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Exponential stability of fractional stochastic differential equations with distributed delay

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Abstract

Equations driven by fractional Brownian motion are attracting more and more attention. This paper considers fractional stochastic differential equations with distributed delay. With the variation-of-constants formula, an explicit expression and asymptotic behavior of the solution are provided, sufficient conditions are derived to guarantee the p th moment exponential stability and almost surely exponential stability.

MSC: 39B82; 60H05

Keywords: fractional Brownian motion; variation-of-constants formula; exponential stability; distributed delay

1 Introduction

Fractional Brownian motion (fBm) was first studied in 1940 by Kolmogorov [1], who defined it in a Hilbert space and named it a Wiener helix. Until 1968, Mandelbrot and Van Ness [2] provided the stochastic integral representation of fBm in terms of a standard Brownian motion, the name fBm was then introduced. As an extension of classic Brownian motion, fBm can give a better description to model natural situations like the temperature at a specific place as a function of time and so on [3]. Because of their extensive applications in finance, economics, biology, *etc.*, fBm is attracting more and more attention.

An fBm with Hurst index $H \in (0,1)$ is a continuous and centered Gaussian process $\{B^H(t), t \geq 0\}$ with covariance function

$$E[B^H(t)B^H(s)] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

For $H = \frac{1}{2}$, the fBm is exactly the standard Brownian motion without memory. For $H \neq \frac{1}{2}$, it is usually divided into two cases, $0 < H < \frac{1}{2}$ and $\frac{1}{2} < H < 1$. The case for $H > \frac{1}{2}$ is called a long memory process, while $H < \frac{1}{2}$ is called a short memory process.

It is known that fBm shares something with Brownian motion, but the critical difference is that the increments of fBm are dependent while that of Brownian motion are independent. Therefore, fBm is neither a Markov process, nor a semimartingale, and the theories of stochastic differential equations (SDEs) driven by standard Brownian motion are invalid in the case of fBm. Recently, there are some studies on existence, uniqueness, and stability of the solutions for fractional SDEs. Ferrante and Rovira [4] proved the existence

and uniqueness for a stochastic delay differential equation driven by a fractional Brownian motion with Hurst parameter $H > 1/2$. Ferrante and Rovira [5] gave a proof of the existence and uniqueness for stochastic delay differential equations with hereditary drift driven by a fractional Brownian motion with Hurst parameter $H > 1/2$. Caraballo *et al.* [6] investigated the existence, uniqueness, and exponential asymptotic behavior of mild solutions to stochastic delay evolution equations perturbed by a fractional Brownian motion with Hurst parameter $H > 1/2$. Based on the research of [6], Boufoussi and Hajji [7] gave a proof of the existence and uniqueness of mild solutions for a neutral stochastic differential equation with finite delay, driven by a fractional Brownian motion in a Hilbert space. Dung [8] gave a sufficient condition for the exponential asymptotic behavior of solutions of a general class of linear fractional SDEs with time-varying delays. However, to the best of our knowledge, fractional SDEs with distributed delay have not been considered till now. In this paper, SDEs driven by an fBm with distributed delay will be introduced and asymptotic behavior of the solution will be analyzed.

In [2], Mandelbrot and Van Ness discussed the following integral representation:

$$B^H(t) = B^H(0) + \frac{1}{\Gamma(H + 1/2)} \left(Z_t + \int_0^t (t-s)^{H-1/2} dW(s) \right), \quad t > 0,$$

where

$$Z_t = \int_{-\infty}^0 [(t-s)^{H-1/2} - (-s)^{H-1/2}] dW(s),$$

$W(t)$ is a standard Brownian motion, Γ represents the gamma function, and $0 < H < 1$ is the Hurst parameter. There are also several other stochastic integral representations for fBm [3].

As to the presentation introduced by Mandelbrot and Van Ness [2], the integral form

$$W^H(t) = \int_0^t (t-s)^{H-1/2} dW(s)$$

is called the Riemann-Liouville fractional integral. There is a relation between the Riemann-Liouville fractional Brownian motion (RLfBm) and fBm [9]. Considering that the process Z_t has absolutely continuous trajectories, it suffices to consider the term $W^H(t)$ instead of $B^H(t)$ [10], thus, $B^H(t)$ will be denoted by $W^H(t)$ along our paper.

This paper is organized as follows: In Section 2, the explicit form of solution to the retarded SDE with an fBm is given. In Section 3, asymptotic behavior of the solution is provided, and sufficient conditions are derived to guarantee the p th moment exponential stability and almost surely exponential stability.

2 Preliminary and explicit solution

In the following sections, we consider retarded SDE driven by an fBm in the form:

$$dX(t) = \left(\int_{-\tau}^0 X(t+\theta) \rho(d\theta) \right) dt + \sigma(t) dW^H(t), \quad t > 0 \tag{2.1}$$

with the initial value $X(t) = \xi(t)$, $t \in [-\tau, 0]$, where $W^H(t)$ is a RLfBm with Hurst parameter $H > \frac{1}{2}$, $\rho(\cdot)$ is a finite signed measure defined on $[-\tau, 0]$.

In order to give the explicit form of (2.1), we first consider the following deterministic retarded differential equation:

$$dY(t) = \left(\int_{-\tau}^0 Y(t + \theta) \rho(d\theta) \right) dt \quad (2.2)$$

with the initial value $Y(t) = \xi(t)$, $t \in [-\tau, 0]$. It is easy to see that the characteristic equation of (2.2) is

$$h(\lambda) := \lambda - \int_{-\tau}^0 e^{\lambda\theta} \rho(d\theta) = 0.$$

Denote $Z(t)$ as the fundamental solution of (2.2) with initial value $Z(0) = 1$ and $Z(\theta) = 0$, $\theta \in [-\tau, 0)$. By the variation-of-constants formula (see, e.g., [11]), the solution of (2.2) has a unique explicit form:

$$Y(t) = Z(t)\xi(0) + \int_{-\tau}^0 \int_{\theta}^0 Z(t + \theta - s) \xi(s) ds \rho(d\theta), \quad t \geq 0.$$

According to Hale [12], for any $\alpha > \alpha_0 := \max\{\operatorname{Re}(\lambda) : h(\lambda) = 0\}$, there exists $k_\alpha > 0$ such that the fundamental solution $Z(t)$ satisfies the inequality

$$|Z(t)| \leq k_\alpha e^{\alpha t}, \quad t \geq -\tau.$$

Before we introduce the explicit representation of (2.1), we first present a lemma that is useful in later parts.

Lemma 2.1 [13] *For every $\epsilon > 0$, denote by $\beta = H - \frac{1}{2}$, define*

$$W^{H,\epsilon}(t) = \int_0^t (t - s + \epsilon)^\beta dW(s).$$

Then $\{W^{H,\epsilon}(t)\}_{t \geq 0}$ is a semimartingale with the following decomposition:

$$W^{H,\epsilon}(t) = \epsilon^\beta W(t) + \int_0^t \phi^\epsilon(s) ds, \quad (2.3)$$

where $\phi^\epsilon(s) = \int_0^s \beta(s + \epsilon - u)^{\beta-1} dW(u)$. Furthermore, $W^{H,\epsilon}(t)$ converges to $W^H(t)$ in $L^2(\Omega)$ uniformly with respect to $t \in [0, T]$ when $\epsilon \rightarrow 0$.

Theorem 2.2 *There is a unique strong solution $\{X(t)\}_{t \geq 0}$ to (2.1) which reads*

$$\begin{aligned} X(t) &= Z(t)\xi(0) + \int_{-\tau}^0 \int_{\theta}^0 Z(t + \theta - s) \xi(s) ds \rho(d\theta) \\ &\quad + \int_0^t Z(t - s) \sigma(s) dW^H(s), \quad t \geq 0, \end{aligned} \quad (2.4)$$

where $Z(t)$ is the fundamental solution of (2.2) with initial value $Z(0) = 1$ and $Z(\theta) = 0$, $\theta \in [-\tau, 0)$.

Proof According to Dung [8], the solution $X(t)$ of (2.1) can be approximated by solving the following equation:

$$dX^\epsilon(t) = \left(\int_{-\tau}^0 X^\epsilon(t+\theta)\rho(d\theta) \right) dt + \sigma(t) dW^{H,\epsilon}(t), \quad t > 0$$

with the initial datum $X^\epsilon(t) = \xi(t)$, $t \in [-\tau, 0]$. By the decomposition (2.3), we can rewrite the above equation as

$$dX^\epsilon(t) = \left(\int_{-\tau}^0 X^\epsilon(t+\theta)\rho(d\theta) + \sigma(t)\phi^\epsilon(t) \right) dt + \sigma(t)\epsilon^\beta dW(t), \quad t > 0.$$

Multiplying this equation by $e^{-\lambda t}$, $\text{Re } \lambda > c$ sufficiently large, integrating from 0 to ∞ , denoting by $\mathcal{L}(X^\epsilon)(\lambda)$ the Laplace transform of $X^\epsilon(t)$, we obtain

$$\begin{aligned} h(\lambda)\mathcal{L}(X^\epsilon)(\lambda) &= \xi(0) + \int_{-\tau}^0 \int_{\theta}^0 e^{\lambda(\theta-t)} \xi(t) dt \rho(d\theta) + \int_0^\infty e^{-\lambda t} \sigma(t) \phi^\epsilon(t) dt \\ &\quad + \int_0^\infty e^{-\lambda t} \sigma(t) \epsilon^\beta dW(t). \end{aligned}$$

An application of the Laplace inversion theorem (see, for example, [12]) yields

$$\begin{aligned} X^\epsilon(t) &= \int_{(c)} e^{\lambda t} h^{-1}(\lambda) \left[\xi(0) + \int_{-\tau}^0 \int_{\theta}^0 e^{\lambda(\theta-t)} \xi(t) dt \rho(d\theta) \right. \\ &\quad \left. + \int_0^\infty e^{-\lambda t} \sigma(t) \phi^\epsilon(t) dt + \int_0^\infty e^{-\lambda t} \sigma(t) \epsilon^\beta dW(t) \right] d\lambda. \end{aligned}$$

Note that

$$\mathcal{L}(Z^\epsilon)(\lambda) = h^{-1}(\lambda),$$

where $Z^\epsilon(t)$ is the fundamental solution of

$$dY^\epsilon(t) = \left(\int_{-\tau}^0 Y^\epsilon(t+\theta)\rho(d\theta) \right) dt$$

with the initial value $Z^\epsilon(0) = 1$ and $Z^\epsilon(\theta) = 0$, $\theta \in [-\tau, 0)$.

With the definition of a convolution of the Laplace transform, we get

$$\begin{aligned} X^\epsilon(t) &= Z^\epsilon(t)\xi(0) + \int_{-\tau}^0 \int_{\theta}^0 Z^\epsilon(t+\theta-s)\xi(s) ds \rho(d\theta) \\ &\quad + \int_0^t Z^\epsilon(t-s)\sigma(s)\phi^\epsilon(s) ds + \int_0^t Z^\epsilon(t-s)\sigma(s)\epsilon^\beta dW(s). \end{aligned}$$

Therefore, by (2.3), it follows that

$$\begin{aligned} X^\epsilon(t) &= Z^\epsilon(t)\xi(0) + \int_{-\tau}^0 \int_{\theta}^0 Z^\epsilon(t+\theta-s)\xi(s) ds \rho(d\theta) \\ &\quad + \int_0^t Z^\epsilon(t-s)\sigma(s) dW^{H,\epsilon}(s), \quad t \geq 0. \end{aligned}$$

Let $\epsilon \rightarrow 0$, the representation of $X(t)$ is obvious. □

3 Exponential behavior

The following two theorems provide the p th moment exponential stability and almost sure exponential stability of the solutions to (2.1).

Theorem 3.1 *Assume that $\alpha_0 < 0$, and there exists $\gamma > 0$ such that*

$$\int_0^\infty e^{2\gamma s} |\sigma(s)|^2 ds < \infty, \tag{3.1}$$

then there exist positive constants K and μ such that for each $p > 0$, and the solution of (2.1) satisfies

$$\mathbb{E}|X(t)|^p \leq K \|\xi\|_\infty^p e^{-\mu pt}, \quad t \geq 0,$$

where $\|\zeta\|_\infty := \sup_{-\tau \leq \theta \leq 0} |\zeta(\theta)|$.

Proof According to the explicit form (2.4), along with the inequality $|a + b + c|^p \leq k_p(|a|^p + |b|^p + |c|^p)$, $p, k_p > 0$, it is easy to get

$$\begin{aligned} \mathbb{E}|X(t)|^p &= \mathbb{E} \left| Z(t)\xi(0) + \int_{-\tau}^0 \int_\theta^0 Z(t + \theta - s)\xi(s) ds \rho(d\theta) \right. \\ &\quad \left. + \int_0^t Z(t-s)\sigma(s) dW^H(s) \right|^p \\ &\leq k_p \mathbb{E} \left(|Z(t)\xi(0)|^p + \left| \int_{-\tau}^0 \int_\theta^0 Z(t + \theta - s)\xi(s) ds \rho(d\theta) \right|^p \right. \\ &\quad \left. + \left| \int_0^t Z(t-s)\sigma(s) dW^H(s) \right|^p \right) \\ &\leq K_1 e^{-\alpha pt} \|\xi\|_\infty^p + k_p \mathbb{E} \left| \int_0^t Z(t-s)\sigma(s) dW^H(s) \right|^p. \end{aligned} \tag{3.2}$$

Denote by

$$I(t) := k_p \mathbb{E} \left| \int_0^t Z(t-s)\sigma(s) dW^H(s) \right|^p.$$

We should remind the reader that fBm is a centered Gaussian process. So, before considering $I(t)$, we can give an estimation on $|\int_0^t Z(t-s)\sigma(s) dW^H(s)|^2$ first. Applying (2.3), we obtain

$$\begin{aligned} &\int_0^t Z(t-s)\sigma(s) dW^H(s) \\ &= \lim_{\epsilon \rightarrow 0} \int_0^t Z(t-s)\sigma(s) dW^{H,\epsilon}(s) \\ &= \beta \int_0^t \int_0^s Z(t-s)\sigma(s)(s-u)^{\beta-1} dW(u) ds \\ &= \beta \int_0^t \int_u^t Z(t-s)\sigma(s)(s-u)^{\beta-1} ds dW(u). \end{aligned}$$

By the Itô isometry formula and the Hölder inequality, we have

$$\begin{aligned} & \mathbb{E} \left| \int_0^t Z(t-s)\sigma(s) dW^H(s) \right|^2 \\ &= \beta^2 \int_0^t \left(\int_u^t Z(t-s)\sigma(s)(s-u)^{\beta-1} ds \right)^2 du \\ &\leq \beta^2 \int_0^t \left(\int_u^t |Z(t-s)\sigma(s)|^2 (s-u)^{\beta-1} ds \right) \left(\int_u^t (s-u)^{\beta-1} ds \right) du \\ &\leq t^\beta \int_0^t |Z(t-s)\sigma(s)|^2 s^\beta ds \leq t^{2\beta} \int_0^t |Z(t-s)\sigma(s)|^2 ds \\ &\leq k_\alpha^2 t^{2\beta} \int_0^t e^{-2\alpha(t-s)} |\sigma(s)|^2 ds. \end{aligned}$$

By virtue of [8, Lemma 3.1] and condition (3.1), letting $\gamma_1 = \alpha \wedge \gamma$, we have

$$\mathbb{E} \left| \int_0^t Z(t-s)\sigma(s) dW^H(s) \right|^2 \leq k_\alpha^2 t^{2\beta} e^{-2\gamma_1 t} \int_0^\infty e^{2\gamma_1 s} |\sigma(s)|^2 ds \leq K_2^2 e^{-2\gamma_2 t},$$

where $K_2^2 = k_\alpha^2 t^{2\beta} e^{-\gamma_1 t} \int_0^\infty e^{2\gamma_1 s} |\sigma(s)|^2 ds$, $\gamma_2 = \frac{1}{2}\gamma_1$. Therefore, the Gaussian property leads to $I(t) \leq K_2^p e^{-\gamma_2 p t}$. An application of $I(t)$ on (3.2) yields

$$\mathbb{E}|X(t)|^p \leq K \|\xi\|_\infty^p e^{-\mu p t}, \quad t \geq 0,$$

where $\mu = \alpha \wedge \gamma_2$. This completes the proof. □

Theorem 3.2 *Assume that (3.1) is satisfied, then there exists a positive constant ν such that the solution of (2.1) has the property*

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log |X(t)| \leq -\nu.$$

Proof For $n-1 \leq t \leq n$, $n \geq 1$, $X(t)$ can be represented as

$$X(t) = X(n-1) + \int_{n-1}^t \left(\int_{-\tau}^0 X(s+\theta)\rho(d\theta) \right) ds + \int_{n-1}^t \sigma(s) dW^H(s).$$

With the inequality $|a+b+c|^2 \leq 3(|a|^2 + |b|^2 + |c|^2)$, we derive from (3.2) that

$$\begin{aligned} \mathbb{E} \left(\sup_{n-1 \leq t \leq n} |X(t)|^2 \right) &\leq 3\mathbb{E}|X(n-1)|^2 + 3\mathbb{E} \left(\int_{n-1}^n \left(\int_{-\tau}^0 X(s+\theta)\rho(d\theta) \right) ds \right)^2 \\ &\quad + 3\mathbb{E} \left(\sup_{n-1 \leq t \leq n} \left| \int_{n-1}^t \sigma(s) dW^H(s) \right|^2 \right) \\ &\leq C_1 \|\xi\|_\infty^2 e^{-2\mu(n-1)} + C_2 e^{-2\mu(n-1)} + 3J(t), \end{aligned}$$

where $J(t) = \mathbb{E}(\sup_{n-1 \leq t \leq n} | \int_{n-1}^t \sigma(s) dW^H(s) |^2)$.

Next, by the Burkhold-Davis-Gundy inequality, (2.3) and (3.1), we arrive at

$$\begin{aligned} J(t) &\leq \mathbb{E} \left(\sup_{n-1 \leq t \leq n} \left| \int_{n-1}^t \sigma(s) dW^H(s) \right|^2 \right) \\ &\leq C_3 \int_{n-1}^n \left(\int_u^n \sigma(s) \beta(s-u)^{\beta-1} ds \right)^2 du \leq C_4 e^{-2\gamma(n-1)}. \end{aligned}$$

Let $\gamma_3 = \mu \wedge \gamma$, we see that

$$\mathbb{E} \left(\sup_{n-1 \leq t \leq n} |X(t)|^2 \right) \leq C e^{-2\gamma_3 n}.$$

Then, with the Chebyshev inequality, for any $\gamma_4 < \gamma_3$,

$$\mathbb{P} \left(\sup_{n-1 \leq t \leq n} |X(t)|^2 > e^{-2\gamma_4 n} \right) \leq e^{2\gamma_4 n} \mathbb{E} \left(\sup_{n-1 \leq t \leq n} |X(t)|^2 \right) \leq C e^{-2(\gamma_3 - \gamma_4)n}.$$

Since $\sum_{n=1}^{\infty} e^{-2(\gamma_3 - \gamma_4)n} < \infty$, by virtue of Borel-Cantelli lemma, there exists an $\Omega_0 \in \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that for every $\omega \in \Omega_0$, there exists an integer $n_0(\omega)$, for $n \geq n_0(\omega)$, $n - 1 \leq t \leq n$,

$$|X(t)|^2 \leq e^{-2\gamma_4 n} \leq e^{-2\gamma_4 t}.$$

The desired conclusion is satisfied with $\nu = \gamma_4$. □

Remark 3.1 This provides a method to ensure the exponential stability of SDEs driven by an fBm, and it can be generalized to fractional SDEs of neutral type.

Competing interests

The author declares to have no competing interests.

Author's contributions

The author carried out the proof and conceived of the study. The author read and approved the final manuscript.

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