Guzowska et al. *Advances in Difference Equations* (2015) 2015:203
DOI 10.1186/s13662-015-0537-0

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Calculus of variations on time scales: applications to economic models

Małgorzata Guzowska[1], Agnieszka B Malinowska[2*] and Moulay Rchid Sidi Ammi[3]

*Correspondence:
a.malinowska@pb.edu.pl
[2]Faculty of Computer Science, Bialystok University of Technology, Białystok, 15-351, Poland
Full list of author information is available at the end of the article

Abstract
The time scale calculus theory can be applicable to any field in which dynamic processes are described by discrete- or continuous-time models. On the other hand, many economic models are dynamic models. Therefore it is natural to relate those two subjects. This work is intended to motivate the use of the calculus of variations and optimal control problems on time scales in the study of economic models. We show that a phenomenon known from the theory of behavioral economics may be described and analyzed by dynamical systems on time scales.

MSC: 49K05; 39A12

Keywords: time scales; dynamic models; optimality conditions; behavioral economics

1 Introduction

The origins of the idea of time scales calculus date back to the late 1980s when S Hilger introduced this notion in his PhD thesis and showed how to unify discrete- and continuous-time dynamical systems [1, 2]. With time this unification aspect has been supplemented by the extension and generalization features. Currently, several researchers are getting interested in the time scale calculus, contributing to its development and showing applications of the theory and methods in biology (see, *e.g.*, [3, 4]), engineering (see, *e.g.*, [5–10]), physics (see, *e.g.*, [11]), and economy (see, *e.g.*, [12–15]).

The calculus of variations on time scales was introduced in 2004 by M Bohner who used the delta derivative and delta integral [16], and it has since then been further developed by several different authors in several different directions, *e.g.*, [17–22]. Many classical results of calculus of variations as necessary or sufficient conditions of optimality have been generalized to arbitrary time scale. The aim of the present work is to apply some of those results to economic models and to show advantages of using time scales tools. We present two economical models: a 'cake-eating' problem and a simple model of household consumption. Both were already considered in the time scale literature, but here the use of them is different from the previous works. We show that besides unification and generalizations aspects the study of dynamical systems on time scales allows one to observe the model behaviors which are different from those known in the classical economic theory but rather coincide with facts of the behavioral economics. Models of the life cycle or, more generally, almost all economic models assume the intertemporal choice model proposed by Samuelson [23] as it is not possible to analyze human decisions concerning consumption and saving without making certain assumptions as regards their time preferences. The models presented in this paper are also based on this concept. However, we have to remember that in many economic models a rationality of people is assumed, their computing skills, iron will and farsightedness, since such assumptions help to solve optimization problems. Yet, the search for increasingly reliable modeling of reality and the desire to use more and more advanced econometrical and statistical techniques force researchers to adopt numerous premises about human behavior, particularly in terms of their time preferences. Therefore, a crucial question arises: do the rules of behavior attributed to individuals reliably reflect this behavior? Studies and tests to verify the accuracy of predictions based on currently dominant theories, *i.e.*, on M Modigliani's and M Friedman's hypotheses as well as on works that support them, indicate that the predictive power of these models is often weak. In fact, the observed human behavior patterns seem to differ from the conclusions drawn from the approach based on the model of a rational consumer who optimizes their decisions across time. For instance, the results of the studies by Shapiro and Slemrod [24] or Parker [25] show that the expected changes in income affect the consumption rate in a short-term period. It means that people's spending on non-durable goods increases in line with the expected rise in income. Such phenomena are rejected in all the life-cycle models, but they remain an integral part of the behavioral models. Relations between temporal choice and behavior patterns can be observed during experiments, but are not taken into account in classical models and approaches. The results obtained in this paper, *i.e.*, the classical economic models developed on non-standard time scales prove that the time scale analysis can explain the phenomena in this part of the behavioral economics which deals with the intertemporal choices.

The paper is organized as follows. In the next section, we recall basic terminology related to the time scale calculus. Section 3 provides a detailed exposition of the 'cake-eating' problem on time scales. In Section 4 we apply the results of time scales optimal control to a simple model of household consumption. We end the paper with some conclusions in Section 5.

2 Time scales

In this section we give a brief exposition of the time scale calculus. For a more complete presentation we refer the reader to the books [26, 27].

A nonempty closed subset of $\mathbb{R}$ is called a *time scale* and it is denoted by $\mathbb{T}$. Thus, $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{N}$, are trivial examples of time scales. Other examples of time scales are: $[-4,3]\cup\mathbb{N}$, $h\mathbb{Z} := \{hz|z\in\mathbb{Z}\}$ for some $h>0$, $q^{\mathbb{N}_0} := \{q^k|k\in\mathbb{N}_0\}$ for some $q>1$, and the Cantor set. We assume that a time scale $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology.

The mapping $\sigma:\mathbb{T}\to\mathbb{T}$, defined by $\sigma(t)=\inf\{s\in\mathbb{T}:s>t\}$ with $\inf\emptyset=\sup\mathbb{T}$ (*i.e.*, $\sigma(M)=M$ if $\mathbb{T}$ has a maximum M) is called the *forward jump operator*. Accordingly, we define the *backward jump operator* $\rho:\mathbb{T}\to\mathbb{T}$ by $\rho(t)=\sup\{s\in\mathbb{T}:s<t\}$ with $\sup\emptyset=\inf\mathbb{T}$ (*i.e.*, $\rho(m)=m$ if $\mathbb{T}$ has a minimum m). The following classification of points is used within the theory: a point $t\in\mathbb{T}$ is called *right-dense*, *right-scattered*, *left-dense* and *left-scattered* if $\sigma(t)=t$, $\sigma(t)>t$, $\rho(t)=t$, and $\rho(t)<t$, respectively. The functions $\mu,\nu:\mathbb{T}\to[0,\infty)$ are defined by $\mu(t)=\sigma(t)-t$ and $\nu(t)=t-\rho(t)$.

Example 1 If $\mathbb{T}=\mathbb{R}$, then $\sigma(t)=\rho(t)=t$ and $\nu(t)=\mu(t)=0$. If $\mathbb{T}=h\mathbb{Z}$, then $\sigma(t)=t+h$, $\rho(t)=t-h$, and $\nu(t)=\mu(t)=h$.

For two points $a, b \in \mathbb{T}$, the time scales interval is defined by $[a,b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}$.

We shall state the definition of differentiability on time scales. Throughout we will frequently write $f^\sigma(t) = f(\sigma(t))$.

Definition 1 We say that a function $f : \mathbb{T} \to \mathbb{R}$ is Δ-*differentiable* at $t \in \mathbb{T}^\kappa := \{t \in \mathbb{T} : t$ non-maximal or left-dense$\}$ if there is a number $f^\Delta(t)$ such that for all $\varepsilon > 0$ there exists a neighborhood U of t (*i.e.*, $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$\left|f^\sigma(t) - f(s) - f^\Delta(t)\big(\sigma(t) - s\big)\right| \leq \varepsilon\left|\sigma(t) - s\right|, \quad \text{for all } s \in U.$$

We call $f^\Delta(t)$ the Δ-*derivative* of f at t.

Example 2 If $\mathbb{T} = \mathbb{R}$, then $f^\Delta(t) = f'(t)$. If $\mathbb{T} = h\mathbb{Z}$, then $f^\Delta(t) = \frac{f(t+h)-f(t)}{h} := \Delta_h f(t)$, where Δ_h is the usual forward difference operator.

Now we shall define Δ-*integration* on time scales.

Definition 2 A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* if it is continuous at right-dense points and if the left-sided limit exists at left-dense points.

We denote the set of all rd-continuous functions by C_{rd} or $C_{\mathrm{rd}}(\mathbb{T})$, and the set of all Δ-differentiable functions with rd-continuous derivative by C^1_{rd} or $C^1_{\mathrm{rd}}(\mathbb{T})$.

Definition 3 A function $F : \mathbb{T} \to \mathbb{R}$ is called a *delta antiderivative* of $f : \mathbb{T}^\kappa \to \mathbb{R}$ provided $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}^\kappa$.

Theorem 4 (Theorem 1.74 in [26]) *Every rd-continuous function has a delta antiderivative.*

Let $f : \mathbb{T}^\kappa \to \mathbb{R}$ be an rd-continuous function and let $F : \mathbb{T} \to \mathbb{R}$ be a delta antiderivative of f. Then the Δ-*integral* is defined by $\int_s^r f(t)\Delta t = F(r) - F(s)$ for all $r, s \in \mathbb{T}$. It satisfies

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t), \quad t \in \mathbb{T}^\kappa.$$

Example 3 Let $a, b \in \mathbb{T}$ with $a < b$. If $\mathbb{T} = \mathbb{R}$, then $\int_a^b f(t)\Delta t = \int_a^b f(t)\,dt$, where the integral on the right-hand side is the classical Riemann integral. If $\mathbb{T} = h\mathbb{Z}$, then $\int_a^b f(t)\Delta t = \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} hf(kh)$.

In order to define the *delta exponential function*, first we introduce the concept of *regressivity*.

Definition 5 A function $p : \mathbb{T}^\kappa \to \mathbb{R}$ is *regressive* provided $1 + \mu(t)p(t) \neq 0$ holds for all $t \in \mathbb{T}^\kappa$. We denote by $\mathcal{R}$ the set of all regressive and rd-continuous functions.

Theorem 6 (Theorem 1.37 in [27]) *Suppose $p \in \mathcal{R}$ and fix $t_0 \in \mathbb{T}$. Then the initial value problem* (*IVP*)

$$y^\Delta = p(t)y, \qquad y(t_0) = 1 \tag{1}$$

has a unique solution on $\mathbb{T}$.

Definition 7 Let $p \in \mathcal{R}$ and $t_0 \in \mathbb{T}$. The exponential function on time scales is defined by the solution of the IVP (1) and is denoted by $e_p(\cdot, t_0)$.

The reader can find several properties of the delta exponential function in [26]. We also recommend this book as a source of material on dynamic equations on time scales via the delta derivative.

3 The 'cake-eating' problem

We will first look at a simple problem which is commonly called the 'cake-eating' problem. Let us first consider a model with the discrete time. Suppose that a consumer has a cake of size f_0 today. She/he only lives for T periods and the time is discrete. She/he can only either consume or save (there is nothing else to eat and no possibility to borrow from anyone else). The cake does not get spoiled. At each point in time, $t = 1, 2, \dots, T$; the consumer has to make a decision on the amount of consumption and saving. So, how would she/he determine the optimal amount of cake consumption (saving) at each point in time? To answer this question, we would need to know the properties of her/his preference, her/his time discount factor, and initial/terminal conditions. Therefore, a discrete maximization problem is

$$\max \sum_{t=1}^{T} \left(\frac{1}{1+\delta} \right)^{t-1} u\bigl(c(t)\bigr)$$

subject to

$$f(t) = f(t-1) - c(t), \qquad f(0) = f_0, \qquad f(T) \geq 0,$$

where c is a consumption function, $\delta \geq 0$ is a discount rate. An individual who cares equally about current and future consumption will have $\delta = 0$. An individual who does not care about future consumption will have $\delta = \infty$. The utility function $u(c(\cdot))$ captures the trade off between consumption today and consumption in the future, and is thus exogenous and subjective. The function u is assumed to be at least C^2, well defined, strictly increasing, and strictly concave; and $\lim_{c \to 0} u'(c) = \infty$, $\lim_{c \to \infty} u'(c) = 0$. This means that the consumer always would like to consume more but each additional unit consumed during the same period generates less utility than the previous unit consumed within the same period. We call this property of utility function the law of diminishing marginal utility (LDMU). In other words, LDMU means that the first unit of consumption of a good or service yields more utility than the second and subsequent units, with a continuing reduction for greater amounts. We can also consider the 'cake-eating' problem with continuous time, that is, a maximization problem of the form

$$\max \int_0^T e^{-\delta t} u\bigl(c(t)\bigr)\, dt$$

subject to

$$c(t) = -f'(t), \qquad f(0) = f_0, \qquad f(T) \geq 0.$$

The discrete-time and continuous-time versions of the 'cake-eating' problem are well known and have been much studied in the literature in different contexts (see, *e.g.*, [28–33]).

Let us now formulate the time scale 'cake-eating' problem.

The 'cake-eating' problem with the delta derivative

$$\max \int_0^T e_{\ominus\delta}(t,0)u\big(c^\sigma(t)\big)\Delta t \tag{2}$$

subject to

$$\tilde{c}(t) = c^\sigma(t) = -f^\Delta(t), \qquad f(0) = f_0, \qquad f(T) \geq 0, \tag{3}$$

where $(\ominus z)(t) := \frac{-z(t)}{1+z(t)\mu(t)}$. As $\delta \geq 0$ and $\mu : \mathbb{T} \to [0,\infty)$, we have $1 + \delta\mu(t) > 0$ for all t. Observe that this model includes the discrete- and continuous-time models as a special cases. Problem (2)-(3) is similar to those that have been studied in the literature (see, *e.g.*, [12, 16, 21]), however, here the terminal condition is in the form $f(T) \geq 0$. Below we shall prove the theorem needed for our purposes, but in a more general case, that is, with the Lagrangian depending explicitly on an unknown function.

Theorem 8 *Let $a, b \in \mathbb{T}$ with $a < b$ and $\hat{y} \in C^1_{\mathrm{rd}}([a,b]_\mathbb{T})$ be a solution to the problem*

$$\max \mathcal{J}_\Delta(y) = \int_a^b L\big(t, y^\sigma(t), y^\Delta(t)\big)\Delta t, \qquad y(a) = y_a, \qquad y(b) \geq y_{\min},$$

where L is rd-continuous in t for any admissible path y, and $L(t,\cdot,\cdot)$ is a C^1 function with respect to the second and third variable uniformly in t. Then $\hat{y}$ satisfies the Euler-Lagrange equation

$$\frac{\Delta}{\Delta t} L_{y^\Delta}\big(t, y^\sigma(t), y^\Delta(t)\big) = L_{y^\sigma}\big(t, y^\sigma(t), y^\Delta(t)\big), \quad t \in \big[a, \rho^2(b)\big]_\mathbb{T}; \tag{4}$$

and the transversality conditions

$$\begin{aligned} &L_{y^\Delta}\big(\rho(b), \hat{y}^\sigma\big(\rho(b)\big), \hat{y}^\Delta\big(\rho(b)\big)\big) + \int_{\rho(b)}^b L_{y^\sigma}\big(t, \hat{y}^\sigma(t), \hat{y}^\Delta(t)\big)\Delta t \leq 0, \qquad \hat{y}(b) \geq y_{\min}, \\ &\big(\hat{y}(b) - y_{\min}\big)\Big(L_{y^\Delta}\big(\rho(b), \hat{y}^\sigma\big(\rho(b)\big), \hat{y}^\Delta\big(\rho(b)\big)\big) \\ &\quad + \int_{\rho(b)}^b L_{y^\sigma}\big(t, \hat{y}^\sigma(t), \hat{y}^\Delta(t)\big)\Delta t\Big) = 0. \end{aligned} \tag{5}$$

Proof We consider the value of $\mathcal{J}_\Delta$ at nearby functions $y(t) = \hat{y}(t) + \varepsilon\eta(t)$, where $\varepsilon \in \mathbb{R}$ is a small parameter, $\eta \in C^1_{\mathrm{rd}}([a,b]_\mathbb{T})$ with $\eta(a) = 0$. Thus, $\mathcal{J}_\Delta(\hat{y} + \varepsilon\eta) \leq \mathcal{J}_\Delta(\hat{y})$ and the function $\phi(\varepsilon) := \mathcal{J}_\Delta(\hat{y} + \varepsilon\eta)$ has an extremum at $\varepsilon = 0$. As $y(b) \geq y_{\min}$ for $\hat{y}(b)$ we can have $\hat{y}(b) > y_{\min}$ or $\hat{y}(b) = y_{\min}$. In the first case $y(b) - \hat{y}(b)$ can take both signs. Therefore, by Theorem 3.2 in [21], we obtain

$$\frac{\Delta}{\Delta t} L_{y^\Delta}\big(t, \hat{y}^\sigma(t), \hat{y}^\Delta(t)\big) = L_{y^\sigma}\big(t, \hat{y}^\sigma(t), \hat{y}^\Delta(t)\big)$$

and the transversality condition in the form

$$L_{y^\Delta}\big(\rho(b),\hat{y}^\sigma\big(\rho(b)\big),\hat{y}^\Delta\big(\rho(b)\big)\big) + \int_{\rho(b)}^{b} L_{y^\sigma}\big(t,\hat{y}^\sigma(t),\hat{y}^\Delta(t)\big)\Delta t = 0. \tag{6}$$

In the second case, $\hat{y}(b) = y_{\min}$, quantity $y(b) - \hat{y}(b)$ is restricted to be nonnegative. Assuming $\eta(b) \geq 0$, $y(b) - \hat{y}(b) \geq 0$ would mean that $\varepsilon \geq 0$. Hence, the first-order necessary optimality condition $\phi'(\varepsilon)|_{\varepsilon=0} = 0$ changes for $\phi'(\varepsilon)|_{\varepsilon=0} \leq 0$. This implies the condition

$$L_{y^\Delta}\big(\rho(b),\hat{y}^\sigma\big(\rho(b)\big),\hat{y}^\Delta\big(\rho(b)\big)\big) + \int_{\rho(b)}^{b} L_{y^\sigma}\big(t,\hat{y}^\sigma(t),\hat{y}^\Delta(t)\big)\Delta t \leq 0 \tag{7}$$

for $\hat{y}(b) = y_{\min}$. Combining (6) and (7) we may write (5). □

By Theorem 8 the necessary optimality conditions for problem (2)-(3) are

$$\big[e_{\ominus\delta}(t,0)u'\big(\tilde{c}(t)\big)\big]^\Delta = 0 \tag{8}$$

and

$$u'\big(\tilde{c}\big(\rho(T)\big)\big) \leq 0, \qquad f(T) \geq 0, \qquad u'\big(\tilde{c}\big(\rho(T)\big)\big)f(T) = 0.$$

Taking the delta derivative in (8) and using the properties of the delta exponential function we obtain

$$\frac{e_{\ominus\delta}(t,0)}{1+\mu(t)\delta}\big[\big(u'\big(\tilde{c}(t)\big)\big)^\Delta - \delta u'\big(\tilde{c}(t)\big)\big] = 0.$$

It means that

$$\frac{[u'(\tilde{c}(t))]^\Delta}{u'(\tilde{c}(t))} = \delta.$$

If $\mathbb{T} = \mathbb{R}$, then

$$\delta = \frac{[u'(c(t))]^\Delta}{u'(c(t))} = \frac{u''(c(t))c'(t)}{u'(c(t))}. \tag{9}$$

Equation (9) coincides with the one obtained for the continuous model. Now let us consider the case when $\mathbb{T}$ is an isolated time scale (it means that all points are isolated). In this case, by taking the delta derivative and rearranging terms we get

$$u'\big(\tilde{c}\big(\sigma(t)\big)\big) = \big(1+\delta\mu(t)\big)u'\big(\tilde{c}(t)\big). \tag{10}$$

Putting $\sigma(t) = t+1$ and $\mu(t) = 1$, we observe that (10) coincides with the one known for the discrete model. However, since in (10) we multiply a discount rate δ by $\mu(t)$, the time scale model allows us to consider more general cases when the lengths of the periods of consumption are different. The model implies that the marginal rate of substitution (*MRS*) between this period t and the next period $\sigma(t)$ is no longer a constant but $MRS_{t,\sigma(t)} = \frac{1}{1+\delta\mu(t)}$.

Example 4 Consider the problem

$$\max \int_0^T e_{\ominus\delta}(t,0)\ln\bigl(\tilde{c}(t)\bigr)\Delta t \tag{11}$$

subject to

$$\tilde{c}(t) = -f^{\Delta}(t), \qquad f(0) = f_0, \qquad f(T) \geq 0, \tag{12}$$

where $u(\tilde{c}) = \ln\tilde{c}$. Then, by Theorem 8, necessary conditions for a solution are

$$\tilde{c}^{\Delta}(t) = -\delta\tilde{c}\bigl(\sigma(t)\bigr), \tag{13}$$

$$f(0) = f_0, \qquad f(T) = 0. \tag{14}$$

Solving (13) (see Theorem 2.74 in [26]) we obtain

$$\tilde{c}(t) = c_0 e_{\ominus\delta}(t,0), \quad c_0 = \tilde{c}(0). \tag{15}$$

On the other hand, combining $\tilde{c}(t) = -f^{\Delta}(t)$ with (14) gives

$$f_0 = f(0) = \int_0^T \tilde{c}(t)\Delta t. \tag{16}$$

Substituting (15) into (16) we get $c_0 = \frac{f_0}{\int_0^T e_{\ominus\delta}(t,0)\Delta t}$. Therefore,

$$\tilde{c}(t) = \frac{f_0}{\int_0^T e_{\ominus\delta}(t,0)\Delta t} e_{\ominus\delta}(t,0). \tag{17}$$

As the function under an integral sign in (11) is concave with respect $\tilde{c}$ for all t we see, by Theorem 3.5 in [21], that a function $\tilde{c}$ given by (17) is the optimal solution to problem (11)-(12). Note that the use of time scales makes the solution to the problem more compact, the optimal consumption path is described by the same expression.

Let us consider the periodic domain as a time scale: $\mathbb{T} = h\mathbb{Z} \cap [0,60]$, then we can rewrite (17) in the form

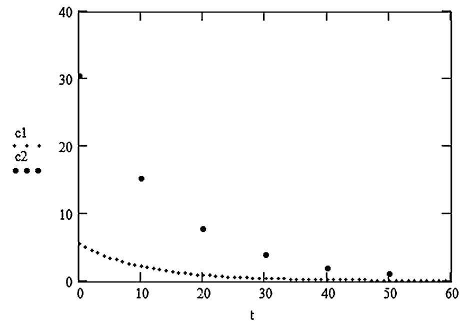
$$\tilde{c}(t) = \frac{\delta}{1+\delta h}\frac{f_0}{1-(\frac{1}{1+\delta h})^{\frac{60}{h}}}\left(\frac{1}{1+\delta h}\right)^{\frac{t}{h}},$$

where we assume that $60 \in \mathbb{T}$. Setting, for example, $\delta = 0.1$ and $f_0 = 60$ we can find the optimal path. Figure 1 shows consumption paths $c2$ on $10\mathbb{Z} \cap [0,60]$ and $c1$ on $\mathbb{Z} \cap [0,60]$.

The next time scale is $\mathbb{T} = (\{2^{\mathbb{N}_0}\} \cup \{0\}) \cap [0,64]$ that will allow us to demonstrate the change in the model dynamics as the breaks between meals are getting longer. In this case we have

$$\tilde{c}(2^k) = \frac{\tilde{c}(0)}{1+\delta}\prod_{s=0}^{k-1}\frac{1}{(1+\delta 2^s)}, \quad k = 0,\dots,5,$$

Figure 1 Consumption paths: $c2$ on $10\mathbb{Z} \cap [0, 60]$, $c1$ on $\mathbb{Z} \cap [0, 60]$.



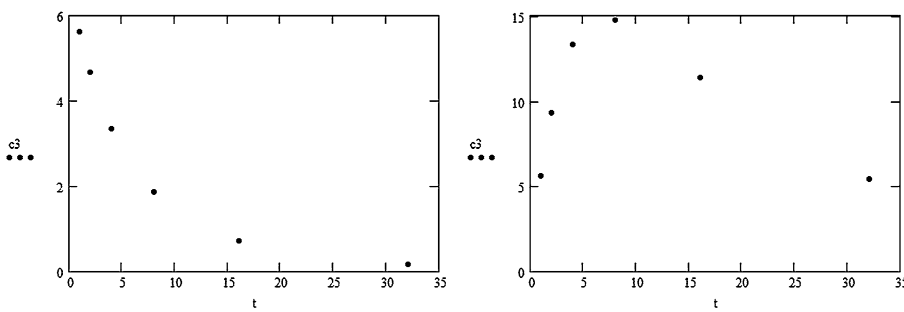


Figure 2 Consumption paths $c3$ on $(\{2^{\mathbb{N}_0}\} \cup \{0\}) \cap [0, 64]$ without scaling (left) and with scaling (right).

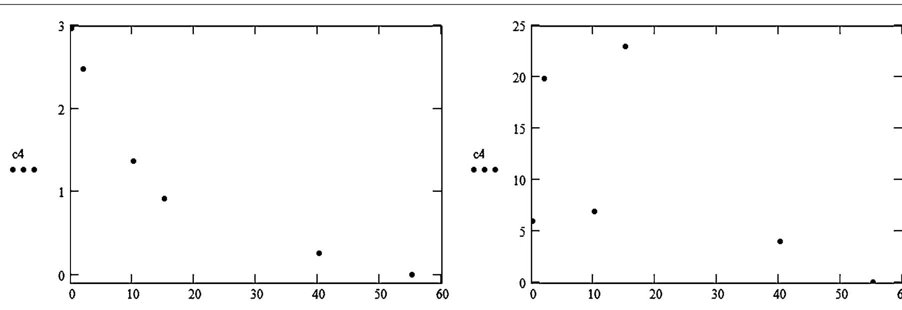


Figure 3 Consumption paths $c4$ on T_1 without scaling (left) and with scaling (right).

where

$$\tilde{c}(0) = \frac{f(0)}{1 + \sum_{k=0}^{5} \frac{2^k}{1+\delta} \prod_{s=0}^{k-1} \frac{1}{(1+\delta 2^s)}}.$$

Setting $\delta = 0.1$ and $f_0 = 60$ we can find the optimal path (see Figure 2).

The next time scale in which μ varies over time is $\mathbb{T}_1 = \{0, 2, 10, 15, 40, 55, 60\}$. Choosing, *e.g.*, $\delta = 0.1$ and $f_0 = 60$ we solve (13)-(14) recursively and obtain $\tilde{c}(0) = 2.97$, $\tilde{c}(2) = 2.48$, $\tilde{c}(10) = 1.37$, $\tilde{c}(15) = 0.92$, $\tilde{c}(40) = 0.26$, $\tilde{c}(55) = 0.10$. Figure 3 shows a consumption path $c4$ on $\mathbb{T}_1$.

Conclusions:

- In the case of the time scales $\mathbb{T} = \mathbb{Z} \cap [0, 60]$ and $\mathbb{T} = 10\mathbb{Z} \cap [0, 60]$ the consumer behavior meets the expectations. Figure 1 shows that the time paths of consumption

Table 1 The percentage share of consumption in the time *t* in relation to the remaining resources on $10\mathbb{Z} \cap [0,60]$

t	$\frac{\tilde{c}(t)}{60-\sum_{\tau\in[0,\rho(t)]}\tilde{c}(\tau)}100$ (%)
0	50.794
10	51.613
20	53.333
30	57.143
40	66.667
50	100

Table 2 The percentage share of consumption in the time *t* in relation to remaining resources on $(\{2^{\mathbb{N}_0}\} \cup \{0\}) \cap [0,64]$ (left) and on T_1 (right)

t	$\mu(t)$	$\frac{\tilde{c}(t)}{60-\sum_{\tau\in[0,\rho(t)]}\tilde{c}(\tau)}100$ (%)	t	$\mu(t)$	$\frac{\tilde{c}(t)}{60-\sum_{\tau\in[0,\rho(t)]}\tilde{c}(\tau)}100$ (%)
1	1	9.348	0	2	9.9
2	2	17.187	2	8	36.718
4	4	29.649	10	5	17.767
8	8	46.827	15	15	84.034
16	16	67.742	40	25	89.245
32	0	100	55	0	100

have the same tendency. We can also observe that consumers eat small amounts when they eat more often and they have large meals when they eat seldom. This observation is also confirmed by the analysis of the percentage share of consumption in the time t in relation to the remaining resources (Table 1), which shows a small variation in time.

- In the case of the time scale $(\{2^{\mathbb{N}_0}\} \cup \{0\}) \cap [0,64]$, where the moment of consumption is step by step increasingly far we can see an increase in the consumption level. We observe a tendency to 'accumulate' food when it is known that a meal will be more and more delayed. It is clearly demonstrated in Figure 2 (right) and in Table 2 (left).
- For the time scale T_1, in which consumption frequency varies over time the 'accumulation' described above can be seen even more clearly. Figure 3 (right) and Table 2 (right) show a considerable growth in consumption in relation to the remaining resources when the moment of the next meal is remote.
- The analysis of the model dynamics on the proposed time scales has allowed for the observation of some theoretical findings known from the behavioral economics [34–38], and which are impossible to observe when analyzing the model dynamics on the traditional homogeneous time scales (that is, with $\mu \equiv \text{const.}$).

Infinite horizon problems can also be modeled using time scales tools. The infinite horizon 'cake-eating' problem with the delta derivative is

$$\max \int_0^{+\infty} e_{\ominus\delta}(t,0)u\bigl(\tilde{c}(t)\bigr)\Delta t \tag{18}$$

subject to

$$\tilde{c}(t) = -f^{\Delta}(t), \qquad f(0) = f_0. \tag{19}$$

The necessary optimality conditions for this problem are split into two parts: the Euler-Lagrange (4), and a transversality condition at infinity. Under certain assumptions the transversality condition for infinite horizon variational problems on time scales was proved in [22]. For problem (18)-(19) the transversality condition is

$$\lim_{T\to+\infty}\inf_{T'\geq T} e_{\ominus\delta}(t,0)u'\big(\tilde{c}(T')\big)f\big(T'\big)=0,$$

provided that assumptions formulated in [22] are satisfied.

4 The household problem

As a second example we consider the household problem. The discrete model is

$$\max\sum_{t=1}^{T}\left(\frac{1}{1+\delta}\right)^{t-1}u\big(c(t)\big)$$

subject to

$$a(t)=(1+r)a(t-1)+y(t)-c(t),\qquad a(0)=a_0,\qquad a(T)=a_T,$$

while the continuous model is

$$\max\int_0^T e^{-\delta t}u\big(c(t)\big)\,dt$$

subject to

$$a'(t)=ra(t)+y(t)-c(t),\qquad a(0)=a_0,\qquad a(T)=a_T,$$

where $\delta\geq 0$ is the discount rate, and u is the utility function. In the budget constraint: $y(t)$ is the exogenous income at the time t, $a(t)$ represents assets/debts that the individual accumulates in a period t. Note that a could be positive or negative, the consumer can either save for the future and borrow against the future at the exogenous interest rate r in any period. Therefore, a_0 can be interpreted as either an inheritance ($a_0>0$) or a debt burden ($a_0<0$) passed down from a previous generation. a_T can be interpreted as either a bequest ($a_T>0$) or a debt burden ($a_T<0$) passed down the next generation.

The household problem with the delta derivative

$$\max\int_0^T e_{\ominus\delta}(t,0)u\big(c^\sigma(t)\big)\Delta t \tag{20}$$

subject to

$$\begin{aligned}&a^\Delta(t)=\frac{r}{1+r\mu(t)}a^\sigma(t)+\frac{1}{1+r\mu(t)}y^\sigma(t)-\frac{1}{1+r\mu(t)}c^\sigma(t),\\&a(0)=a_0,\qquad a(T)=a_T.\end{aligned} \tag{21}$$

Again, this model includes the discrete and continuous model as a special cases. We shall use Theorem 3.4 in [21] in order to write the necessary optimality conditions for problem

(20)-(21). In what follows we assume that σ is a Δ-differentiable function. Note that, in particular, the differential calculus, the difference calculus, the h-calculus ($\mathbb{T} = h\mathbb{Z} := \{hz : z \in \mathbb{Z}\}$, for some $h > 0$), and the q-calculus ($\mathbb{T} = q^{\mathbb{N}_0} := \{q^k : k \in \mathbb{N}_0\}$) satisfy this condition.

Theorem 9 [21] *Let $(\tilde{x}, \tilde{u})$ be a normal extremizer for the problem*

$$\mathcal{L}[x,u] = \int_a^b f\big(t, x^\sigma(t), u^\sigma(t)\big)\Delta t$$

subject to

$$x^\Delta(t) = g\big(t, x^\sigma(t), u^\sigma(t)\big),$$
$$x(a) = x_a, \qquad x(b) = x_b,$$

where L is rd-continuous in t, $L(t,\cdot,\cdot)$, and $g(t,\cdot,\cdot)$ are C^1 functions with respect to the second and third variable uniformly in t. Then there exists a function $\tilde{p}$ such that the triple $(\tilde{x}, \tilde{u}, \tilde{p})$ satisfies the Hamiltonian system

$$x^\Delta(t) = H_{p^\sigma}\big(t, x^\sigma(t), u^\sigma(t), p^\sigma(t)\big),$$
$$\big(p^\sigma(t)\big)^\Delta = -H_{x^\sigma}\big(t, x^\sigma(t), u^\sigma(t), p^\sigma(t)\big),$$

and the stationary condition

$$H_{u^\sigma}\big(t, x^\sigma(t), u^\sigma(t), p^\sigma(t)\big) = 0,$$

for all $t \in [a, \rho^2(b)]_{\mathbb{T}}$, where the Hamiltonian $H(t,x,v,p) : [a, \rho(b)] \times \mathbb{R}^3 \to \mathbb{R}$ is defined by

$$H\big(t, x^\sigma, u^\sigma, p^\sigma\big) = f\big(t, x^\sigma, u^\sigma\big) + p^\sigma g\big(t, x^\sigma, u^\sigma\big).$$

For problem (20)-(21) the Hamiltonian is

$$H\big(t, a^\sigma, y^\sigma, c^\sigma, p^\sigma\big) = e_{\ominus\delta} u\big(c^\sigma\big) + p^\sigma\left(\frac{r}{1+r\mu}a^\sigma + \frac{1}{1+r\mu}y^\sigma - \frac{1}{1+r\mu}c^\sigma\right)$$

and the necessary optimality conditions are

$$a^\Delta(t) = \frac{r}{1+r\mu(t)}a^\sigma(t) + \frac{1}{1+r\mu(t)}y^\sigma(t) - \frac{1}{1+r\mu(t)}c^\sigma(t), \tag{22}$$

$$\big(p^\sigma(t)\big)^\Delta = -\frac{r}{1+r\mu(t)}p^\sigma(t), \tag{23}$$

$$e_{\ominus\delta}(t,0)u'\big(c^\sigma(t)\big) - \frac{1}{1+r\mu(t)}p^\sigma(t) = 0. \tag{24}$$

Combining (23) with (24), by the product rule and properties of the delta exponential function, we derive the Euler-Lagrange equation for problem (20)-(21)

$$\big[u'\big(c^\sigma(t)\big)\big]^\Delta = \frac{\delta - r - r\mu^\Delta(t)}{1+r\mu^\sigma(t)}u'\big(c^\sigma(t)\big), \tag{25}$$

where we assume that μ is delta differentiable. Consider the case when $\mathbb{T} = \mathbb{R}$. Then $\mu(t) = 0$ for all t and the delta derivative is the usual derivative with respect to t. Therefore, (25) gives the Euler-Lagrange equation for the continuous model

$$u''\bigl(c(t)\bigr)c'(t) = (\delta - r)u'\bigl(c(t)\bigr).$$

In the case when $\mathbb{T} = \mathbb{Z}$ we have $\mu(t) = 1$ for all t and the delta derivative is the usual forward difference operator. Hence, (25) can be rewritten as

$$u'\bigl(c(t+2)\bigr) = \frac{\delta + 1}{1 + r}u'\bigl(c(t+1)\bigr),$$

which is the same expression as we could obtain for the discrete model using the Bellman equation. It follows that the time scales models unify discrete and continuous models in a more general framework. Moreover, (25) demonstrates another potential use of time scales models: when μ varies over time, the growth rate of marginal utility fluctuates. Hence, we see that generally the growth rate of marginal utility is not a constant as in the continuous or discrete time setting.

Using (23) and (24) we derive

$$u'\bigl(c^\sigma(t)\bigr) = \frac{C}{1 + r\mu(t)}\frac{e_{\ominus r}(t,0)}{e_{\ominus\delta}(t,0)} = \frac{C}{1 + r\mu(t)}e_{r\ominus\delta}(t,0), \tag{26}$$

where C is a constant. Combining (22), (26) with initial and terminal conditions we can find the time paths of consumption and assets.

In [15] the authors developed a technique for a dynamic optimization problem in which the objective function and constraints can be on different time scales. Such a technique can also be used here. Suppose consumption $c(t)$ takes place on some time scale $\mathbb{T}_c$, income $y(t)$ arrives on a time scale $\mathbb{T}_y$, and assets/debts accrue on $\mathbb{T}_a$. We define $\mathbb{T} = \mathbb{T}_c \cup \mathbb{T}_y \cup \mathbb{T}_a$, and

$$m(t) = \max\{\tau \le t : \tau \in \mathbb{T}_c\},$$

$$r(t) = \begin{cases} r, & t \in \mathbb{T}_a, \\ 0, & \text{otherwise}, \end{cases}$$

$$i(t) = \begin{cases} 1, & t \in \mathbb{T}_y, \\ 0, & \text{otherwise}. \end{cases}$$

Then problem (20)-(21) takes the form

$$\max \int_0^T e_{\ominus\delta}\bigl(m(t),0\bigr)u\bigl(c^\sigma\bigl(m(t)\bigr)\bigr)\Delta t \tag{27}$$

subject to

$$\begin{aligned} a^\Delta(t) &= \frac{r(t)}{1 + r(t)\mu(t)}a^\sigma(t) + \frac{i(t)}{1 + r\mu(t)}y^\sigma(t) - \frac{1}{1 + r(t)\mu(t)}c^\sigma\bigl(m(t)\bigr), \\ a(0) &= a_0, \qquad a(T) = a_T. \end{aligned} \tag{28}$$

Therefore,

(i) if $\mathbb{T}_y = \emptyset$, then the budget constraint is

$$a^{\Delta}(t) = \frac{r(t)}{1 + r(t)\mu(t)} a^{\sigma}(t) - \frac{1}{1 + r(t)\mu(t)} c^{\sigma}\big(m(t)\big);$$

(ii) if $\mathbb{T}_y = \emptyset$ and $\mathbb{T}_a = \emptyset$, then model (27)-(28) coincides with the delta 'cake-eating' problem (2)-(3).

Clearly, we can use Theorem 9 to derive necessary optimality conditions for problem (27)-(28).

Example 5 Consider the problem

$$\max \int_0^T e_{\ominus\delta}\big(m(t), 0\big) \ln\big(c^{\sigma}\big(m(t)\big)\big) \Delta t$$

subject to

$$a^{\Delta}(t) = \frac{r(t)}{1 + r(t)\mu(t)} a^{\sigma}(t) + \frac{i(t)}{1 + r\mu(t)} y^{\sigma}(t) - \frac{1}{1 + r(t)\mu(t)} c^{\sigma}\big(m(t)\big),$$
$$a(0) = 100, \qquad a(T) = 0,$$

with $\delta = 0.1$ on $\mathbb{T} = \mathbb{T}_c \cup \mathbb{T}_y \cup \mathbb{T}_a$, where $\mathbb{T}_c = 2\mathbb{Z} \cap [0, 12]$, $\mathbb{T}_y = \emptyset$, and $\mathbb{T}_a = 3\mathbb{Z} \cap [0, 12]$. The adopted time scales can illustrate a situation when over a year a consumer lives off their savings ($a(0) = 100$, $y = 0$) and interests (we will consider $r = 0.03$ and $r = 0.05$) which he/she receives every three months.

This example demonstrates a phenomenon described by such researchers as J Parker, MD Shapiro, and J Slemrod [24, 25], who observed that the expected changes in income influence the rate of short-term consumption, *i.e.*, that spending increases when we expect revenue. This phenomenon can be clearly seen (Figure 4 and Table 3) in the time $t = 2$, $t = 6$, and $t = 8$ when consumption increases significantly. In the periods of time when a consumer does not expect any extra income, their consumption goes down ($t = 4$, $t = 10$).

The change in the interest rate from 0.03 to 0.05 does not affect the above mentioned behavior, but it indicates the higher propensity to save (see Table 3). After increasing r, the dynamics of consumer spending is the same as for $r = 0.03$. However, a greater rate of interest makes the consumer try to save more money for the last period.

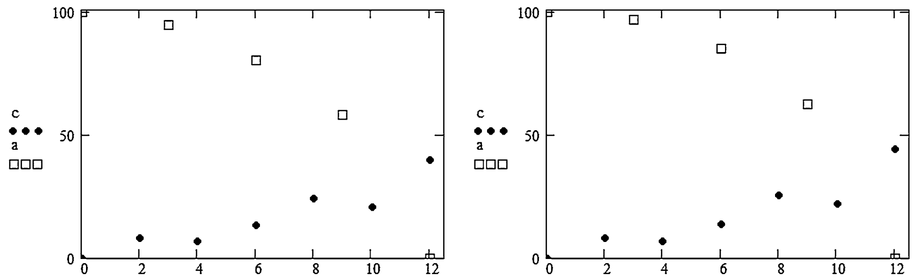


Figure 4 Assets *a* and consumption *c*, in Example 5, with *r* = 0.03 (left), *r* = 0.05 (right).

Table 3 Consumption $c(t)$, in Example 5, with $r = 0.03$, $r = 0.05$

t	$c(t)$ with $r = 0.03$	$c(t)$ with $r = 0.05$
0	0	0
2	7.905	7.905
4	6.785	6.917
6	13.077	13.834
8	23.975	25.362
10	20.578	22.192
12	39.66	44.384

5 Conclusions

Dynamic optimization in economics appeared in the 1920s with the work of Hotelling and Ramsey (see, *e.g.*, [39, 40]). There are three major approaches to dynamic optimization problems: dynamic programming, calculus of variations, and optimal control theory. In this paper we have examined the last two approaches but in the more general framework, using the time scale theory. Economists model time as continuous or discrete. The time scale theory allows us to handle discrete and continuous models as being two pieces of the same framework. Moreover, as was shown in this paper the time scale approach enhances economic modeling by the possibility of working with more complex time domains. This possibility allows one to illustrate and confirm the theories dealing with preferences concerning the time and intermediate choices, which were discussed before in the neoclassical economic theory [23] and the behavioral economics [34–38].

Competing interests
The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' contributions
The authors contributed equally in writing this paper, and they read and approved the final manuscript.

Author details
[1]Faculty of Economics and Management, University of Szczecin, Szczecin, 71-101, Poland. [2]Faculty of Computer Science, Bialystok University of Technology, Białystok, 15-351, Poland. [3]Department of Mathematics, Faculty of Sciences and Techniques, Moulay Ismail University, BP 509, Errachidia, Morocco.

Acknowledgements
The authors sincerely thank the referees for their valuable suggestions and comments. AB Malinowska was supported by the Bialystok University of Technology grant S/WI/02/2011.

Received: 22 February 2015 Accepted: 10 June 2015 Published online: 04 July 2015

References
1. Aulbach, B, Hilger, S: A unified approach to continuous and discrete dynamics. In: Qualitative Theory of Differential Equations (Szeged, 1988). Colloq. Math. Soc. János Bolyai, vol. 53, pp. 37-56. North-Holland, Amsterdam (1990)
2. Hilger, S: Analysis on measure chains-a unified approach to continuous and discrete calculus. Results Math. **18**(1-2), 18-56 (1990)
3. Bohner, M, Warth, H: The Beverton-Holt dynamic equation. Appl. Anal. **86**, 1007-1015 (2007)
4. Bohner, M, Fan, M, Zhang, J: Periodicity of scalar dynamic equations on time scales and applications to population models. J. Math. Anal. Appl. **330**, 1-9 (2007)
5. Bartosiewicz, Z, Kotta, U, Pawłuszewicz, E, Wyrwas, M: Control systems on regular time scales and their differential rings. Math. Control Signals Syst. **22**(3), 185-201 (2013)
6. Belikov, J, Kotta, U, Tonso, M: Realization of nonlinear MIMO system on homogeneous time scales. Eur. J. Control **23**, 48-54 (2015)
7. Casagrande, D, Kotta, U, Tonso, M, Wyrwas, M: Transfer equivalence and realization of nonlinear input-output delta-differential equations on homogeneous time scales. IEEE Trans. Autom. Control **55**(11), 2601-2606 (2010)
8. Kaparin, V, Kotta, U, Wyrwas, M: Observable space of the nonlinear control system on a homogeneous time scale. Proc. Est. Acad. Sci. **63**(1), 11-25 (2014)
9. Kotta, U, Bartosiewicz, Z, Pawłuszewicz, E, Wyrwas, M: Irreducibility, reduction and transfer equivalence of nonlinear input-output equations on homogeneous time scales. Syst. Control Lett. **58**(9), 646-651 (2009)
10. Seiffertt, J, Sanyal, S, Wunsch, DC: Hamilton-Jacobi-Bellman equations and approximate dynamic programming on time scales. IEEE Trans. Syst. Man Cybern., Part B, Cybern. **38**(4), 918-923 (2008)

11. Bekker, M, Bohner, M, Herega, A, Voulov, H: Spectral analysis of a q-difference operator. J. Phys. A **43**(14), 145207 (2010)
12. Atici, FM, Biles, DC, Lebedinsky, A: An application of time scales to economics. Math. Comput. Model. **43**(7-8), 718-726 (2006)
13. Atici, FM, McMahan, CS: A comparison in the theory of calculus of variations on time scales with an application to the Ramsey model. Nonlinear Dyn. Syst. Theory **9**(1), 1-10 (2009)
14. Atici, FM, Uysal, F: A production-inventory model of HMMS on time scales. Appl. Math. Lett. **21**(3), 236-243 (2008)
15. Atici, FM, Biles, DC, Lebedinsky, A: A utility maximisation problem on multiple time scales. Int. J. Dyn. Syst. Differ. Equ. **3**, 39-46 (2011)
16. Bohner, M: Calculus of variations on time scales. Dyn. Syst. Appl. **13**(3-4), 339-349 (2004)
17. Bartosiewicz, Z, Martins, N, Torres, DFM: The second Euler-Lagrange equation of variational calculus on time scales. Eur. J. Control **17**(1), 9-18 (2011)
18. Ferreira, RAC, Malinowska, AB, Torres, DFM: Optimality conditions for the calculus of variations with higher-order delta derivatives. Appl. Math. Lett. **24**(1), 87-92 (2011)
19. Hilscher, R, Zeidan, V: Calculus of variations on time scales: weak local piecewise C^1_{rd} solutions with variable endpoints. J. Math. Anal. Appl. **289**(1), 143-166 (2004)
20. Hilscher, R, Zeidan, V: Weak maximum principle and accessory problem for control problems on time scales. Nonlinear Anal. **70**(9), 3209-3226 (2009)
21. Malinowska, AB, Torres, DFM: Natural boundary conditions in the calculus of variations. Math. Methods Appl. Sci. **33**, 1712-1722 (2010)
22. Malinowska, AB, Martins, N, Torres, DFM: Transversality conditions for infinite horizon variational problems on time scales. Optim. Lett. **5**(1), 41-53 (2011)
23. Samuelson, PA: A note on measurement of utility. Rev. Econ. Stud. **4**(2), 151-161 (1937)
24. Shapiro, MD, Slemrod, J: Consumer response to the timing of income: evidence from a change in tax withholding. Am. Econ. Rev. **85**(1), 274-283 (1995)
25. Parker, JA: The reaction of household consumption to predictable changes in social security taxes. Am. Econ. Rev. **89**(4), 959-973 (1999)
26. Bohner, M, Peterson, A: Dynamic Equations on Time Scales: An Introduction with Applications. Birkhäuser Boston, Boston (2001)
27. Bohner, M, Peterson, A: Advances in Dynamic Equations on Time Scales. Birkhäuser Boston, Boston (2003)
28. Adda, J, Cooper, RW: Dynamic Economics: Quantitative Methods and Applications. MIT Press, Cambridge (2003)
29. Anderson, P: The Economics of Business Valuation: Towards a Value Functional Approach. Stanford University Press, Stanford (2013)
30. Gale, D: On optimal development in a multi-sector model. Rev. Econ. Stud. **34**, 1-18 (1967)
31. Hotelling, H: The economics of exhaustible resources. J. Polit. Econ. **39**, 137-175 (1931)
32. Kemp, MC: Further generalizations of the cake-eating problem under uncertainty. Theory Decis. **8**(4), 363-367 (1977)
33. Romer, P: Cake eating, chattering, and jumps: existence results for variational problems. Econometrica **54**(4), 897-908 (1986)
34. Friedman, M: A Theory of Consumption Function. Princeton University Press, Princeton (1957)
35. Friedman, M, Kuznets, R: Income from Independent Professional Practices. National Bureau of Economic Research, New York (1945)
36. Modigliani, F: The life cycle hypothesis of saving, the demand for wealth and the supply of capital. Soc. Res. **33**(2), 160-217 (1966)
37. Modigliani, F, Ando, A: The 'life cycle' hypothesis of saving: aggregate implications and tests. Am. Econ. Rev. **LIII**, 55-84 (1963)
38. Modigliani, F, Brumberg, R: Utility Analysis and Aggregate Consumption Functions: An Empirical Test and Its Meaning. Unpublished dissertation, John Hopkins University (1953)
39. Chiang, AC: Elements of Dynamic Optimization. McGraw-Hill, Singapore (1992)
40. Sethi, SP, Thompson, GL: Optimal Control Theory. Kluwer Academic, Boston (2000)