t-\tau_l)\alpha_2\right] \\
&\quad + \sum_{i=1}^{2m+1}\sum_{0\leq k\leq j\leq 2m+1} D_{ikj}u_i(t-\tau_k)u(t-\tau_j),
\end{aligned} \tag{3.3}$$

where $\tau_0 = 0$.

Using (3.3) and the formulas obtained in [3], we deduce the second order of the BT bifurcation normal form as follows.

Theorem 3.1 *Let* (H_1) *and* (H_2) *hold. Then the delay differential system* (3.1) *can be reduced to the following two-dimensional system of ODE on the center manifold at* $(u_t,\alpha) = (0,0)$:

$$\begin{aligned}
\dot{z}_1 &= z_2, \\
\dot{z}_2 &= k_1z_1 + k_2z_2 + \eta_1z_1^2 + \eta_2z_1z_2 + h.o.t.,
\end{aligned} \tag{3.4}$$

where

$$
\begin{aligned}
k_1 &= \psi_2^0\left(A_1 + \sum_{l=1}^{2m+1} B_{l1}\right)\phi_1^0\alpha_1 + \psi_2^0\left(A_2 + \sum_{l=1}^{2m+1} B_{l2}\right)\phi_1^0\alpha_2, \\
k_2 &= \left\{\psi_1^0\left(A_1 + \sum_{l=1}^{2m+1} B_{l1}\right)\phi_1^0 + \psi_2^0\left[\left(A_1 + \sum_{l=1}^{2m+1} B_{l1}\right)\phi_2^0 - \sum_{l=1}^{2m+1}\tau_l B_{l1}\phi_1^0\right]\right\}\alpha_1 \\
&\quad + \left\{\psi_1^0\left(A_2 + \sum_{l=1}^{2m+1} B_{l2}\right)\phi_1^0 + \psi_2^0\left[\left(A_2 + \sum_{l=1}^{2m+1} B_{l2}\right)\phi_2^0 - \sum_{l=1}^{2m+1}\tau_l B_{l2}\phi_1^0\right]\right\}\alpha_2, \\
\eta_1 &= \psi_2^0\left(\sum_{i=1}^{2m+1}\sum_{0\le k\le j\le 2m+1} D_{ikj}\phi_1^0\phi_{i1}^0\right), \\
\eta_2 &= 2\psi_1^0\left(\sum_{i=1}^{2m+1}\sum_{0\le k\le j\le m} D_{ikj}\phi_1^0\phi_{i1}^0\right) + \psi_2^0\left\{\sum_{i=1}^{2m+1}\sum_{0\le k\le j\le 2m+1} D_{ikj}\left(\phi_1^0\phi_{2i}^0 + \phi_2^0\phi_{i1}^0\right)\right. \\
&\quad \left. - \sum_{i=1}^{2m+1}\sum_{0\le k\le j\le 2m+1}(\tau_k + \tau_j)D_{ikj}\phi_1^0\phi_{i1}^0\right\}.
\end{aligned}
$$

If $\eta_1 \neq 0$ and $\eta_2 \neq 0$ hold, the bifurcation curves related to the perturbation parameters α_1, α_2 are as follows [9, 22, 23]:

TB: $k_1 = 0$ (transcritical bifurcation occurs),

H_0: $k_2 = 0$, $k_1 < 0$ (Hopf bifurcation from the zero equilibrium point),

H_1: $k_2 = \frac{\eta_2}{\eta_1}k_1$, $k_1 > 0$ (a Hopf bifurcation from the non-trivial equilibrium),

H_c^0: $k_2 = \frac{\eta_2}{7\eta_1}k_1$, $k_1 < 0$ (a homoclinic bifurcation with the zero equilibrium point),

H_c^1: $k_2 = \frac{6\eta_2}{7\eta_1}k_1$, $k_1 > 0$ (a homoclinic bifurcation from the non-trivial equilibrium).

A numerical example is given in Section 4 (see Figures 3-5).

If $f_i''(0) = g_i''(0) = 0$, then $\eta_1 = \eta_2 = 0$, system (3.1) is degenerate. To determine the dynamics near BT bifurcation we need to calculate the higher-order normal form. As [1, 22] we have

$$
g_3^1(z,0,\mu) = \left(I - P_{I,3}^1\right)\tilde{f}_3^1(z,0,\mu) = \mathrm{Proj}_{\mathrm{Im}(M_3^1)^c}\tilde{f}_3^1(z,0,\mu), \tag{3.5}
$$

where

$$
\begin{aligned}
\tilde{f}_3^1(z,0,\mu) = {}& f_3^1(z,0,\mu) + \frac{3}{2}\left[\left(D_z f_2^1\right)(z,0,\mu)U_2^1(z,\mu) + \left(D_y f_2^1\right)(z,0,\mu)U_2^2(z,\mu)\right. \\
& \left. - \left(D_z U_2^1\right)(z,\mu)g_2^1(z,0,\mu)\right].
\end{aligned}
$$

It is easy to obtain

$$
f_3^1(z,0,\mu) = \begin{pmatrix}
\psi_{11}[a_0 f_1'''\varphi_1^3(-\tau_s) - b_0 g_{2m}'''\varphi_{2m}^3(-\tau_{2m})] \\
\quad + \psi_{12}[-a_0 f_2'''\varphi_2^3(-\tau_s) + b_0 g_1'''\varphi_1^3(-\tau_1)] \\
\quad + \sum_{i=3}^{2m}\psi_{1i}[(-1)^{i+1}a_0 f_i'''\varphi_i^3(-\tau_s) + (-1)^i a_{i-1,i}g_{i-1}'''\varphi_{i-1}^3(-\tau_{i-1})] \\
\psi_{21}[a_0 f_1'''\varphi_1^3(-\tau_s) - a_{2m,1}g_{2m}'''\varphi_{2m}^3(-\tau_{2m})] \\
\quad + \psi_{22}[-a_0 f_2'''\varphi_2^3(-\tau_s) + b_0 g_1'''\varphi_1^3(-\tau_1)] \\
\quad + \sum_{i=3}^{2m}\psi_{2i}[(-1)^{i+1}a_0 f_i'''\varphi_i^3(-\tau_s) + (-1)^i a_{i-1,i}g_{i-1}'''\varphi_{i-1}^3(-\tau_{i-1})]
\end{pmatrix}, \tag{3.6}
$$

where $f_i''' = f_i'''(0)$ and $g_i''' = g_i'''(0)$, $i = 1,2,3,\dots,2m$, $\varphi_1(\theta) = z_1 + \theta z_2$, $\varphi_j(\theta) = \phi_{j1}z_1 + (\phi_{j2} + \theta\phi_{j1})z_2$, $j = 2,3,\dots,2m$.

To obtain the third-order normal form, one needs the decomposition

$$\mathrm{V}_3^4\left(R^2\right) = \mathrm{Im}\left(M_3^1\right) \oplus \mathrm{Im}\left(M_3^1\right)^c.$$

Then the canonical basis in $\mathrm{V}_3^4 X(R^2)$ has 40 elements: $((z,\alpha)^3, 0)^T$, $(0,(z,\alpha)^3)^T$, and for the bases of $\mathrm{Im}(M_3^1)$ and $\mathrm{Im}(M_3^1)^c$ one can refer to [22]. By the definition of $\mathrm{Proj}_{\mathrm{Im}(M_3^1)^c}$ we have

$$\begin{aligned}
&\mathrm{Proj}_{\mathrm{Im}(M_3^1)^c}\, p = \begin{cases} p, & p \in \mathrm{Im}(M_3^1)^c, \\ 0, & p \in \mathrm{Im}(M_3^1), \end{cases} \qquad \mathrm{Proj}_{\mathrm{Im}(M_3^1)^c}\begin{pmatrix} z_1^3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 3z_1^2 z_2 \end{pmatrix}, \\
&\mathrm{Proj}_{\mathrm{Im}(M_3^1)^c}\begin{pmatrix} z_1\alpha_i^2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ z_2\alpha_i^2 \end{pmatrix}, \qquad \mathrm{Proj}_{\mathrm{Im}(M_3^1)^c}\begin{pmatrix} z_1\alpha_1\alpha_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha_1\alpha_2 z_2 \end{pmatrix}, \\
&\mathrm{Proj}_{\mathrm{Im}(M_3^1)^c}\begin{pmatrix} z_1^2\alpha_i \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2z_1 z_2\alpha_i \end{pmatrix}.
\end{aligned}$$

Together with (3.6) and by [22, 23] the third-order normal form of system (3.1) can be written as

$$\begin{aligned}
\dot{z}_1 &= z_2, \\
\dot{z}_2 &= k_1 z_1 + k_2 z_2 + c z_1^3 + d z_1^2 z_2 + h.o.t.,
\end{aligned} \tag{3.7}$$

where k_1 and k_2 are the same as in (3.4), and

$$\begin{aligned}
c = {}& \frac{1}{6}\Bigg[\psi_{21}\left(a_0 f_1''' - b_0 g_{2m}''' \phi_{2m,1}^3\right) + \psi_{22}\left(b_0 g_1''' - a_0 f_1''' \phi_{21}^3\right) \\
& + \sum_{i=3}^{2m} \psi_{2i}\left((-1)^{i+1} a_0 f_i''' \phi_{i1}^3 + (-1)^i a_{i-1,i} g_{i-1}''' \phi_{i-1,1}^3\right)\Bigg], \\
d = {}& \frac{1}{2}\Bigg\{a_0 f_1'''\left(\psi_{11} - \psi_{21}\tau_s - \psi_{12}\phi_{21}^3 - \phi_{21}^2(\phi_{22} - \tau_s\phi_{21})\psi_{22}\right) + b_0 g_1'''(\psi_{12} - \tau_1\psi_{22}) \\
& - b_0 g_{2m}''' \phi_{2m,1}^2\left(\psi_{11}\phi_{2m,1} + (\phi_{2m,2} - \tau_{2m}\phi_{2m,1})\psi_{21}\right) \\
& + \sum_{i=3}^{2m}\Big[(-1)^{i+1} a_0 f_1''' \phi_{i1}^2\left(\psi_{1i}\phi_{i1} + \psi_{2i}(\phi_{i2} - \tau_s\phi_{i1})\right) \\
& + (-1)^i a_{i-1,i} g_{i-1}''' \phi_{i-1,1}^2\left(\psi_{1i}\phi_{i-1,1} + (\phi_{i-1,2} - \tau_{i-1}\phi_{i-1,1})\psi_{2i}\right)\Big]\Bigg\}.
\end{aligned}$$

Let $\bar{t} = -\frac{|c|}{d}t$, $w_1 = \frac{d}{\sqrt{|c|}}z_1$, $w_2 = -\frac{d^2}{|c|\sqrt{|c|}}z_2$. Then system (3.7) becomes

$$\begin{aligned}
\dot{w}_1 &= w_2, \\
\dot{w}_2 &= \nu_1 w_1 + \nu_2 w_2 + s w_1^3 - w_1^2 w_2 + h.o.t.,
\end{aligned} \tag{3.8}$$

where $\nu_1 = (\frac{d}{c})^2 k_1$, $\nu_2 = -\frac{d}{|c|} k_2$, $s = \mathrm{sgn}(c)$. From [15] we know the bifurcation of system (3.8) is related to the sign of s. If $s = 1$, we have

S: $\nu_1 = 0$, $\nu_2 \in R$ (a pitchfork bifurcation),

H: $\nu_2 = 0$, $\nu_1 < 0$ (a Hopf bifurcation at the trivial equilibrium),

T: $\nu_2 = -\frac{1}{5}\nu_1$, $\nu_1 < 0$ (a heteroclinic bifurcation).

In Section 4, we show a numerical example under the case of $s = 1$ (see Figure 9).

If $s = -1$, we have

S: $\nu_1 = 0$, $\nu_2 \in R$ (a pitchfork bifurcation),

H: $\nu_1 = \nu_2$, $\nu_1 > 0$ (a Hopf bifurcation at the non-trivial equilibrium),

T: $\nu_2 = \frac{4}{5}\nu_1$, $\nu_1 > 0$ (a homoclinic bifurcation),

H_d: $\nu_2 = d_0\nu_1$, $\nu_1 > 0$, $d_0 \approx 0.752$ (a double cycle bifurcation).

4 Numerical simulation

To verify our main results in the previous sections, in this section, we choose system parameters and functions $f_i(u_i)$, $g_i(u_i)$ satisfying the conditions obtained in Sections 2 and 3 and give some numerical examples and simulations.

First, we take $n = 2$, $f_i(u_i) = \frac{(1+\epsilon)(\mathrm{e}^{u_i}-1)}{\mathrm{e}^{u_i}+\epsilon}$, $g_i(u_i) = \frac{\tanh(d_i u_i)}{d_i}$ [10], $i = 1,2$, $\epsilon = 1.5$, $d_1 = 2$, $d_2 = 1$, $\tau_1 = 6$, $\tau_2 = 0 = \tau_3 = 0$, $a_{2,1} = a_{1,2} = \frac{\sqrt{3}}{3}$, $a_0 = \frac{2\sqrt{3}}{3}$. One can verify that all the conditions in (H_1) and (H_2) are satisfied. Moreover, $f_i(0) = 0$, $f_i'(0) = 1$, $f_i''(0) \neq 0$, $g_i(0) = 0$, $g_i'(0) = 1$, $i = 1,2$. Thus, the coefficients in (3.4) are $k_1 = 0.845299\alpha_1 - 0.845299\alpha_2$, $k_2 = -0.0450298362\alpha_1 + 5.11682660736522\alpha_2$, $\eta_1 = 0.07320508073$, $\eta_2 = 0.1076706575$. The corresponding bifurcation curves of system (3.1) with $m = 1$ is obtained. (See Figure 3.)

If we take $(\alpha_1, \alpha_2) = (-0.01, -0.001)$ and initial conditions $(u_1(0), u_2(0)) = (0.001, 0.001)$, then, in Figure 4(a), one can see that the equilibrium $(0,0)$ is a locally stable focus. However, when $(\alpha_1, \alpha_2) = (-0.01, 0.007935)$, the origin loses its stability, and a periodic solution is bifurcated from the origin (see Figure 4(b)).

Under initial values $(u_1(0), u_2(0)) = (-0.02, -0.00001)$, if we take parameters $(\alpha_1, \alpha_2) = (0.01, 0.00053)$, then system (3.1) has a locally stable non-trivial equilibrium which, how-

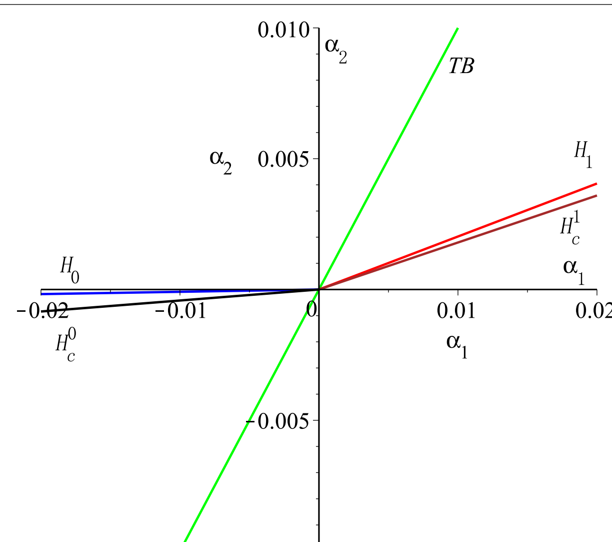


Figure 3 The bifurcation set and phase portraits for (3.4).

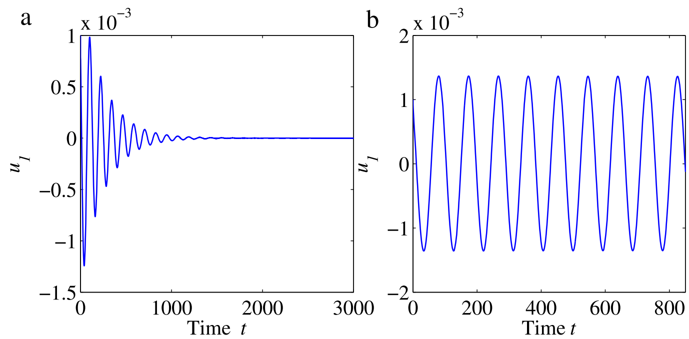


Figure 4 The Hopf bifurcation from the zero equilibrium. (a) The time series with parameters $(\alpha_1,\alpha_2) = (-0.01,-0.001)$ and initial conditions $(u_1(0),u_2(0)) = (0.001,0.001)$. **(b)** The time series with parameters $(\alpha_1,\alpha_2) = (-0.01,0.007935)$ and initial conditions $(u_1(0),u_2(0)) = (0.001,0.001)$.

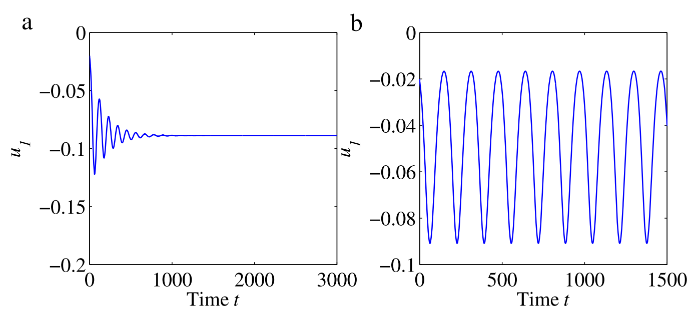


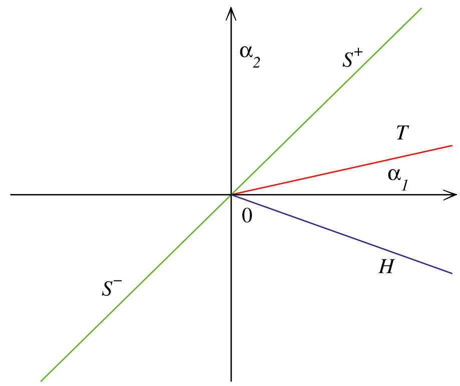
Figure 5 The Hopf bifurcation from the non-trivial equilibrium. (a) The time series with parameters $(\alpha_1,\alpha_2) = (0.01,0.000053)$ and initial conditions $(u_1(0),u_2(0)) = (-0.02,-0.00001)$. **(b)** The time series with parameters $(\alpha_1,\alpha_2) = (0.01,0.006053)$ and initial conditions $(u_1(0),u_2(0)) = (-0.02,-0.00001)$.

ever, becomes unstable when the parameters (α_1,α_2) cross the Hopf bifurcation curve H_1 to another side. One can see a periodic solution is bifurcated from the non-trivial equilibrium as shown in Figure 5.

Second, when $f_i''(0) = g_i''(0) = 0$, we also give an example with $m = 2$ where $\tau_1 = \tau_2 = \tau_3 = \tau_4 = 2$, $\tau_s = 0$, $a_{23} = 0.7$, $a_{34} = \frac{\sqrt{2}}{2}$, $a_{41} = \frac{\sqrt{2}}{2}$, $a_0 = \frac{\sqrt{6}}{2}$, $a_{12} = \frac{\sqrt{2}}{2}$, $f_i(u_i) = g_i(u_i) = \tanh(u_i)$, $i = 1,2,3,4$. One can verify $s = 1$, thus, with the parameters α_1 and α_2 changing in the small neighborhood of $(0,0)$, system (3.1) can undergo a pitchfork bifurcation, a Hopf bifurcation, and a heteroclinic bifurcation. The corresponding bifurcation diagram is exhibited in the parameter plane (α_1,α_2) (see Figure 6).

It can be seen that when $(\alpha_1,\alpha_2) = (-0.001,-0.001)$, three trajectory curves with different initial values consistently converge to the origin $(0,0,0,0)$, *i.e.*, the zero equilibrium is a locally asymptotically stable focus under the given parameters α_1 and α_2. Keeping the initial conditions fixed, we move the perturbation parameter (α_1,α_2) until they across the pitchfork bifurcation curve S to another side, then the origin becomes unstable. Simultaneously, two local stable non-zero equilibria are bifurcated from the origin, which leads to system (3.1) undergoing a pitchfork bifurcation. In Figure 7, we see that system (3.1) has two local stable foci when $(\alpha_1,\alpha_2) = (-0.001,-0.006)$.

Figure 6 The bifurcation set and phase portraits for (3.7) with $m = 2$.



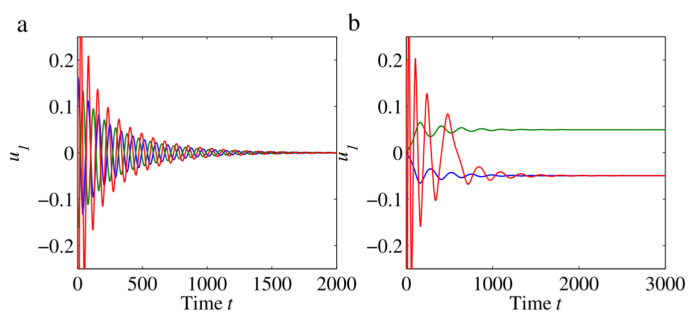


Figure 7 Pitchfork bifurcation is shown with initial values $(u_1(0), u_2(0), u_3(0), u_4(0)) = (0.00053, 0.004, 0.0002, 0.0004)$ (green curve), $(-0.00053, -0.004, -0.0002, -0.0004)$ (blue curve), $(0.5, 0.4, 0.3, 0.4)$ (red curve). (a) The time series when $(\alpha_1, \alpha_2) = (-0.001, -0.001)$. **(b)** The time series when $(\alpha_1, \alpha_2) = (-0.001, -0.006)$.

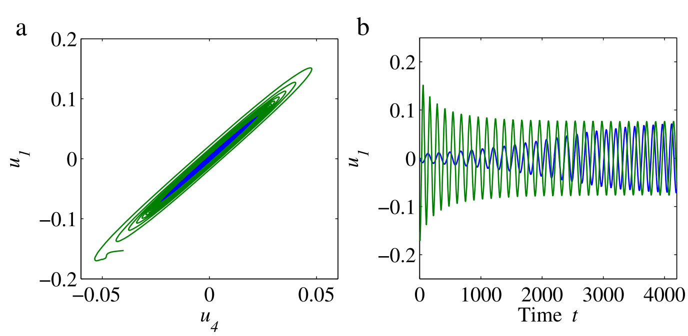


Figure 8 Hopf bifurcation from the origin with initial values $(u_1(0), u_2(0), u_3(0), u_4(0)) = (0.00053, 0.0004, 0.0002, 0.0004)$ (blue curve), $(-0.153, -0.04, -0.02, -0.04)$ (green curve), and parameters $(\alpha_1, \alpha_2) = (-0.001, 0.005)$. (a) The phase portrait in the plane (u_4, u_1). **(b)** The time series in the plane (t, u_1).

Above the line S, we let (α_1, α_2) pass the Hopf bifurcation curve H, and take $(\alpha_1, \alpha_2) = (-0.001, 0.005)$, then the zero equilibrium loses stability, which yields a stable periodic solution as shown in Figure 8.

When the parameters (α_1, α_2) are located under the bifurcation of S^+, and near S^+, all solutions will approach the outer stable periodic solution excluding equilibria and the stable manifold of the trivial equilibrium (see Figure 9(a)). However, when (α_1, α_2) are chosen

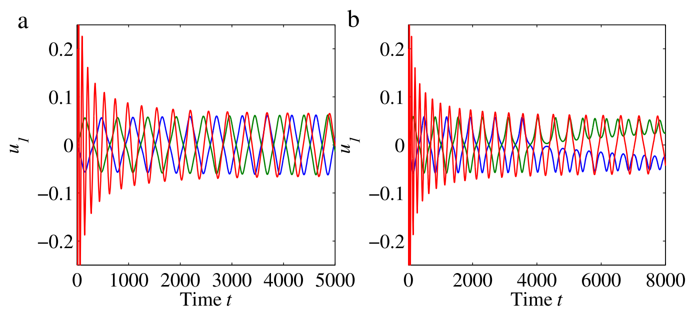


Figure 9 Initial values $(u_1(0), u_2(0), u_3(0), u_4(0)) = (0.00053, 0.004, 0.0002, 0.0004)$ (blue curve), $(0.00053, 0.004, 0.0002, 0.0004)$ (green curve), $(0.5, 0.4, 0.05, 0.4)$ (red curve). (a) $(\alpha_1, \alpha_2) = (0.001, 0.002)$. **(b)** $(\alpha_1, \alpha_2) = (0.001, 0.0018)$.

at the upper-left side of the curve S, then the solutions of system (3.1) are attracted to the corresponding non-zero equilibria if the initial conditions are close to one of the two non-zero equilibria (see the green and blue curves in Figure 9(b)). But if the initial conditions are chosen sufficiently far from the two non-zero equilibria, then the solutions approach the outer stable periodic solution (see the red curve in Figure 9).

Appendix

The bases of P and its dual space P^* have the following representations:

$$\begin{aligned} &P = \operatorname{span}\Phi, \qquad \Phi(\theta) = \bigl(\phi_1(\theta), \phi_2(\theta)\bigr), \\ &P^* = \operatorname{span}\Psi, \qquad \Psi(s) = \operatorname{col}\bigl(\psi_1(s), \psi_2(s)\bigr), \end{aligned}$$

where $\phi_1(\theta) = \phi_1^0 \in R^n\backslash\{0\}$, $\phi_2(\theta) = \phi_2^0 + \phi_1^0\theta$, $\phi_2^0 \in R^n$, and $\psi_2(s) = \psi_2^0 \in R^{n*}\backslash\{0\}$, $\psi_1(s) = \psi_1^0 - s\psi_2^0$, $\psi_1^0 \in R^{n*}$, which satisfy

$$\begin{aligned} &(1) \quad \Biggl(A + \sum_{l=1}^{2m+1} B_l\Biggr)\phi_1^0 = 0, \\ &(2) \quad \Biggl(A + \sum_{l=1}^{2m+1} B_l\Biggr)\phi_2^0 = \Biggl(\sum_{l=1}^{2m+1} \tau_l B_l + I\Biggr)\phi_1^0, \\ &(3) \quad \psi_2^0\Biggl(A + \sum_{l=1}^{2m+1} B_l\Biggr) = 0, \\ &(4) \quad \psi_1^0\Biggl(A + \sum_{l=1}^{2m+1} B_l\Biggr) = \psi_2^0\Biggl(\sum_{l=1}^{2m+1} \tau_l B_l + I\Biggr), \\ &(5) \quad \psi_2^0\phi_2^0 + \psi_2^0\sum_{l=1}^{2m+1} \tau_l B_l\phi_2^0 - \frac{1}{2}\psi_2^0\sum_{l=1}^{2m+1} \tau_l^2 B_l\phi_1^0 = 1, \\ &(6) \quad \psi_1^0\phi_2^0 + \psi_1^0\sum_{l=1}^{2m+1} \tau_l B_l\phi_2^0 - \frac{1}{2}\psi_1^0\sum_{l=1}^{2m+1} \tau_l^2 B_l\phi_1^0 \\ &\qquad - \frac{1}{2}\psi_2^0\sum_{l=1}^{2m+1} \tau_l^2 B_l\phi_2^0 + \frac{1}{6}\psi_2^0\sum_{l=1}^{2m+1} \tau_l^3 B_l\phi_1^0 = 0. \end{aligned}$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
YL, XL, RW conceived and designed the research; YL wrote the paper; YL, RW, ZL revised the manuscript; YL, XL, SL implemented numerical simulations; YL, SL, RW, ZL participated in the discussions. All authors read and approved the final manuscript.

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