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On the hyper-order of solutions of two class of complex linear differential equations

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Abstract
We investigate the hyper-order of solutions of two class of complex linear differential equations. We investigate the growth of solutions of higher order and certain second order linear differential equations, and we obtain some results which improve and extend some previous results in complex oscillations.

MSC: 30D35; 34M10

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1 Introduction and results

We shall assume that reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions (*e.g.* see [1, 2]). In this paper, we use $\rho(f)$, $\tau(f)$ to denote the order and type of an entire function $f(z)$, use $\lambda(f)$ ($\overline{\lambda}(f)$) to denote the exponent of convergence of zeros (distinct zeros) of $f(z)$, and use $\rho_2(f)$ to denote the hyper-order of $f(z)$ (see [3]), which is defined to be

$$\rho_2(f) = \varlimsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}.$$

The hyper-exponent of convergence of zeros and distinct zeros of $f(z)$ are, respectively, defined to be (see [4])

$$\lambda_2(f) = \varlimsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}, \qquad \overline{\lambda}_2(f) = \varlimsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}.$$

In addition, we use M to denote a positive constant, not necessarily the same at each occurrence. We denote the linear measure of a set $E \subset (0,+\infty)$ by $mE = \int_E dt$ and the logarithmic measure of E by $m_l E = \int_E dt/t$, respectively. The upper and the lower logarithmic density of E are defined by

$$\overline{\log\operatorname{dens}}\, E = \varlimsup_{r\to\infty} \frac{m_l(E\cap[1,r])}{\log r}, \qquad \underline{\log\operatorname{dens}}\, E = \varliminf_{r\to\infty} \frac{m_l(E\cap[1,r])}{\log r}.$$

For the second order linear differential equation

$$f'' + A(z)f' + B(z)f = 0, \tag{1.1}$$

where $A(z)$ and $B(z) \not\equiv 0$ are entire functions of finite order, it is well known that every solution $f \not\equiv 0$ of (1.1) is of infinite order if $\rho(A) < \rho(B)$ or $\rho(B) < \rho(A) \leq 1/2$ (see [5–7]). For the case of $\rho(A) > 1/2$ and $\rho(B) < \rho(A)$, the possibility of solutions of infinite order of (1.1) remains open, many authors have studied the problem (*e.g.* see [8–11]). In 2000, Laine and Wu proved the following.

Theorem A (see [10]) *Suppose that* $\rho(B) < \rho(A) < \infty$ *and that* $T(r,A) \sim \log M(r,A)$ *as* $r \to \infty$ *outside a set of finite logarithmic measure. Then every nonconstant solution* f *of* (1.1) *is of infinite order.*

For the higher order linear differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_0(z)f = F(z), \tag{1.2}$$

there are similar results as follows.

Theorem B (see [12]) *Let* $A_j(z)$ $(j = 0, \ldots, k-1)$, $F(z) \not\equiv 0$ *be entire functions. Suppose that there exists some* $d \in \{1, \ldots, k-1\}$ *such that* $\max\{\rho(F), \rho(A_j) : j \neq d\} = \rho < \rho(A_d) < \infty$ *and* $T(r, A_d) \sim \log M(r, A_d)$ *as* $r \to \infty$ *outside a set of upper logarithmic density less than* $(\rho(A_d) - \rho)/\rho(A_d)$. *Then every transcendental solution* $f(z)$ *of* (1.2) *satisfies* $\overline{\lambda}(f) = \lambda(f) = \rho(f) = \infty$.

Theorem C (see [12]) *Let* $A_j(z)$ $(j = 0, \ldots, k-1)$, $F(z) \equiv 0$ *be entire functions. Suppose that there exists some* $d \in \{1, \ldots, k-1\}$ *such that* $\max\{\rho(A_j) : j \neq 0, d\} < \rho(A_0) \leq \frac{1}{2}$ *and that* $A_d(z)$ *has a finite deficient value. Then every solution* $f(z) \not\equiv 0$ *of* (1.2) *satisfies* $\rho(A_0) \leq \rho_2(f) \leq \rho(A_d)$.

Then a natural question is: Can we estimate the hyper-order of the solutions of (1.1) and (1.2) under the same condition in Theorems A and B? And: Can we estimate the hyper-order of the solutions of (1.2) in Theorem C if $\rho(A_0) > \frac{1}{2}$? Theorems 1.1 and 1.2 below give answers to the above questions.

At the same time, many authors have investigated the growth of solutions of (1.1) and its non-homogeneous linear differential equation

$$f'' + A(z)f' + B(z)f = F(z), \tag{1.3}$$

when $\rho(A) = \rho(B)$ and obtained the following results.

Theorem D (see [8]) *Let* $P(z)$ *and* $Q(z)$ *be nonconstant polynomials such that* $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$, $Q(z) = b_n z^n + b_{n-1}z^{n-1} + \cdots + b_1 z + b_0$ *for some complex numbers* a_i, b_i $(i = 0, \ldots, n)$ *with* $a_n b_n \neq 0$ *and let* $A_1(z)$ *and* $A_0(z) \not\equiv 0$ *be entire functions satisfying* $\rho(A_1) < n$ *and* $\rho(A_0) < n$. *Then the following statements hold*:

(i) *If either* $\arg a_n \neq \arg b_n$ *or* $a_n = cb_n$ *with* $0 < c < 1$, *then every nonconstant solution* f *of*

$$f'' + A_1(z)e^{P(z)}f' + A_0(z)e^{Q(z)}f = 0 \tag{1.4}$$

has infinite order with $\rho_2(f) \geq n$.

(ii) *Let* $a_n = b_n$ *and* $\deg(P - Q) = m \geq 1$, *and let the orders of* $A_1(z)$ *and* $A_0(z)$ *be less than* m. *Then every nonconstant solution* f *of* (1.4) *has infinite order with* $\rho_2(f) \geq m$.
(iii) *Let* $a_n = cb_n$ *with* $c > 1$ *and* $\deg(P - cQ) = m \geq 1$. *Suppose* $\rho(A_1) < m$ *and* $0 < \rho(A_0) < \frac{1}{2}$. *Then every nonconstant solution of* (1.4) *has infinite order with* $\rho_2(f) \geq \rho(A_0)$.
(iv) *Let* $a_n = cb_n$ *with* $c > 1$ *and let* $P - cQ$ *be a constant. Suppose that* $\rho(A_1) < \rho(A_0) < \frac{1}{2}$. *Then every nonconstant solution of* (1.4) *has infinite order with* $\rho_2(f) \geq \rho(A_0)$.

Theorem E (see [13]) *Let* a, b *be nonzero complex numbers and* $a \neq b$, $Q(z)$ *be a nonconstant polynomial or* $Q(z) = h(z)e^{bz}$, *where* $h(z) \not\equiv 0$ *is a polynomial. Then every solution* $f \not\equiv 0$ *of the equation*

$$f'' + e^{az}f' + Q(z)f = 0 \tag{1.5}$$

has infinite order and $\rho_2(f) = 1$.

Theorem F (see [14]) *Suppose that* $A_0 \not\equiv 0$, $A_1 \not\equiv 0$, F *are entire functions of order less than one, and the complex constants* a, b *satisfy* $ab \neq 0$ *and* $b \neq a$. *Then every nontrivial solution* f *of*

$$f'' + A_1(z)e^{az}f' + A_0(z)e^{bz}f = F(z) \tag{1.6}$$

is of infinite order.

Theorem G (see [15]) *Let* $P(z) = a_nz^n + \cdots + a_0$, $Q(z) = b_nz^n + \cdots + b_0$ *be polynomials of degree* $n \geq 1$ *where* a_i, b_i $(i = 0, 1, \ldots, n)$ *are complex numbers, and let* $A_0(z) \not\equiv 0$, $A_1(z) \not\equiv 0$, $F(z)$ *be entire functions with order less than* n. *If* $a_n \neq b_n$, *then every solution* $f \not\equiv 0$ *of*

$$f'' + A_1(z)e^{P(z)}f' + A_0(z)e^{Q(z)}f = F(z) \tag{1.7}$$

is of infinite order. Furthermore, if $F(z) \not\equiv 0$, *then every solution* f *of* (1.7) *satisfies* $\overline{\lambda}(f) = \lambda(f) = \rho(f) = \infty$.

Theorem D left us a question: Can we have $\rho_2(f) = n$ (n is a positive integer) for every nontrivial solution of (1.4) if $a_n \neq b_n$? Theorem E tells us that the question holds if $n = 1$. Many authors investigated the above question but none of them solve the question completely, and Theorem 1.3 completely solves this question. In the following, we give our results.

Theorem 1.1 *Let* A_j $(j = 0, \ldots, k-1)$, $F(z)$ *be entire functions. Suppose that there exists some* $d \in \{1, \ldots, k-1\}$ *such that* $\max\{\rho(A_j), \rho(F) : j \neq d\} \leq \rho(A_d) < \infty$, $\max\{\tau(A_j) : \rho(A_j) = \rho(A_d), \tau(F)\} < \tau(A_d)$ *and that* $T(r, A_d) \sim \log M(r, A_d)$ *as* $r \to \infty$ *outside a set of* r *of finite logarithmic measure. Then we have*:
(i) *Every transcendental solution* f *of* (1.2) *satisfies* $\rho_2(f) = \rho(A_d)$, *and* (1.2) *may have polynomial solutions* f *of degree* $< d$.
(ii) *If* $F(z) \not\equiv 0$, *then every transcendental solution* f *of* (1.2) *satisfies* $\overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) = \rho(A_d)$.

(iii) *If* $d = 1$, *then every nonconstant solution* f *of* (1.2) *satisfies* $\rho_2(f) = \rho(A_1)$. *Furthermore, if* $F(z) \not\equiv 0$, *then every nonconstant solution* f *of* (1.2) *satisfies* $\overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) = \rho(A_1)$.

Theorem 1.2 *Let* A_j $(j = 0, \ldots, k-1)$, $F(z) \equiv 0$ *be entire functions satisfying* $\max\{\rho(A_j) : j \neq 0, d\} < \rho(A_0) < \rho(A_d) < \infty$. *Suppose that* $T(r, A_0) \sim \log M(r, A_0)$ *as* $r \to \infty$ *outside a set of* r *of finite logarithmic measure and that* A_d *has a finite deficient value. Then every solution* $f \not\equiv 0$ *of* (1.2) *satisfies* $\rho(A_0) \leq \rho_2(f) \leq \rho(A_d)$.

Theorem 1.3 *Let* $P(z)$, $Q(z)$, $A_0(z)$, $A_1(z)$, $F(z)$ *satisfy the hypotheses of Theorem* G. *Then we have*:

(1) *If* $a_n \neq b_n$, $F(z) \equiv 0$, *then every solution* $f \not\equiv 0$ *of* (1.7) *satisfies* $\rho_2(f) = n$.
(2) *If* $a_n = cb_n$ $(c < 0)$, $F(z) \not\equiv 0$, *then every solution* f *of* (1.7) *satisfies* $\overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) = n$.

Remark 1.1 Theorems 1.1 and 1.2 are improvements of Theorems A-C. Theorem 1.3 is an improvement of Theorems D, E and a supplement to Theorems F, G.

2 Lemmas

Lemma 2.1 (see [16], p.399) *Let* $A_j(z)$ $(j = 0, \ldots, k-1)$, $F(z)$ *be entire functions satisfying* $\max\{\rho(A_j), \rho(F) : j = 0, \ldots, k-1\} \leq \rho < \infty$. *Then every solution* f *of* (1.2) *satisfies* $\rho_2(f) \leq \rho$.

Lemma 2.2 (see [17]) *Let* $f(z)$ *be a transcendental meromorphic function, and let* $\alpha > 1$ *be a given constant. Then for any given constant and for any given* $\varepsilon > 0$:

(i) *There exist a constant* $B > 0$ *and a set* $E_1 \subset (0, \infty)$ *having finite logarithmic measure such that, for all* z *satisfying* $|z| = r \notin E_1$, *we have*

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \leq B\left[\frac{T(\alpha r, f)}{r}(\log r)^{\alpha} \log T(\alpha r, f)\right]^{j-i} \quad (0 \leq i < j). \tag{2.1}$$

(ii) *There exist a set* $H_1 \subset [0, 2\pi)$ *that has linear measure zero and a constant* $B > 0$ *that depends only on* α, *for any* $\theta \in [0, 2\pi) \setminus H_1$, *there exists a constant* $R_0 = R_0(\theta) > 1$ *such that, for all* z *satisfying* $\arg z = \theta$ *and* $|z| = r > R_0$, *we have*

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \leq B\big[T(\alpha r, f) \log T(\alpha r, f)\big]^{j-i} \quad (0 \leq i < j). \tag{2.2}$$

Remark 2.1 We use $E_2 \subset (0, \infty)$ to denote a set of r of finite logarithmic measure throughout this paper, not necessarily the same at each occurrence.

Lemma 2.3 (see [18]) *Let* $f(z)$ *be a transcendental entire function, and let* $z_r = re^{i\theta_r}$ *be a point satisfying* $|f(z_r)| = M(r, f)$, *then there exists a constant* $\delta_r > 0$ (*depending on* r) *such that, for all* z *satisfying* $|z| = r \notin E_2$ *and* $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$, *we have*

$$\left|\frac{f(z)}{f^{(j)}(z)}\right| \leq 2r^{j} \quad (j \in \mathbb{N}). \tag{2.3}$$

Lemma 2.4 (see [19]) *Let* $f(z)$ *be an entire function satisfying* $0 < \rho(f) = \rho < \infty$, $0 < \tau(f) = \tau < \infty$. *Then for any* $\beta < \tau$, *there exists a set* $E_3 \subset [1, +\infty)$ *that has an infinite logarithmic* *measure such that, for all* $r \in E_3$, *we have*

$$\log M(r,f) > \beta r^{\rho}. \tag{2.4}$$

Lemma 2.5 *Let* $f(z)$ *be a transcendental entire function satisfying* $0 < \rho(f) = \rho < \infty$, $\tau(f) = \tau > 0$, *and* $T(r,f) \sim \log M(r,f)$ *as* $r \to \infty$ *outside a set of r of finite logarithmic measure. Then for any* $\beta < \tau$, *there exists a set* $E_3 \subset (0,\infty)$ *having infinite logarithmic measure and a set* $H_2 \subset [0,2\pi)$ *with linear measure zero such that, for all z satisfying* $|z| = r \in E_3$ *and* $\arg z = \theta \in [0,2\pi)\backslash H_2$, *we have*

$$\left|f\left(re^{i\theta}\right)\right| > \exp\left\{\beta r^{\rho}\right\}. \tag{2.5}$$

Proof Since $m(r,f) \sim \log M(r,f)$ as $r \to \infty$ ($r \notin E_2$), by the definition of $m(r,f)$, we see that there exists a set $H_2 \subset [0,2\pi)$ with linear measure zero such that for all z satisfying $\arg z = \theta \in [0,2\pi)\backslash H_2$ and for any $\varepsilon > 0$, we have

$$\left|f\left(re^{i\theta}\right)\right| > M(r,f)^{1-\varepsilon} \quad (r \notin E_2). \tag{2.6}$$

Otherwise, we find that there exists a set $H \subset [0,2\pi)$ with positive linear measure, *i.e.*, $mH > 0$ such that, for all z satisfying $\arg z = \theta \in H$ and for any $\varepsilon > 0$, one has

$$\left|f\left(re^{i\theta}\right)\right| \le M(r,f)^{1-\varepsilon} \quad (r \notin E_2).$$

Then, for all $r \notin E_2$, we have

$$\begin{aligned} m(r,f) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|f\left(re^{i\theta}\right)\right| d\theta \\ &= \frac{1}{2\pi}\int_H \log^+\left|f\left(re^{i\theta}\right)\right| d\theta + \frac{1}{2\pi}\int_{[0,2\pi)\backslash H} \log^+\left|f\left(re^{i\theta}\right)\right| d\theta \\ &\le \frac{(1-\varepsilon)mH}{2\pi}\log M(r,f) + \frac{2\pi - mH}{2\pi}\log M(r,f) \\ &= \frac{2\pi - \varepsilon \cdot mH}{2\pi}\log M(r,f). \end{aligned} \tag{2.7}$$

Since $\varepsilon > 0$, $mH > 0$, (2.7) is a contradiction with $m(r,f) \sim \log M(r,f)$.

For any $\beta < \tau$, we choose β_1 satisfying $\beta < \beta_1 < \tau$, by Lemma 2.4, there exists a set $E_3 \subset (0,\infty)$ having infinite logarithmic measure such that, for all $|z| = r \in E_3$, we have

$$M(r,f) > \exp\left\{\beta_1 r^{\rho}\right\}. \tag{2.8}$$

By (2.6) and (2.8), for any given $0 < \varepsilon < 1 - \frac{\beta}{\beta_1}$ and, for all z satisfying $|z| = r \in E_3\backslash E_2$ and $\arg z = \theta \in [0,2\pi)\backslash H_2$, we have

$$\left|f\left(re^{i\theta}\right)\right| > M(r,f)^{1-\varepsilon} > \exp\left\{(1-\varepsilon)\beta_1 r^{\rho}\right\} > \exp\left\{\beta r^{\rho}\right\}.$$

Therefore we complete the proof of Lemma 2.5. □

Remark 2.2 The following lemma is a special case of Lemma 29 in [12].

Lemma 2.6 (see [18]) *Let $f(z)$ be a transcendental entire function satisfying $0 < \rho(f) = \rho < \infty$ and $T(r,f) \sim \log M(r,f)$ as $r \to \infty$ outside a set of r of finite logarithmic measure. Then for any given $\varepsilon > 0$, there exists a set $E_4 \subset (0,\infty)$ with positive upper logarithmic density and a set $H_2 \subset [0,2\pi)$ with linear measure zero such that, for all z satisfying $r \in E_4$ and $\arg z = \theta \in [0,2\pi)\backslash H_2$, we have*

$$\left|f\left(re^{i\theta}\right)\right| > \exp\left\{r^{\rho-\varepsilon}\right\}. \tag{2.9}$$

Lemma 2.7 (see [20]) *Let $f(z)$ be a meromorphic function of finite order ρ, for any given $\xi > 0$ and l $(0 < l < \frac{1}{2})$, there exist a constant $K(\rho,\xi)$ and a set $E_\xi \subset (0,\infty)$ of lower logarithmic density greater than $1-\xi$ such that, for all $r \in E_\xi$ and for J of length l, we have*

$$r\int_J \left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right| d\theta < K(\rho,\xi)\left(l\log\frac{1}{l}\right)T(r,f). \tag{2.10}$$

Lemma 2.8 (see [21]) *Let A_j $(j = 0,\dots,k-1)$, $F \not\equiv 0$ be entire functions. If f is a solution of (1.2) satisfying $\max\{\rho_2(F), \rho_2(A_j) : j = 0,\dots,k-1\} < \rho_2(f)$, then*

$$\overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f). \tag{2.11}$$

Lemma 2.9 (see [22]) *Suppose that $P(z) = a_n z^n + \cdots + a_0$ is a polynomial with degree $n \geq 1$, $a_n \in \mathbb{C}$, and that $A(z)$ $(\not\equiv 0)$ is an entire function with $\rho(A) < n$. Set $g(z) = A(z)e^{P(z)}$, $z = re^{i\theta}$, $\delta(P,\theta) = \delta(a_n z^n,\theta) = \operatorname{Re}\{a_n e^{in\theta}\}$. Then for any given $\varepsilon > 0$, there exists a set $H_3 \subset [0,2\pi)$ of linear measure zero such that for any $\theta \in [0,2\pi)\backslash H_3$, there is a constant $R(\theta) > 0$ such that for $|z| = r > R(\theta)$, we have:*

(i) *If $\delta(P,\theta) > 0$, then*

$$\exp\left\{(1-\varepsilon)\delta(P,\theta)r^n\right\} < \left|g\left(re^{i\theta}\right)\right| < \exp\left\{(1+\varepsilon)\delta(P,\theta)r^n\right\}. \tag{2.12}$$

(ii) *If $\delta(P,\theta) < 0$, then*

$$\exp\left\{(1+\varepsilon)\delta(P,\theta)r^n\right\} < \left|g\left(re^{i\theta}\right)\right| < \exp\left\{(1-\varepsilon)\delta(P,\theta)r^n\right\}. \tag{2.13}$$

3 Proofs of Theorems 1.1-1.3

Proof of Theorem 1.1 (i) By Lemma 2.1, we know that every solution f of (1.2) satisfies $\rho_2(f) \leq \rho(A_d)$. In the following, we show that every transcendental solution $f(z)$ of (1.2) satisfies $\rho_2(f) \geq \rho(A_d)$. Suppose that $f(z)$ is a transcendental solution of (1.2). By (1.2), we have

$$|A_d| \leq \left|\frac{f^{(k)}}{f^{(d)}}\right| + \cdots + \left|A_{d+1}\frac{f^{(d+1)}}{f^{(d)}}\right| + \left|\frac{f}{f^{(d)}}\right|\left(\left|A_{d-1}\frac{f^{(d-1)}}{f}\right| + \cdots + |A_0| + \left|\frac{F}{f}\right|\right). \tag{3.1}$$

For each sufficiently large circle $|z| = r$, we take a point $z_r = re^{i\theta_r}$ satisfying $|f(z_r)| = M(r, f) > 1$. By Lemma 2.3, there exist a constant $\delta_r > 0$ and a set E_2 such that, for all z satisfying

$|z| = r \notin E_2$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$, we have

$$\left|\frac{f(z)}{f^{(d)}(z)}\right| \leq 2r^d. \tag{3.2}$$

By Lemma 2.2, there exist a set $H_1 \subset [0, 2\pi)$ having linear measure zero and a constant $B > 0$ such that, for all z satisfying $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \backslash H_1$ and for sufficiently large r, we have

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \leq B\bigl[T(2r, f)\bigr]^{2k} \quad (0 \leq i < j \leq k). \tag{3.3}$$

We choose α_1, α_2 satisfying $\max\{\tau(A_j) : \rho(A_j) = \rho(A_d), \tau(F)\} < \alpha_1 < \alpha_2 < \tau(A_d)$, since $|f(z) - f(z_r)| < \varepsilon$ and $|f(z_r)| \to \infty$ as $r \to \infty$, for all sufficiently large $|z| = r \notin E_2$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$, we have

$$\bigl|A_j(z)\bigr| \leq \exp\bigl\{\alpha_1 r^{\rho(A_d)}\bigr\} \quad (j \neq d), \qquad \left|\frac{F(z)}{f(z)}\right| \leq \bigl|F(z)\bigr| \leq \exp\bigl\{\alpha_1 r^{\rho(A_d)}\bigr\}. \tag{3.4}$$

Since $T(r, A_d) \sim \log M(r, A_d)$ as $r \to \infty$ $(r \notin E_2)$, by Lemma 2.5, for any $\alpha_2 < \tau(A_d)$, there exist a set $E_3 \subset (0, \infty)$ having infinite logarithmic measure and a set $H_2 \subset [0, 2\pi)$ with linear measure zero such that for all z satisfying $|z| = r \in E_3$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \backslash H_2$, we have

$$\bigl|A_d(z)\bigr| > \exp\bigl\{\alpha_2 r^{\rho(A_d)}\bigr\}. \tag{3.5}$$

Substituting (3.2)-(3.5) into (3.1), for all z satisfying $|z| = r \in E_3 \backslash E_2$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \backslash (H_1 \cup H_2)$, we have

$$\exp\bigl\{\alpha_2 r^{\rho(A_d)}\bigr\} \leq (k+1)B\bigl[T(2r, f)\bigr]^{2k} \cdot 2r^d \cdot \exp\bigl\{\alpha_1 r^{\rho(A_d)}\bigr\}. \tag{3.6}$$

From (3.6), we have $\rho_2(f) \geq \rho(A_d)$. Therefore every transcendental solution $f(z)$ of (1.2) satisfies $\rho_2(f) = \rho(A_d)$. If $f(z)$ is a polynomial solution of (1.2) with $\deg f \geq d$, then by a simple estimation on both sides of (1.2), we have $\rho(f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_0 f) = \rho(F) < \rho(A_d)$, this is a contradiction, therefore each polynomial solution f of (1.2) must satisfy $\deg f < d$.

(ii) If $F \not\equiv 0$, by Lemma 2.8, we have that every transcendental solution $f(z)$ of (1.2) satisfies $\overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) = \rho(A_d)$.

(iii) If $d = 1$, it is easy to see that (1.2) cannot have polynomial solutions, and by (i) and (ii), we see that every nonconstant solution $f(z)$ of (1.2) satisfies $\rho_2(f) = \rho(A_1)$ and $\overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) = \rho(A_1)$ if $F \not\equiv 0$. □

Proof of Theorem 1.2 Suppose that $A_d(z)$ has $a \in \mathbb{C}$ as a finite deficient value and satisfying $\delta(a, A_d) = 2\beta > 0$. Then by the definition of deficiency, for sufficiently large r, we have $m(r, \frac{1}{A_d - a}) > \beta T(r, A_d)$. Hence, for sufficiently large r, there exists a point $z_r = re^{i\theta_r}$ satisfying $|z_r| = r$ and

$$\log\bigl|A_d(z_r) - a\bigr| < -\beta T(r, A_d). \tag{3.7}$$

Without loss of generality, we assume that $a = 0$. Set $z_r = re^{i\theta_r}$, by Lemma 2.7, for any given ξ $(0 < \xi < 1)$ and for any given l $(0 < l < \frac{1}{2})$, there exists a set $E_\xi \subset (0,\infty)$ of lower logarithmic density greater than $1 - \xi$ such that, for all z satisfying $|z| = r \in E_\xi$ and $\arg z = \theta \in [\theta_r, \theta_r + l]$, we have

$$r \int_{\theta_r}^{\theta_r + l} \left| \frac{A'_d(re^{i\theta})}{A_d(re^{i\theta})} \right| d\theta < K\bigl(\rho(A_d), \xi\bigr) \left(l \log \frac{1}{l} \right) T(r, A_d). \tag{3.8}$$

We choose l sufficiently small such that $K(\rho(A_d), \xi)(l \log \frac{1}{l}) < \beta$, then, for all $\theta \in [\theta_r, \theta_r + l]$, we have

$$\begin{aligned} \log\bigl|A_d\bigl(re^{i\theta}\bigr)\bigr| &= \log\bigl|A_d\bigl(re^{i\theta_r}\bigr)\bigr| + \int_{\theta_r}^{\theta} \frac{d}{dt} \log\bigl|A_d\bigl(re^{it}\bigr)\bigr| \, dt \\ &\leq -\beta T(r, A_d) + r \int_{\theta_r}^{\theta} \left| \frac{A'_d(re^{it})}{A_d(re^{it})} \right| |dt| \\ &\leq -\beta T(r, A_d) + K\bigl(\rho(A_d), \xi\bigr) \left(l \log \frac{1}{l} \right) T(r, A_d) \leq 0. \end{aligned} \tag{3.9}$$

In general, if $a \neq 0$, then $A_d(z) - a$ has zero as a deficient value, and using the reasoning above to $A_d(z) - a$, we have

$$\log\bigl|A_d\bigl(re^{i\theta}\bigr) - a\bigr| \leq 0 \tag{3.10}$$

holds, for all $|z| = r \in E_\xi$ and $\arg z = \theta \in [\theta_r, \theta_r + l]$. From (3.10), we have

$$\bigl|A_d\bigl(re^{i\theta}\bigr)\bigr| \leq |a| + 1 \tag{3.11}$$

holds, for all z satisfying $|z| = r \in E_\xi$ and $\arg z = \theta \in [\theta_r, \theta_r + l]$. Let $f \not\equiv 0$ be a solution of (1.2). By (1.2), we have

$$\bigl|A_0(z)\bigr| \leq \left| \frac{f^{(k)}(z)}{f(z)} \right| + \cdots + \left| A_d(z) \frac{f^{(d)}(z)}{f(z)} \right| + \cdots + \left| A_1(z) \frac{f'(z)}{f(z)} \right|. \tag{3.12}$$

By Lemma 2.2, there exists a set $H_1 \subset [0, 2\pi)$ having linear measure zero and a constant $B > 0$ such that, for all z satisfying $\arg z = \theta \in [\theta_r, \theta_r + l] \backslash H_1$ and for all sufficiently large r, we have

$$\left| \frac{f^{(j)}(z)}{f(z)} \right| \leq B\bigl[T(2r, f)\bigr]^{2k} \quad (j = 1, \ldots, k). \tag{3.13}$$

Since $T(r, A_0) \sim \log M(r, A_0)$ as $r \to \infty$ $(r \notin E_2)$, by Lemma 2.6, for any given $\varepsilon > 0$, there exists a set $E_4 \subset (0, \infty)$ with positive upper logarithmic density and a set $H_2 \subset [0, 2\pi)$ with linear measure zero such that, for all z satisfying $|z| = r \in E_4$ and $\arg z = \theta \in [\theta_r, \theta_r + l] \backslash H_2$, we have

$$\bigl|A_0\bigl(re^{i\theta}\bigr)\bigr| \geq \exp\bigl\{r^{\rho(A_0) - \varepsilon}\bigr\}. \tag{3.14}$$

Set $\max\{\rho(A_j), j \neq 0, d\} = b < \rho(A_0)$, then for any given ε $(0 < 2\varepsilon < \rho(A_0) - b)$ and, for all sufficiently large $|z| = r$, we have

$$\bigl|A_j(z)\bigr| < \exp\bigl\{r^{b + \varepsilon}\bigr\} \quad (j \neq 0, d). \tag{3.15}$$

Substituting (3.11), (3.13)-(3.15) into (3.12), for all $|z| = r \in (E_\xi \cap E_4)\backslash E_2$ and $\arg z = \theta \in [\theta_r, \theta_r + l]\backslash(H_1 \cup H_2)$, we have

$$\exp\{r^{\rho(A_0)-\varepsilon}\} \leq kB[T(2r,f)]^{2k} \cdot \exp\{r^{b+\varepsilon}\}, \tag{3.16}$$

where $(E_\xi \cap E_4)\backslash E_2$ is a set having positive upper logarithmic density. From (3.16), we have $\rho_2(f) \geq \rho(A_0)$. On the other hand, by Lemma 2.1, we see that $\rho_2(f) \leq \rho(A_d)$ holds, for all solutions of (1.2). Therefore, each solution $f \not\equiv 0$ of (1.2) satisfies $\rho(A_0) \leq \rho_2(f) \leq \rho(A_d)$. □

Proof of Theorem 1.3 (1) By Lemma 2.1, it is easy to see that every solution $f \not\equiv 0$ of (1.7) satisfies $\rho_2(f) \leq n$ and that (1.7) has no polynomial solutions by the assumption. In the following, we need to show that every transcendental solution f of (1.7) satisfies $\rho_2(f) \geq n$. We divide the proof into two parts: (i) $\arg a_n \neq \arg b_n$ or $a_n = cb_n$ $(0 < c < 1)$, (ii) $a_n = cb_n$ $(c > 1)$.

(i) $\arg a_n \neq \arg b_n$ or $a_n = cb_n$ $(0 < c < 1)$. By Theorem D(i), every solution $f \not\equiv 0$ of (1.7) satisfies $\rho_2(f) \geq n$.

(ii) $a_n = cb_n$ $(c > 1)$. We have $\delta(a_n z^n, \theta) = c\delta(b_n z^n, \theta)$ $(c > 1)$. For each sufficiently large circle $|z| = r$, if $z_r = re^{i\theta_r}$ is a point satisfying $|f(z_r)| = M(r,f)$, then we affirm that for any given (sufficiently small in general) $\delta_r > 0$, we have $[\theta_r - \delta_r, \theta_r + \delta_r] \cap \{\theta : \delta(a_n z^n, \theta) > 0\} \neq \emptyset$ and $m([\theta_r - \delta_r, \theta_r + \delta_r] \cap \{\theta : \delta(a_n z^n, \theta) > 0\}) > 0$. On the contrary, if $z_r = re^{i\theta_r}$ is a point satisfying $|f(z_r)| = M(r,f)$, and there exists a $\delta_1 > 0$ (depending on r, in the same way as the following δ_j, $j = 2,3,4,5$) such that $[\theta_r - \delta_1, \theta_r + \delta_1] \cap \{\theta : \delta(a_n z^n, \theta) > 0\} = \emptyset$, *i.e.*, $[\theta_r - \delta_1, \theta_r + \delta_1] \subset \{\theta : \delta(a_n z^n, \theta) \leq 0\}$, we will get a contradiction. In fact, we can choose a $\delta_2 > 0$ $(\delta_2 < \delta_1)$ such that $[\theta_r - \delta_2, \theta_r + \delta_2] \subset \{\theta : \delta(a_n z^n, \theta) < 0\}$, by (1.7), we have

$$|A_1 e^{P(z)}|\left|\frac{f'(z)}{f''(z)}\right| + |A_0 e^{Q(z)}|\left|\frac{f(z)}{f''(z)}\right| \geq 1. \tag{3.17}$$

On each sufficiently large circle $|z| = r$, we take a point $z_r = re^{i\theta_r}$ such that $|f(z_r)| = M(r,f)$ and $[\theta_r - \delta_2, \theta_r + \delta_2] \subset \{\theta : \delta(a_n z^n, \theta) < 0\}$. By Lemma 2.3, there exists a constant $\delta_3 = \min\{\delta, \delta_2\} > 0$ such that for all z satisfying $|z| = r \notin E_2$ and $\arg z = \theta \in [\theta_r - \delta_3, \theta_r + \delta_3]$, we have

$$\left|\frac{f(z)}{f''(z)}\right| \leq 2r^2, \qquad \left|\frac{f'(z)}{f''(z)}\right| \leq 2r. \tag{3.18}$$

Since $\max\{\rho(A_0), \rho(A_1)\} < n$, by Lemma 2.9, for any given $\varepsilon > 0$, there exists a set $H_3 \subset [0, 2\pi)$ of linear measure zero such that, for all z satisfying $|z| = r \notin E_2$ and $\arg z = \theta \in [\theta_r - \delta_3, \theta_r + \delta_3]\backslash H_3$, we have

$$|A_0 e^{Q(z)}|\left|\frac{f(z)}{f''(z)}\right| \leq 2r^2 \cdot \exp\{(1-\varepsilon)\delta(b_n z^n, \theta)r^n\} \to 0 \quad (r \to \infty), \tag{3.19}$$

$$|A_1 e^{P(z)}|\left|\frac{f'(z)}{f''(z)}\right| \leq 2r \cdot \exp\{(1-\varepsilon)\delta(a_n z^n, \theta)r^n\} \to 0 \quad (r \to \infty). \tag{3.20}$$

Substituting (3.18)-(3.20) into (3.17), we get $1 \leq 0$, this is a contradiction. Therefore, for each sufficiently large circle $|z| = r \notin E_2$, if $z_r = re^{i\theta_r}$ is a point satisfying $|f(z_r)| = M(r,f)$, then for any given (sufficiently small) $\delta_r > 0$, we have $[\theta_r - \delta_r, \theta_r + \delta_r] \cap \{\theta : \delta(a_n z^n, \theta) > 0\} \neq \emptyset$ and $m([\theta_r - \delta_r, \theta_r + \delta_r] \cap \{\theta : \delta(a_n z^n, \theta) > 0\}) > 0$. Then by (1.7), we have

$$\left|A_1 e^{P(z)}\right| \leq \left|\frac{f''(z)}{f'(z)}\right| + \left|A_0 e^{Q(z)}\right| \left|\frac{f(z)}{f'(z)}\right|. \tag{3.21}$$

On each sufficiently large circle $|z| = r \notin E_2$, we choose a point $z_r = re^{i\theta_r}$ such that $|f(z_r)| = M(r,f)$ and $m([\theta_r - \delta_r, \theta_r + \delta_r] \cap \{\theta : \delta(a_n z^n, \theta) > 0\}) > 0$. By Lemma 2.2 and Lemma 2.3, for all z satisfying $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \backslash H_1$, we have

$$\left|\frac{f''(z)}{f'(z)}\right| < B\bigl[T\bigl(2r,f'\bigr)\bigr] \leq B\bigl[T(2r,f) + M\{\log r\}\bigr], \qquad \left|\frac{f(z)}{f'(z)}\right| \leq 2r \quad (r \notin E_2). \tag{3.22}$$

Since $\delta(a_n z^n, \theta) = c\delta(b_n z^n, \theta) > 0$, by Lemma 2.9, for any given ε $(0 < \varepsilon < \frac{c-1}{c+1})$, there exists a set $H_3 \subset [0, 2\pi)$ of linear measure zero such that, for all z satisfying $|z| = r \notin E_2$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \backslash (H_1 \cup H_3)$, we have

$$\begin{aligned} &\left|A_0 e^{Q(z)}\right| \leq \exp\bigl\{(1+\varepsilon)\delta_r\bigl(b_n z^n, \theta\bigr) r^n\bigr\}, \\ &\left|A_1 e^{P(z)}\right| \geq \exp\bigl\{(1-\varepsilon)c \cdot \delta_r\bigl(b_n z^n, \theta\bigr) r^n\bigr\}. \end{aligned} \tag{3.23}$$

Substituting (3.22) and (3.23) into (3.21), for any given ε $(0 < \varepsilon < \frac{c-1}{c+1})$ and for sufficiently large $r \notin E_2$, we have

$$\begin{aligned} &\exp\bigl\{(1-\varepsilon)c \cdot \delta_r\bigl(b_n z^n, \theta\bigr) r^n\bigr\} \\ &\quad \leq B\bigl[T(2r,f) + M\{\log r\}\bigr] + 2r \cdot \exp\bigl\{(1+\varepsilon)\delta_r\bigl(b_n z^n, \theta\bigr) r^n\bigr\}. \end{aligned} \tag{3.24}$$

By (3.24), we have $\rho_2(f) \geq n$.

Combining (i) and (ii), we have every solution $f \not\equiv 0$ of (1.7) satisfies $\rho_2(f) = n$.

(2) By Lemma 2.1, it is easy to see that every solution f of (1.7) satisfies $\rho_2(f) \leq n$. It is easy to know that (1.7) has no polynomial solutions by the assumption. In the following, we need to show that every transcendental solution f of (1.7) satisfies $\rho_2(f) \geq n$. Since $a_n = cb_n$ $(c < 0)$, then we have $\{\theta : \delta(a_n z^n, \theta) > 0\} \cap \{\theta : \delta(b_n z^n, \theta) > 0\} = \emptyset$ and $\{\theta : \delta(a_n z^n, \theta) > 0\} \cup \{\theta : \delta(b_n z^n, \theta) > 0\} \cup H_3 = [0, 2\pi)$, where $H_3 \subset [0, 2\pi)$ is a set of linear measure zero. For each sufficiently large circle $|z| = r$, we have if $z_r = re^{i\theta_r}$ is a point satisfying $|f(z_r)| = M(r,f)$, for any given $\delta_4 > 0$, set $I = [\theta_r - \delta_4, \theta_r + \delta_4]$, then we have either $m(I \cap \{\theta : \delta(a_n z^n, \theta) > 0\}) > 0$ or $m(I \cap \{\theta : \delta(a_n z^n, \theta) < 0\}) > 0$. We divide the proof into two cases: (i) $m(I \cap \{\theta : \delta(a_n z^n, \theta) > 0\}) > 0$, (ii) $m(I \cap \{\theta : \delta(a_n z^n, \theta) < 0\}) > 0$.

(i) $m(I \cap \{\theta : \delta(a_n z^n, \theta) > 0\}) > 0$. Suppose that $f(z)$ is a transcendental solution of (1.7), by (1.7), we have

$$\left|A_1 e^{P(z)}\right| \leq \left|\frac{f''(z)}{f'(z)}\right| + \left|A_0 e^{Q(z)}\right| \left|\frac{f(z)}{f'(z)}\right| + \left|\frac{F(z)}{f(z)}\frac{f(z)}{f'(z)}\right|. \tag{3.25}$$

On each sufficiently large circle $|z| = r$, we choose a point $z_r = re^{i\theta_r}$ satisfying $|f(z_r)| = M(r,f)$. By Lemma 2.2 and Lemma 2.3, there exists a constant $\delta_5 = \min\{\delta_r, \delta_4\} > 0$ such that, for all z satisfying $|z| = r \notin E_2$ and $\arg z = \theta \in [\theta_r - \delta_5, \theta_r + \delta_5]\backslash H_1$, we have

$$\begin{aligned} &\left|\frac{f''(z)}{f'(z)}\right| \le B\bigl[T(2r,f) + M\{\log r\}\bigr], \qquad \left|\frac{f(z)}{f'(z)}\right| \le 2r, \\ &\left|\frac{F(z)}{f(z)}\right| \le \bigl|F(z)\bigr| \le \exp\bigl\{r^{\alpha}\bigr\} \quad (\alpha < n). \end{aligned} \tag{3.26}$$

Since $\max\{\rho(A_0), \rho(A_1)\} < n$, by Lemma 2.9, for any given $\varepsilon > 0$, there exists a set $H_3 \subset [0, 2\pi)$ of linear measure zero such that, for all z satisfying $\arg z = \theta \in [\theta_r - \delta_5, \theta_r + \delta_5]\backslash H_3$, we have

$$\bigl|A_0 e^{Q(z)}\bigr|\left|\frac{f(z)}{f'(z)}\right| \le 2r \cdot \exp\bigl\{(1-\varepsilon)\delta_r\bigl(b_n z^n, \theta\bigr) r^n\bigr\} \to 0 \quad (r \to \infty), \tag{3.27}$$

$$\bigl|A_1 e^{P(z)}\bigr| \ge \exp\bigl\{(1-\varepsilon)\delta_r\bigl(a_n z^n, \theta\bigr) r^n\bigr\} \quad (r \to \infty). \tag{3.28}$$

Substituting (3.26)-(3.28) into (3.25), for any given ε (> 0) and for sufficiently large $r \notin E_2$, we have

$$\exp\bigl\{(1-\varepsilon)\delta_r\bigl(a_n z^n, \theta\bigr) r^n\bigr\} \le B\bigl[T(2r,f) + M\{\log r\}\bigr] + M + 2r \cdot \exp\bigl\{r^{\alpha}\bigr\}. \tag{3.29}$$

By (3.29), we have $\rho_2(f) \ge n$.

(ii) $m(I \cap \{\theta : \delta(a_n z^n, \theta) < 0\}) > 0$. Replacing $|A_1 e^{P(z)}|$ with $|A_0 e^{Q(z)}|$ on the left side of (3.25) and by the same reasoning in case (i), we can obtain $\rho_2(f) \ge n$ for every transcendental solution of (1.7).

Combining (i) and (ii), every solution f of (1.7) satisfies $\rho_2(f) = n$. Since $F(z) \not\equiv 0$, by Lemma 2.8, every solution f of (1.7) satisfies $\overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) = n$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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