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The existence of positive mild solutions for fractional differential evolution equations with nonlocal conditions of order $1 < \alpha < 2$

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Abstract
In this paper, we investigate the existence of positive mild solutions of fractional evolution equations with nonlocal conditions of order $1 < \alpha < 2$ by using Schauder's fixed point theorem and Krasnoselskii's fixed point theorem.

Keywords: positive mild solutions; fractional evolution equations; solution operators; accretive operator; fixed point theorem

1 Introduction

The differential equations involving fractional derivatives have recently been studied extensively, because they have proved to be valuable in various fields of science and engineering. Indeed, we can find numerous applications in electrochemistry, electromagnetism, biology, and hydrogeology. For example space-fractional diffusion equations have been used in groundwater hydrology to model the transport of passive-tracers carried by fluid flow in a porous medium [1, 2], or to model activator-inhibitor dynamics with anomalous diffusion [3]. In particular, there has been a significant development in fractional evolution equations. The existence of solutions for fractional evolution equations has been studied by many authors during recent years. Many excellent results were obtained in this field [3–11].

Shu and Wang [4] studied the existence of mild solutions for the fractional differential equations with nonlocal conditions in a Banach space X:

$$\begin{cases} {}^cD_t^\alpha u(t) = Au(t) + f(t,u(t)) + \int_0^t q(t-s)g(s,u(s))\,ds, & t \in [0,T], \\ u(0) + m(u) = u_0 \in X, \qquad u'(0) + n(u) = u_1 \in X, \end{cases}$$

where D_t^α is the Caputo fractional derivative of order $1 < \alpha < 2$. By using the contraction mapping principle and Krasnoselskii's fixed point theorem, they obtained the existence of solutions for the equation.

Mu [5] considered the existence of mild solutions to the following semilinear fractional evolution equations:

$$\begin{cases} D^\alpha u(t) + Au(t) = f(t,u(t)), & t \in I, \\ u(0) = x_0 \in X, \end{cases} \tag{1.1}$$

where D^α is the Caputo fractional derivative of order $0 < \alpha < 1$. $-A$ is the infinitesimal generator of an analytic semigroup $T(t) = e^{At}|_{t\geq 0}$, and $f : I \times X \to X$ is continuous.

As is well known, a mild solution to system (1.1) satisfies the following equation:

$$u(t) = S_\alpha(t)x_0 + \int_0^t T_\alpha(t-s)f\bigl(s, u(s)\bigr)\,ds.$$

Since $0 < \alpha < 1$, we can combine the probability density function and semigroup theory to describe the corresponding solution operators $S_\alpha(t)$, $T_\alpha(t)$ (see [12]), where

$$T_\alpha(t) = \alpha\int_0^\infty \theta\phi_\alpha(\theta)t^{\alpha-1}T\bigl(t^\alpha\theta\bigr)\,d\theta, \qquad S_\alpha(t) = \int_0^\infty \phi_\alpha(\theta)T\bigl(t^\alpha\theta\bigr)\,d\theta,$$

where $\phi_\alpha(\theta)$ is the probability density function defined on $(0, \infty)$ such that its Laplace transform is

$$\int_0^\infty e^{-\theta x}\phi_\alpha(\theta)\,d\theta = \sum_{j=0}^\infty \frac{(-x)^j}{\Gamma(1+\alpha j)}, \quad x > 0.$$

Thus, it is obvious that whatever operator A is, $T(t) = e^{At}$, $T_\alpha(t)$, $S_\alpha(t)$ are always positive if $-A$ is the infinitesimal generator of an analytic semigroup $T(t)$.

In [6], Mu and Fan investigate the existence and uniqueness of positive mild solutions of the following periodic boundary value problem for the fractional evolution equations in an ordered Banach space in [6]:

$$\begin{cases} D^\alpha u(t) + Au(t) = f(t, u(t)), & t \in [0, \omega], \\ u(0) = u(\omega), \end{cases}$$

where D^α is the Caputo fractional derivative of order $0 < \alpha < 1$. First, using the probability density function and semigroup theory the authors established the mild solutions of the associated linear fractional evolution equations. Then they estimated the spectral radius of resolvent operators accurately. With the aid of the estimation, the existence and uniqueness results of positive mild solutions are obtained by using the monotone iterative technique.

However, for $1 < \alpha < 2$, the existence of a positive mild solution for a fractional differential evolution equation still is an untreated topic in the literature. On the one hand, whether $K_\alpha(t)$, $T_\alpha(t)$ is positive is still unknown, meanwhile $S_\alpha(t)$ is not positive; see Remark 2.3. On the other hand, we do not know if we can still use the probability density function together with semigroup theory to describe the corresponding solution operators $S_\alpha(t)$, $K_\alpha(t)$, and $T_\alpha(t)$. So, it is difficult for us to investigate the positive mild solutions of fractional differential evolution equation in the way as [5] did. In [11], Bai and Lü investigate the positive solutions for nonlinear fractional differential equation by means of some fixed point theorem on cone. Given these, using the fixed point theorem on a cone, we investigate the existence of positive mild solution of such a fractional order differential equation.

Motivated by the above, in this paper, we study the existence of positive mild solutions of the following fractional differential evolution equation in an ordered Banach space X

with positive cone P:

$$\begin{cases} D_t^\alpha u(t) = Au(t) + f(t,u(t)) + \int_0^t q(t-s)g(s,u(s))\,ds, & t \in J = [0,T], \\ u(0) + m(u) = u_0 \in X, \qquad u'(0) + n(u) = u_1 \in X, \end{cases} \tag{1.2}$$

where D^α is the Caputo fractional derivatives of $1 < \alpha < 2$, A is a sectorial operator of type (M,θ,α,μ) defined from domain $D(A) \subset X$ into X, the nonlinear map $f,g : J \times X \to P$ are continuous, $q : [0,T] \to R^+$ is an integrable function on $[0,T]$, and $q = \max_{t\in[0,T]} \int_0^t |q(t-s)|\,ds$. $m,n : X \to X$ are also continuous.

The rest of this paper is organized as follows. In Section 2, we first present some basic definitions and theorems to be used. Then we investigate the properties of solution operators by means of classical Mittag-Leffler function. The main results of this article are given in Section 3. In Section 4, an example is considered to illustrate the applications of the main results presented in Section 3. Finally, in Section 5, we make a conclusion of this paper.

2 Preliminaries

In this part, we will introduce some basic definitions and theorems that are used throughout this paper.

2.1 Definitions and theorems

If X is an ordered Banach space with the norm $\|\cdot\|$. Let P be a cone in which defined a partial ordering in X by $x \le y$ if and only if $y - x \in P$. P is said to be normal if there exists a positive constant N such that $\theta \le x \le y$ implies $\|x\| \le N\|y\|$, where θ denotes the zero element of X, and the smallest N is called the normal constant of P. P is called solid if its interior P is nonempty. If $x \le y$ and $x \ne y$, we write $x < y$. If P is solid and $y - x \in \dot{P}$, we write $x \ll y$. For details on cone theory, see [13].

Besides, if X is an ordered Banach space, then $C(J,X)$ is also an ordered Banach space with the partial order $\le$ induced by the positive cone $K = \{x \in C(J,X) : x(t) \ge \theta, \text{for all } t \in J\}$.

Throughout this paper, we assume that P is a positive cone of ordered Banach space X, then $K = \{x \in C(J,X) : x(t) \ge \theta, \text{for all } t \in J\}$ also is the positive cone of Banach space $C(J,X)$.

In general, the Mittag-Leffler function is defined as [14]:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} = \frac{1}{2\pi i} \int_{H_\alpha} e^\mu \frac{\mu^{\alpha-\beta}}{\mu^\alpha - z}\,d\mu, \quad \alpha,\beta > 0, z \in C,$$

where H_α denotes a Hankel path, a contour starting and ending at $-\infty$, and encircling the disc $|\mu| \le |z|^{\frac{1}{\alpha}}$ counterclockwise.

Theorem 2.1 ([15]) *Let A be a densely defined operator in X satisfying the following conditions*:

(1) *For some $0 < \theta < \pi/2$, $\mu + S_\theta = \{\mu + \lambda : \lambda \in C, |\operatorname{Arg}(-\lambda)| < \theta\}$.*

(2) *There exists a constant M such that*

$$\left\|(\lambda I - A)^{-1}\right\| \le \frac{M}{|\lambda - \mu|}, \quad \lambda \notin \mu + S_\theta.$$

Then A is the infinitesimal generator of a semigroup $T(t)$ satisfying $\|T(t)\| \leq C$. Moreover, $T(t) = \frac{1}{2\pi i}\int_c e^{\lambda t}R(\lambda, A)\,d\lambda$ with c being a suitable path $\lambda \notin \mu + S_\theta$ for $\lambda \in c$.

Definition 2.1 ([4]) Let $A : D(A) \subseteq X \to X$ be a closed linear operator. A is said to be a sectorial operator of type (M, θ, α, μ) if there exist $0 < \theta < \pi/2$, $M > 0$, and $\mu \in R$ such that the α-resolvent of A exists outside the sector

$$\mu + S_\theta = \{\mu + \lambda^\alpha : \alpha \in C, |\operatorname{Arg}(-\lambda^\alpha)| < \theta\}$$

and

$$\|(\lambda^\alpha I - A)^{-1}\| \leq \frac{M}{|\lambda^\alpha - \mu|}, \quad \lambda \notin \mu + S_\theta.$$

Remark 2.1 ([4]) If A is a sectorial operator of type (M, θ, α, μ), then it is not difficult to see that A is the infinitesimal generator of a α-resolvent family $\{T_\alpha(t)\}_{t\geq 0}$ in a Banach space, where $T_\alpha(t) = \frac{1}{2\pi i}\int_C e^{\lambda t}R(\lambda^\alpha, A)\,d\lambda$.

Definition 2.2 ([4]) A function $x \in C([0,T], X)$ is called a mild solution of problem (1.2) if it satisfies the operator equation

$$\begin{aligned} u(t) = {} & S_\alpha(t)\big[u_0 - m(u)\big] + K_\alpha(t)\big[u_1 - n(u)\big] \\ & + \int_0^t T_\alpha(t-s)\left[f\big(s, u(s)\big) + \int_0^s q(s-\tau)g\big(\tau, u(\tau)\big)\,d\tau\right] ds. \end{aligned}$$

Definition 2.3 ([5]) Let $R(t)_{(t\geq 0)}$ be an α-resolvent solution operator in X. If $R(t)x \geq \theta$ for every $x \geq \theta$, $x \in X$, and $t \geq 0$, then $R(t)_{(t\geq 0)}$ is called to be positive.

Definition 2.4 ([7]) Let $A : D(A) \to X$ be a linear operator. Operator A is said to be non-negative if and only if it satisfies both of the following conditions:

(1) There exists $K \geq 0$ such that for every value of $\lambda > 0$ and every $u \in D(A)$, we have

$$\lambda\|u\| \leq K\|\lambda u + Au\|.$$

(2) $R(\lambda I + A) = X$ for every value of $\lambda > 0$.

Definition 2.5 ([7]) If A is a linear operator and satisfies condition (1) in Definition 2.4 for $K = 1$, then A is said to be accretive. In addition, A is said to be m-accretive if condition (2) is also satisfied.

Definition 2.6 Operator A is a sectorial accretive operator of type (M, θ, α, μ) if and only if A is a sectorial operator of type (M, θ, α, μ) and A is accretive.

Remark 2.2 ([7]) Assume that A is a Hilbert space with inner product $(\cdot;\cdot)$. Then the necessary and sufficient condition for A to be accretive is $\operatorname{Re}(Au, u) \geq 0$ for every $u \in D(A)$. Particularly, if X is a real Hilbert space and A is positive, then we obtain $(Au, u) \geq 0$ for every $u \in D(A)$. Note that an ordered Banach space is a real space, implying that if X is an ordered Banach space and A is accretive, then $(Au, u) \geq 0$ for every $u \in D(A)$.

Now, we state two well-known fixed point theorems, which are needed to prove our main results.

Theorem 2.2 (Schauder's fixed point theorem) *Let Y be a nonempty, closed, bounded, and convex subset of a Banach space X, and suppose that $T : Y \to Y$ is a compact operator. Then T has at least one fixed point in Y.*

Theorem 2.3 (Krasnoselskii's fixed point theorem) *Let M be a closed convex and nonempty subset of a Banach space X. Let A, B be the operators such that* (i) $Ax + By \in M$ *whenever $x, y \in M$*; (ii) *A is compact and continuous*; (iii) *B is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.*

2.2 Properties of solution operators

Lemma 2.1 *If A is a sectorial operator of type (M,θ,α,μ), then we have*

$$S_\alpha(t) = \frac{1}{2\pi i}\int_c e^{\lambda t}\lambda^{\alpha-1}R(\lambda^\alpha, A)\,d\lambda = E_{\alpha,1}(At^\alpha) = \sum_{k=0}^{\infty}\frac{(At^\alpha)^k}{\Gamma(1+\alpha k)},$$

$$T_\alpha(t) = \frac{1}{2\pi i}\int_c e^{\lambda t}R(\lambda^\alpha, A)\,d\lambda = t^{\alpha-1}E_{\alpha,\alpha}(At^\alpha) = t^{\alpha-1}\sum_{k=0}^{\infty}\frac{(At^\alpha)^k}{\Gamma(\alpha+\alpha k)}$$

and

$$K_\alpha(t) = \frac{1}{2\pi i}\int_c e^{\lambda t}\lambda^{\alpha-2}R(\lambda^\alpha, A)\,d\lambda = tE_{\alpha,2}(At^\alpha) = t\sum_{k=0}^{\infty}\frac{(At^\alpha)^k}{\Gamma(2+\alpha k)}.$$

Proof We note that

$$\frac{1}{\Gamma(s)} = \frac{1}{2\pi i}\int_c e^\zeta \zeta^{-s}\,d\zeta, \quad \operatorname{Re} s > 0. \tag{2.1}$$

Applying the transformation $\zeta = \eta^{\frac{1}{\alpha}}$ to (2.1) gives

$$\frac{1}{\Gamma(s)} = \frac{1}{2\pi\alpha i}\int_c e^{\eta^{\frac{1}{\alpha}}}\eta^{-\frac{s}{\alpha}+\frac{1}{\alpha}-1}\,d\eta.$$

Since A is a sectorial operator of type (M,θ,α,μ), it follows from the inequality

$$\left\|(\lambda^\alpha I - A)^{-1}\right\| \le \frac{M}{|\lambda^\alpha - \mu|}$$

that A is the infinitesimal generator of α-resolvent families $\{S_\alpha(t)\}_{t\ge 0}$, $\{K_\alpha(t)\}_{t\ge 0}$, and $\{T_\alpha(t)\}_{t\ge 0}$ (see [4]). Hence, using the transformation $t^{-\alpha}\eta = \lambda^\alpha$ (*i.e.*, $t^{-\alpha}\,d\eta = \alpha\lambda^{\alpha-1}\,d\lambda$ and $e^{\eta^{\frac{1}{\alpha}}} = e^{t\lambda}$), we obtain

$$\begin{aligned} E_{\alpha,1}(At^\alpha) &= \sum_{k=0}^{\infty}\frac{(At^\alpha)^k}{\Gamma(1+\alpha k)} \\ &= \frac{1}{2\pi\alpha i}\sum_{k=0}^{\infty}\left\{\int_c e^{\eta^{\frac{1}{\alpha}}}\eta^{-k-1}\,d\eta\right\}(At^\alpha)^k \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi\alpha i}\int_c e^{\eta^{\frac{1}{\alpha}}}\eta^{-1}\left\{\sum_{k=0}^{\infty}\left(At^{\alpha}\eta^{-1}\right)^k\right\}d\eta \\
&= \frac{1}{2\pi\alpha i}\int_c e^{\eta^{\frac{1}{\alpha}}}t^{-\alpha}\left(t^{-\alpha}\eta I - A\right)^{-1}d\eta \\
&= \frac{1}{2\pi i}\int_c e^{\lambda t}\lambda^{\alpha-1}R\left(\lambda^{\alpha},A\right)d\lambda \\
&= S_{\alpha}(t).
\end{aligned}
$$

Similarly, we show that

$$
\begin{aligned}
t^{\alpha-1}E_{\alpha,\alpha}\left(At^{\alpha}\right) &= t^{\alpha-1}\sum_{k=0}^{\infty}\frac{(At^{\alpha})^k}{\Gamma(\alpha+\alpha k)} \\
&= \frac{t^{\alpha-1}}{2\pi\alpha i}\sum_{k=0}^{\infty}\left\{\int_c e^{\eta^{\frac{1}{\alpha}}}\eta^{-k+\frac{1}{\alpha}-2}d\eta\right\}\left(At^{\alpha}\right)^k \\
&= \frac{t^{\alpha-1}}{2\pi\alpha i}\int_c e^{\eta^{\frac{1}{\alpha}}}\eta^{\frac{1}{\alpha}-2}\left\{\sum_{k=0}^{\infty}\left(At^{\alpha}\eta^{-1}\right)^k\right\}d\eta \\
&= \frac{1}{2\pi\alpha i}\int_c e^{\eta^{\frac{1}{\alpha}}}t^{-1}\eta^{\frac{1}{\alpha}-1}\left(t^{-\alpha}\eta I - A\right)^{-1}d\eta \\
&= \frac{1}{2\pi i}\int_c e^{\lambda t}R\left(\lambda^{\alpha},A\right)d\lambda \\
&= T_{\alpha}(t)
\end{aligned}
$$

and

$$
\begin{aligned}
tE_{\alpha,2}\left(At^{\alpha}\right) &= t\sum_{k=0}^{\infty}\frac{(At^{\alpha})^k}{\Gamma(2+\alpha k)} \\
&= \frac{t}{2\pi\alpha i}\sum_{k=0}^{\infty}\left\{\int_c e^{\eta^{\frac{1}{\alpha}}}\eta^{-k-\frac{1}{\alpha}-1}d\eta\right\}\left(At^{\alpha}\right)^k \\
&= \frac{t}{2\pi\alpha i}\int_c e^{\eta^{\frac{1}{\alpha}}}\eta^{-\frac{1}{\alpha}-1}\left\{\sum_{k=0}^{\infty}\left(At^{\alpha}\eta^{-1}\right)^k\right\}d\eta \\
&= \frac{1}{2\pi\alpha i}\int_c e^{\eta^{\frac{1}{\alpha}}}t^{-\alpha+1}\eta^{-\frac{1}{\alpha}}\left(t^{-\alpha}\eta I - A\right)^{-1}d\eta \\
&= \frac{1}{2\pi i}\int_c \lambda^{\alpha-2}e^{\lambda t}R\left(\lambda^{\alpha},A\right)d\lambda \\
&= K_{\alpha}(t).
\end{aligned}
$$

□

Remark 2.3 The Mittag-Leffler function $E_{\alpha,1}(x)$ is well known to have a finite number of real zeros in the range $1 < \alpha < 2$. Reference [16] deduces that operator $S_{\alpha}(t)$ is non-positive by Lemma 2.1.

Remark 2.4 It follows from Definition 2.6 and Remark 2.2 that if X is an ordered Banach space and A is a sectorial accretive operator of type (M,θ,α,μ), then the α-resolvent families $\{T_{\alpha}(t)\}_{t\geq 0}$, $\{S_{\alpha}(t)\}_{t\geq 0}$, and $\{K_{\alpha}(t)\}_{t\geq 0}$ are all positive.

Theorem 2.4 ([4]) *Let A be a sectorial operator of type (M,θ,α,μ). Then the following estimates on $\|S_\alpha(t)\|$ hold.*

(i) *Suppose $\mu \geq 0$. Given $\phi \in (0,\pi)$, we have*

$$\begin{aligned}\|S_\alpha(t)\| \leq{}& \frac{K_1(\theta,\phi)Me^{[K_1(\theta,\phi)(1+\mu t^\alpha)]^{\frac{1}{\alpha}}}[(1+\frac{\sin\phi}{\sin(\phi-\theta)})^{\frac{1}{\alpha}}-1]}{\pi\sin^{1+\frac{1}{\alpha}}\theta}(1+\mu t^\alpha)\\&+\frac{\Gamma(\alpha)M}{\pi(1+\mu t^\alpha)|\cos\frac{\pi-\phi}{\alpha}|^\alpha\sin\theta\sin\phi},\end{aligned}$$

for $t > 0$, where $K_1(\theta,\phi) = \max\{1, \frac{\sin\theta}{\sin(\theta-\phi)}\}$.

(ii) *Suppose $\mu < 0$. Given $\phi \in (0,\pi)$, we have*

$$\|S_\alpha(t)\| \leq \left(\frac{eM[(1+\sin\phi)^{\frac{1}{\alpha}}-1]}{\pi|\cos\phi|^{1+\frac{1}{\alpha}}}+\frac{\Gamma(\alpha)M}{\pi|\cos\phi||\cos\frac{\pi-\phi}{\alpha}|^\alpha}\right)\frac{1}{1+|\mu|t^\alpha},$$

for $t > 0$.

Theorem 2.5 ([4]) *Let A be a sectorial operator of type (M,θ,α,μ). Then the following estimates on $\|T_\alpha(t)\|$, $\|K_\alpha(t)\|$ hold.*

(i) *Suppose $\mu \geq 0$. Given $\phi \in (0,\pi)$, we have*

$$\begin{aligned}\|T_\alpha(t)\| \leq{}& \frac{Me^{[K_1(\theta,\phi)(1+\mu t^\alpha)]^{\frac{1}{\alpha}}}[(1+\frac{\sin\phi}{\sin(\phi-\theta)})^{\frac{1}{\alpha}}-1]}{\pi\sin\theta}(1+\mu t^\alpha)^{\frac{1}{\alpha}}t^{\alpha-1}\\&+\frac{Mt^{\alpha-1}}{\pi(1+\mu t^\alpha)|\cos\frac{\pi-\phi}{\alpha}|^\alpha\sin\theta\sin\phi},\\ \|K_\alpha(t)\| \leq{}& \frac{MK_1(\theta,\phi)e^{[K_1(\theta,\phi)(1+\mu t^\alpha)]^{\frac{1}{\alpha}}}[(1+\frac{\sin\phi}{\sin(\phi-\theta)})^{\frac{1}{\alpha}}-1]}{\pi\sin^{\frac{\alpha+2}{\alpha}}\theta}(1+\mu t^\alpha)^{\frac{\alpha-1}{\alpha}}t^{\alpha-1}\\&+\frac{M\alpha\Gamma(\alpha)}{\pi(1+\mu t^\alpha)|\cos\frac{\pi-\phi}{\alpha}|^\alpha\sin\theta\sin\phi},\end{aligned}$$

for $t > 0$, where $K_1(\theta,\phi) = \max\{1, \frac{\sin\theta}{\sin(\theta-\phi)}\}$.

(ii) *Suppose $\mu < 0$. Given $\phi \in (0,\pi)$, we have*

$$\begin{aligned}\|T_\alpha(t)\| &\leq \left(\frac{eM[(1+\sin\phi)^{\frac{1}{\alpha}}-1]}{\pi|\cos\phi|}+\frac{M}{\pi|\cos\phi||\cos\frac{\pi-\phi}{\alpha}|}\right)\frac{t^{\alpha-1}}{1+|\mu|t^\alpha},\\ \|K_\alpha(t)\| &\leq \left(\frac{eMt[(1+\sin\phi)^{\frac{1}{\alpha}}-1]}{\pi|\cos\phi|^{1+\frac{2}{\alpha}}}+\frac{\alpha\Gamma(\alpha)M}{\pi|\cos\phi||\cos\frac{\pi-\phi}{\alpha}|}\right)\frac{1}{1+|\mu|t^\alpha},\end{aligned}$$

for $t > 0$.

3 Main result

Because of the estimation on $\|S_\alpha(t)\|$, $\|K_\alpha(t)\|$, and $\|T_\alpha(t)\|$ in Theorem 2.4 and Theorem 2.5, it is easy to see they are bounded. So we make the following assumptions:

(H_1): There exist positive numbers $\widetilde{M}$ such that for any $t \in J$, we have

$$\sup_{t\in J}\|S_\alpha(t)\| \le \widetilde{M}, \qquad \sup_{t\in J}\|K_\alpha(t)\| \le \widetilde{M}, \qquad \sup_{t\in J}\|T_\alpha(t)\| \le \widetilde{M}.$$

(H_2): The linear operator A is a sectorial accretive operator of type (M,θ,α,μ) and generates compact α-resolvent families $\{T_\alpha(t)\}_{t\ge 0}$, $\{S_\alpha(t)\}_{t\ge 0}$, and $\{K_\alpha(t)\}_{t\ge 0}$.

(H_3): $f,g : J \times X \to P$ is jointly continuous and for any $k > 0$ there exist positive functions $\mu_k, \nu_k \in L([0,T],R^+)$ such that

$$\sup_{\|u\|\le k}\|f(t,u)\| \le \mu_k(t), \qquad \sup_{\|u\|\le k}\|g(t,u)\| \le \nu_k(t).$$

(H_4): $u_0 - m(u), u_1 - n(u) \in C(X,P)$ and there exist positive numbers a, b, c, d such that

$$\|m(u)\| \le a\|u\| + b, \qquad \|n(u)\| \le c\|u\| + d, \quad \text{for all } u \in X.$$

Theorem 3.1 *Assume that* (H_1)-(H_4) *hold, if* $\widetilde{M}(a+c) < 1$, *then problem* (1.2) *has at least one positive mild solution on* J.

Proof Using Definition 2.2, the mild solution of nonlocal fractional differential evolution equation (1.2) can be expressed as

$$\begin{aligned} u(t) &= S_\alpha(t)\bigl[u_0 - m(u)\bigr] + K_\alpha(t)\bigl[u_1 - n(u)\bigr] \\ &\quad + \int_0^t T_\alpha(t-s)\left[f\bigl(s,u(s)\bigr) + \int_0^s q(s-\tau)g\bigl(\tau,u(\tau)\bigr)\,d\tau\right]ds. \end{aligned}$$

Choose

$$r \ge \frac{\widetilde{M}(T\|\mu_r\|_{L^\infty(J,R^+)} + Tq\|\nu_r\|_{L^\infty(J,R^+)} + \|u_0\| + \|u_1\| + b + d)}{1 - \widetilde{M}(a+c)}$$

and consider $Q = \{u \in K : \|u\| \le r\}$. Define the operator $\Gamma : Q \to C(J,X)$ by

$$\begin{aligned} (\Gamma u)(t) &= S_\alpha(t)\bigl[u_0 - m(u)\bigr] + K_\alpha(t)\bigl[u_1 - n(u)\bigr] \\ &\quad + \int_0^t T_\alpha(t-s)\left[f\bigl(s,u(s)\bigr) + \int_0^s q(s-\tau)g\bigl(\tau,u(\tau)\bigr)\,d\tau\right]ds. \end{aligned}$$

Step 1: we prove that $\Gamma Q \subseteq Q$.

For any $u \in Q$, based on assumptions (H_3) and (H_4), for $t \in J$, we have

$$u_0 - m(u) \ge \theta, \qquad u_1 - n(u) \ge \theta, \tag{3.1}$$

$$f\bigl(t,u(t)\bigr) \ge \theta, \qquad \int_0^t q(t-s)g\bigl(s,u(s)\bigr)\,ds \ge \theta. \tag{3.2}$$

In view of (H_2), we note that A is a sectorial accretive operator of type (M,θ,α,μ) and generates compact and positive α-resolvent families $\{S_\alpha(t)\}_{t\ge 0}$, $\{K_\alpha(t)\}_{t\ge 0}$, and $\{T_\alpha(t)\}_{t\ge 0}$. Then we have

$$S_\alpha(t)\bigl[u_0 - m(u)\bigr] \ge \theta, \qquad K_\alpha(t)\bigl[u_1 - n(u)\bigr] \ge \theta, \tag{3.3}$$

$$\int_0^t T_\alpha(t-s)\left[f\big(s,u(s)\big)+\int_0^s q(a-\tau)g\big(\tau,u(\tau)\big)\,d\tau\right]ds\geq\theta. \tag{3.4}$$

Consequently, we obtain

$$(\Gamma u)\geq\theta,\quad \text{for } u\in\Omega. \tag{3.5}$$

Also we have

$$\begin{aligned}\big\|(\Gamma u)(t)\big\| &\leq \big\|S_\alpha(t)\big\|\cdot\big\|u_0-m(u)\big\|+\big\|K_\alpha(t)\big\|\cdot\big\|u_1-n(u)\big\|\\
&\quad+\int_0^t\big\|T_\alpha(t-s)\big\|\cdot\left\|f\big(s,x(s)\big)+\int_0^s q(s-\tau)g\big(\tau,u(\tau)\big)\,d\tau\right\|ds\\
&\leq\widetilde{M}\bigg(\|u_0\|+\big\|m(u)\big\|+\|u_1\|+\big\|n(u)\big\|+T\big\|f\big(t,u(t)\big)\big\|\\
&\quad+T\int_0^t\big|q(t-s)\big|\cdot\big\|g\big(s,u(s)\big)\big\|\,ds\bigg)\\
&\leq\widetilde{M}\big(\|u_0\|+a\|u\|+b+\|u_1\|+c\|u\|+d\\
&\quad+T\|\mu_r\|_{L^\infty(J,R^+)}+Tq\|\nu_r\|_{L^\infty(J,R^+)}\big)\\
&\leq r.\end{aligned} \tag{3.6}$$

Combining (3.5) with (3.6), we obtain

$$\Gamma Q\subseteq Q,\quad \text{for all } u\in Q.$$

Step 2: continuity of F.

Let $\{u_n\}$ be a sequence in Q such that $\|u_n-\overline{u}\|\to 0$. Noting that f, g, m, n are continuous, as $n\to\infty$ we have

$$m(u_n)\to m(u),\qquad n(u_n)\to n(u), \tag{3.7}$$

$$f\big(t,u_n(t)\big)\to f\big(t,u(t)\big),\qquad g\big(t,u_n(t)\big)\to g\big(t,u(t)\big). \tag{3.8}$$

For all $t\in J$, we get

$$\begin{aligned}\big\|(\Gamma u_n)(t)-(\Gamma u)(t)\big\| &\leq \big\|S_\alpha(t)\big\|\cdot\big\|m(u_n)-m(u)\big\|+\big\|K_\alpha(t)\big\|\cdot\big\|n(u_n)-n(u)\big\|\\
&\quad+\int_0^t\big\|T_\alpha(t-s)\big\|\cdot\Big[\big\|f\big(s,u_n(s)\big)-f\big(s,u(s)\big)\big\|\\
&\quad+\int_0^s\big|q(s-\tau)\big|\cdot\big\|g\big(\tau,u_n(\tau)\big)-g\big(\tau,u(\tau)\big)\big\|\,d\tau\Big]ds\\
&\leq\widetilde{M}\big\|m(u_n)-m(u)\big\|+\widetilde{M}\big\|n(u_n)-n(u)\big\|\\
&\quad+\widetilde{M}T\big\|f\big(t,u_n(t)\big)-f\big(t,u(t)\big)\big\|\\
&\quad+\widetilde{M}Tq\big\|g\big(t,u_n(t)\big)-g\big(t,u(t)\big)\big\|.\end{aligned}$$

From (3.7) and (3.8), we obtain $\lim_{n\to\infty}(\Gamma u_n)(t)=(\Gamma u)(t)$, that is, the operator Γ is continuous.

Step 3: compactness of Γ.

To this end, we use the Arzela-Ascoli theorem. We prove that $\{(\Gamma u)(t) : u \in Q\}$ is relatively compact in X. First, we prove that $\{(\Gamma u)(t) : u \in Q\}$ is uniformly bounded. We have

$$\begin{aligned}
\|(\Gamma u)(t)\| &\le \|S_\alpha(t)\| \cdot \|u_0 - m(u)\| + \|K_\alpha(t)\| \cdot \|u_1 - n(u)\| \\
&\quad + \int_0^t \|T_\alpha(t-s)\| \cdot \biggl[\bigl\|f(s,u(s))\bigr\| + \int_0^s \bigl|q(\tau - s)\bigr| \cdot \bigl\|g(\tau, u(\tau))\bigr\| \,d\tau\biggr] ds \\
&\le \widetilde{M}\biggl(\|u_0\| + \bigl\|m(u)\bigr\| + \|u_1\| + \bigl\|n(u)\bigr\| + T\bigl\|f(t,u(t))\bigr\| \\
&\quad + T\int_0^t \bigl|q(t-s)\bigr| \bigl\|g(s,u(s))\bigr\| \,ds\biggr) \\
&\le \widetilde{M}\bigl(\|u_0\| + a\|u\| + b + \|u_1\| + c\|u\| + d \\
&\quad + T\|\mu_r\|_{L^\infty(J,R^+)} + Tq\|\nu_r\|_{L^\infty(J,R^+)}\bigr) \\
&\le r \\
&\le \infty.
\end{aligned}$$

Now, let us prove that $\Gamma(Q)$ is equicontinuous. The function $\{(\Gamma u)(t) : u \in Q\}$ are equicontinuous at $t = 0$. For $0 < t_1 < t_2 \le T$ and $u \in Q$, we have

$$\begin{aligned}
\bigl\|(\Gamma u)(t_2) - (\Gamma u)(t_1)\bigr\| &\le \bigl\|S_\alpha(t_2) - S_\alpha(t_1)\bigr\| \cdot \bigl\|u_0 - m(u)\bigr\| \\
&\quad + \bigl\|K_\alpha(t_2) - K_\alpha(t_1)\bigr\| \cdot \bigl\|u_1 - n(u)\bigr\| \\
&\quad + \int_0^{t_1} \bigl\|T_\alpha(t_2 - s) - T_\alpha(t_1 - s)\bigr\| \biggl[\bigl\|f(s,u(s))\bigr\| \\
&\quad + \int_0^s \bigl|q(\tau - s)\bigr| \cdot \bigl\|g(\tau, u(\tau))\bigr\| \,d\tau\biggr] ds \\
&\quad + \int_{t_1}^{t_2} \bigl\|T_\alpha(t_2 - s)\bigr\| \biggl[\bigl\|f(s,u(s))\bigr\| \\
&\quad + \int_0^s \bigl|q(\tau - s)\bigr| \cdot \bigl\|g(\tau, u(\tau))\bigr\| \,d\tau\biggr] ds \\
&\le I_1 + I_2 + I_3,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \bigl\|S_\alpha(t_2) - S_\alpha(t_1)\bigr\| \cdot \bigl\|u_0 - m(u)\bigr\| + \bigl\|K_\alpha(t_2) - K_\alpha(t_1)\bigr\| \cdot \bigl\|u_1 - n(u)\bigr\|, \\
I_2 &= \int_0^{t_1} \bigl\|T_\alpha(t_2 - s) - T_\alpha(t_1 - s)\bigr\| \cdot \biggl[\bigl\|f(s,u(s))\bigr\| + \int_0^s q(\tau - s)\bigl\|g(\tau, u(\tau))\bigr\| \,d\tau\biggr] ds, \\
I_3 &= \int_{t_1}^{t_2} \bigl\|T_\alpha(t_2 - s)\bigr\| \cdot \biggl[\bigl\|f(s,u(s))\bigr\| + \int_0^s \bigl|q(\tau - s)\bigr| \bigl\|g(\tau, u(\tau))\bigr\| \,d\tau\biggr] ds.
\end{aligned}$$

The continuity of functions $t \to \|S_\alpha(t)\|$, $t \to \|K_\alpha(t)\|$ for $t \in (0, T]$, allows us to conclude that $\lim_{t_1 \to t_2} I_1 = 0$. Indeed, we have

$$I_2 \le \int_{t_1}^{t_2} \bigl\|T_\alpha(t_2 - s) - T_\alpha(t_1 - s)\bigr\| \cdot \bigl[\|\mu_r\|_{L^\infty(J,R^+)} + q\|\nu_r\|_{L^\infty(J,R^+)}\bigr] ds.$$

Therefore, the continuity of function $t \to \|T_\alpha(t)\|$ for $t \in (0, T]$ also allows us to conclude that $\lim_{t_1 \to t_2} I_2 = 0$. We have

$$\begin{aligned} I_3 &\le \int_{t_1}^{t_2} \|T_\alpha(t_2 - s)\| \big(\|\mu_r\|_{L^\infty(J,R^+)} + q\|\nu_r\|_{L^\infty(J,R^+)}\big)\, ds \\ &\le \widetilde{M}\big(\|\mu_r\|_{L^\infty(J,R^+)} + q\|\nu_r\|_{L^\infty(J,R^+)}\big)|t_2 - t_1|. \end{aligned}$$

Consequently, $\lim_{t_1 \to t_2} I_3 = 0$.

Thus, for $t \in J$, $\{(\Gamma u)(t) : u \in Q\}$ is a family of equicontinuous function. We have proved that $\{\Gamma(Q)\}$ is relatively compact. Hence by the Arzela-Ascoli theorem, Γ is compact. Schauder's fixed point theorem allows us to conclude that Γ has at least one fixed point on J. As $(\Gamma u)(t) \ge \theta$ when $u \in Q$, problem (1.2) has at least one positive mild solution on J. □

In the following, we give an existence result in the case where (H_4) is not satisfied. We need the following assumptions.

(H_5): $u_0 - m(u), u_1 - n(u) : X \to P$ are continuous and bounded on X.
(H_6): There exist positive numbers l_1, l_2 such that for any $t \in J$, $u, v \in X$ we have

$$\big\|f(t,u) - f(t,v)\big\| \le l_1\|u - v\|, \qquad \big\|g(t,u) - g(t,v)\big\| \le l_2\|u - v\|.$$

Theorem 3.2 *Assume that* (H_1)-(H_3), (H_5)-(H_6) *hold. If* $\widetilde{M}T(l_1 + ql_2) < 1$, *then problem* (1.2) *has at least one positive mild solution on* J.

Proof Choose

$$R \ge \widetilde{M}\big(\big\|u_0 - m(u)\big\| + \big\|u_1 - n(u)\big\| + T\|\mu_R\|_{L^\infty(J,R^+)} + Tq\|\nu_R\|_{L^\infty(J,R^+)}\big)$$

and consider $\Omega = \{u \in K : \|u\| \le R\}$. Define operators S, T on Ω by

$$\begin{aligned} (Su)(t) &= S_\alpha(t)\big[u_0 - m(u)\big] + K_\alpha(t)\big[u_1 - n(u)\big], \\ (Tu)(t) &= \int_0^t T_\alpha(t - s)\bigg[f\big(s, u(s)\big) + \int_0^s q(s - \tau)g\big(\tau, u(\tau)\big)\, d\tau\bigg]\, ds. \end{aligned}$$

Firstly, we prove that when $u, v \in \Omega$, we have $Su + Tv \in \Omega$.
Similar to (3.3) and (3.4), for $u, v \in \Omega$, we obtain

$$\begin{aligned} &S_\alpha(t)\big[u_0 - m(u)\big] \ge \theta, \qquad K_\alpha(t)\big[u_1 - n(u)\big] \ge \theta, \\ &\int_0^t T_\alpha(t - s)\bigg[f\big(s, v(s)\big) + \int_0^s q(s - \tau)g\big(\tau, v(\tau)\big)\, d\tau\bigg]\, ds \ge \theta. \end{aligned}$$

Consequently, we have

$$(Su)(t) + (Tv)(t) \ge \theta, \quad \text{for } u, v \in \Omega. \tag{3.9}$$

Following from (H_1), (H_3), and (H_5), we also have

$$\begin{aligned}\|(Su)(t)+(Tv)(t)\| &\le \|S_\alpha(t)\|\cdot\|u_0-m(u)\|+\|K_\alpha(t)\|\cdot\|u_1-n(u)\|\\&\quad+\int_0^t\|T_\alpha(t-s)\|\cdot\bigg\|f\big(s,u(s)\big)\\&\quad+\int_0^s q(s-\tau)g\big(\tau,u(\tau)\big)\,d\tau\bigg\|\,ds\\&\le \widetilde{M}\big(\|u_0-m(u)\|+\|u_1-n(u)\|\\&\quad+T\|\mu_R\|_{L^\infty(J,R^+)}+Tq\|\nu_R\|_{L^\infty(J,R^+)}\big)\\&\le R.\end{aligned}\tag{3.10}$$

Combining (3.9) with (3.10), we have $Su+Tv\in\Omega$, for $u,v\in\Omega$.

Secondly, we prove that the operator T is a contraction.

For any $u,v\in\Omega$, we get

$$\begin{aligned}\|(Tu)(t)-(Tv)(t)\| &\le \int_0^t\|T_\alpha(t-s)\|\cdot\|f\big(s,u(s)\big)-f\big(s,v(s)\big)\|\,ds\\&\quad+\int_0^t\|T_\alpha(t-s)\|\cdot\bigg(\int_0^s|q(s-\tau)|\cdot\|g\big(\tau,u(\tau)\big)\\&\quad-g\big(\tau,v(\tau)\big)\|\,d\tau\bigg)\,ds\\&\le \widetilde{M}Tl_1\|u(t)-v(t)\|+\widetilde{M}Tl_2q\|u(t)-v(t)\|\\&\le \widetilde{M}T(l_1+ql_2)\|u-v\|.\end{aligned}$$

Since $\widetilde{M}T(l_1+ql_2)<1$, operator T is a contraction.

Thirdly, we prove that S is continuous.

Let $u_n,u\in\Omega$, $\|u_n(t)-u(t)\|\to 0$ as $n\to\infty$. Noting that m, n are continuous, we have

$$m(u_n)\to m(u),\qquad n(u_n)\to n(u),\quad \text{as } n\to\infty.\tag{3.11}$$

Then

$$\begin{aligned}\|(Su_n)(t)-(Su)(t)\| &\le \|S_\alpha(t)\|\cdot\|m(u_n)-m(u)\|\\&\quad+\|K_\alpha(t)\|\cdot\|n(u_n)-n(u)\|\\&\le \widetilde{M}\big(\|m(u_n)-m(u)\|+\|n(u_n)-n(u)\|\big).\end{aligned}$$

Following from (3.11), we have $\lim_{n\to\infty}(Su_n)(t)=(Su)(t)$. That is, operator S is continuous.

Lastly, we prove S is compact.

To this end, we use the Ascoli-Arzela theorem. We prove that $\{(Su)(t):u\in\Omega\}$ is relatively compact for $t\in J$. For $u\in\Omega$, we have

$$\begin{aligned}\|(Su)(t)\| &\le \|S_\alpha(t)\|\cdot\|u_0-m(u)\|\\&\quad+\|K_\alpha(t)\|\cdot\|u_1-n(u)\|\end{aligned}$$

$$\begin{aligned}&\le \widetilde{M}\big(\|u_0 - m(u)\| + \|u_1 - n(u)\|\big)\\&< \infty.\end{aligned}$$

This shows operator S is uniformly bounded.

Now, let us prove that $S(\Omega)$ is equicontinuous. Obviously, the function $(Su)(t)$ is equicontinuous at $t = 0$. For $0 < t_1 < t_2 \le T$, $u \in \Omega$, we have

$$\begin{aligned}\big\|(Su)(t_2) - (Su)(t_1)\big\| \le{}& \big\|S_\alpha(t_2) - S_\alpha(t_1)\big\| \cdot \big\|u_0 - m(u)\big\|\\&+ \big\|K_\alpha(t_2) - K_\alpha(t_1)\big\| \cdot \big\|u_1 - n(u)\big\|.\end{aligned}$$

In view of (H_5), $\|u_0 - m(u)\|$, $\|u_1 - n(u)\|$ are bounded, so the continuity of functions $t \to \|S_\alpha(t)\|$, $t \to \|K_\alpha(t)\|$ for $t \in (0, T]$, allows us to conclude that

$$\lim_{t_1 \to t_2} (Su)(t_1) = (Su)(t_2).$$

In short, we have proved that $\{S(\Omega)\}$ is relatively compact for $\{(Su)(t) : u \in \Omega\}$ is a family of equicontinuous function. Hence by the Arzela-Ascoli theorem, S is compact. As all the conditions of Krasnoselskii's fixed point theorem are satisfied, we conclude that Cauchy problem (1.2) has at least one mild solution on J. Given that $Su + Tu \ge \theta$ for $u \in \Omega$, we learn that Cauchy problem (1.2) has at least one positive mild solution on J. □

4 Example

We consider the following fractional differential equation:

$$\begin{cases} D^\alpha u(t,x) = \frac{\partial^2 u(t,x)}{\partial x^2} + \frac{e^t |u(t,x)|}{(24+e^t)(1+|u(t,x)|)} + \int_0^t e^{t-s} \frac{e^s}{\sqrt{48}+|u(t,x)|}\, ds, \\ u(0,x) - \frac{|u(t,x)|}{8+|u(t,x)|} = 0, \qquad \frac{du(t,x)}{dt}\big|_{t=0} - \frac{|u(t,x)|}{8+|u(t,x)|} = 0, \\ u(t,0) = u(t,\pi) = 0, \qquad u'(t,0) = u'(t,\pi) = 0, \end{cases} \tag{4.1}$$

where $t \in J = [0,1]$, $0 \le x \le \pi$, $1 < \alpha < 2$, let $X = L^2([0,\pi])$. Then the fractional differential equation (4.1) has at least one positive mild solution on J.

Proof As $X = L^2([0,\pi])$, then the positive cone of X is $P = \{u \in C(J,X) : u(t,x) \ge 0, \text{a.e. } (t,x) \in J \times X\}$. The operator $A : D(A) \subset X \to X$ is given by

$$Ax = x'' \quad \text{with } D(A) := \big\{x \in X : x' \in X, x(0) = x(\pi) = 0\big\}.$$

It is well known that A is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\ge 0}$ on X. Furthermore, A has discrete spectrum with eigenvalues $-n^2$, $n \in N$, and corresponding normalized eigenfunctions given by $z_n(x) = (\frac{\pi}{2})^{1/2} \sin(nx)$ [17]. In addition, $\{z_n : n \in N\}$ is an orthonormal basis of X and

$$T(t) = \sum_{n=1}^{\infty} e^{-n^2 t} \langle x, z_n \rangle z_n, \quad \text{for } x \in X, t \ge 0.$$

It follows from this representation that $T(t)$ is compact for every $t > 0$ and that $\|T(t)\| \le e^{-t}$ for every $t \ge 0$ [18].

As indicated in [9], the operator $A = \Delta$ is a sectorial operator of type (M,θ,α,μ) and generates compact α-resolvent families $\{S_\alpha(t)\}_{t\geq 0}$, $\{K_\alpha(t)\}_{t\geq 0}$, and $\{T_\alpha(t)\}_{t\geq 0}$. Since it was proved in [19] that $A = \Delta$ is an m-accretive operator on X with dense domain, assumption (H_2) is satisfied.

In this situation,

$$f(t,u) = \frac{e^t|u|}{(24+e^t)(1+|u|)}, \qquad g(t,u) = \frac{e^t}{\sqrt{48}+|u|},$$
$$m(u) = -\frac{|u|}{8+|u|}, \qquad n(u) = -\frac{|u|}{8+|u|}, \qquad q(t-s) = e^{t-s}.$$

From the estimates on the norms of operators of Theorem 2.4 and Theorem 2.5, we can obtain $\widetilde{M} = 3$ (see [4]). Moreover, for $t \in J$, $u, v \in R$ we have

$$\|f(t,u) - f(t,v)\| = \frac{e^t}{24+e^t}\left\|\frac{u}{1+u} - \frac{v}{1+v}\right\| \leq \frac{e^t}{24+e^t}\|u-v\| \leq \frac{1}{8}\|u-v\|,$$
$$\|g(t,u) - g(t,v)\| = e^t\left\|\frac{1}{\sqrt{48}+u} - \frac{1}{\sqrt{48}+v}\right\| \leq \frac{e^t}{48}\|u-v\| \leq \frac{1}{16}\|u-v\|.$$

So, we have $l_1 = \frac{1}{8}$, $l_2 = \frac{1}{16}$. Meanwhile,

$$\|f(t,u)\| \leq \frac{e^t}{24+e^t} < \frac{1}{8}, \qquad \|g(t,u)\| \leq \frac{e^t}{\sqrt{48}} < \frac{3}{\sqrt{48}} = \frac{\sqrt{3}}{4}.$$

Hence, $\|\mu_1\| = \frac{1}{8}$, $\|\nu_1\| = \frac{\sqrt{3}}{4}$, which means assumptions (H_3) and (H_6) are satisfied. We have

$$\max \int_0^t |q(t-s)|\, ds = \max_{t\in[0,1]} \int_0^t e^{t-s}\, ds = \max_{t\in[0,1]} e^t - 1 \leq 2.$$

For $u \in R$, we have

$$\|u_0 - m(u)\| \leq \left\|\frac{u}{8+u}\right\| \leq \frac{1}{8}, \qquad \|u_1 - n(u)\| \leq \frac{1}{8}.$$

Then assumption (H_5) is satisfied.

Consequently,

$$\widetilde{M}T(l_1 + ql_2) = 3\left(\frac{1}{8} + 2 \times \frac{1}{16}\right) = \frac{3}{4} < 1.$$

It is not difficult to conclude that all the conditions of Theorem 3.2 are satisfied. Hence, the nonlocal fractional differential equation (4.1) has at least one positive mild solution. $\square$

5 Conclusion

In this paper, we discussed the existence of positive mild solution of a kind of fractional differential evolution equation with nonlocal conditions of order $1 < \alpha < 2$. Firstly, we investigated the properties of solution operators by means of the classical Mittag-Leffler function. Secondly, we obtain the existence result by applying Schauder's fixed point theorem and Krasnoselskii's fixed point theorem under some special conditions.

Based on this work, our future work will be devoted to the study of the existence of positive mild solutions of impulsive fractional differential evolution equations of order $1 < \alpha < 2$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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