Advances in Difference Equations
a SpringerOpen Journal

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Master-slave synchronization of chaotic systems with a modified impulsive controller

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Abstract

This paper investigates global exponential synchronization of chaotic systems by designing a novel impulsive controller. The novel impulsive controller is a combination of current and past error states, which is a modification of the normal impulsive one. Some global exponential stability criteria are derived for the error system by utilizing the stability analysis of impulsive differential equations and differential inequalities and, moreover, the exponential convergence rate can be specified. An illustrative example is given to show the effectiveness of the modified impulsive control scheme.

Keywords: impulsive synchronization; chaotic systems; impulsive controller; global exponential synchronization

1 Introduction

Synchronization of chaotic systems has become an active research area because of its potential applications in different industrial areas [1–3]. Communication security scheme is one of the hottest fields based on chaos synchronization. In this secure communication scheme, the message signals are injected to a chaotic carrier in the transmitter and then are masked or encrypted. The resulting masked signals are transmitted across a public channel to the receiver. To recover the message in the receiver, the synchronization between the chaotic systems at the transmitter and receiver ends is required. Since Pecora and Carroll [4] originally proposed the synchronization of the drive and response systems with different initial states in 1990, many synchronization techniques such as coupling control [5], adaptive control [6], feedback control [7], fuzzy control [8], observer-based control [9], *etc.* have been developed in the literature.

Most recently, the impulsive control techniques have been reported and developed to be an interesting method [10–14]. In addition, Yang and Cao [15] investigated the exponential synchronization of the complex dynamical networks with a coupled delay and impulsive control. Guan *et al.* [16] derived the synchronization of complex dynamical networks with time-varying delays via impulsive distributed control. In [17], the authors analyzed the robustness of impulsive synchronization coupled by linear delayed impulses. The main ideas of these impulses are to use samples of the state variables of the drive system at discrete moments and to synchronize the response system discretely. Once the error system of the two coupled systems is asymptotically stable, they are said to be synchronized. Generally speaking, these impulses are samples of the state variables of the drive system at current discrete moments to drive the response system. However, we can also design a novel impulse using not only current instantaneous errors, but also the previous time instants of errors. By using such a technique, we can increase the impulse distances and reduce the control cost. Although the idea is relatively well defined in control theory, it brings difficulties and challenges to determine the stability of the impulsive differential equation due to a combination of current and past error states. In [18], the authors investigated the synchronization of hyper-chaotic systems with such a modified impulsive controller. Based on the above discussion, we design a more general impulsive controller than the one in [18] and give a new approach to investigate the synchronization of the drive and response system.

The main contributions of this paper are three-fold: (1) An effective modified impulsive controller is designed for the global exponential synchronization of coupled chaotic systems. (2) Due to the additional integral term of the errors corresponding to each impulse, equipped with the definitions and results, we establish a uniform comparison system for this case and derive a sufficient condition in this paper. (3) Global exponential synchronization of the chaotic systems with the proposed impulsive controller can be simultaneously realized. In other words, by adding the summation term in the error dynamics, one could achieve the same effect by increasing the impulse distance and reducing the control cost.

The outline of this paper is listed as follows. In Section 2, model description and some preliminaries are introduced. In Section 3, based on the stability analysis of impulsive functional differential equations, the criteria for the synchronization are derived. In Section 4, a numerical example is given to illustrate the effectiveness and feasibility of the synchronization criteria. Finally, concluding remarks are made in Section 5.

Notation We list some mathematical notations used throughout this paper as follows. Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space and $N = \{0, 1, 2, \ldots\}$. Let $\|\cdot\|$ be the Euclidean norm and I be the identity matrix. Denote $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ as the maximal and minimal eigenvalues of P, respectively. For a sequence t_k, $k \in N$ satisfying $0 \leq t_0 < t_1 < \cdots < t_k < t_{k+1} < \cdots$, let $\Delta_k = t_k - t_{k-1}$, $k = 1, 2, \ldots$, $\Delta_{\sup} \triangleq \sup_{k\in N}\{\Delta_k\}$, $\Delta_{\inf} \triangleq \inf_{k\in \mathbb{N}}\{\Delta_k\}$.

2 Model description and some preliminaries

A chaos-based communication system usually consists of two chaotic systems at the transmitter and receiver ends, which are called the master system and the slave system. At the transmitter end, the master system is

$$\dot{x}(t) = Ax(t) + \Phi_1\bigl(x(t)\bigr), \tag{2.1}$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in \mathbb{R}^n$ is the state variable, $A \in \mathbb{R}^{n\times n}$ is a constant matrix, and $\Phi_1(x) : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function.

Generally speaking, all the chaotic systems such as Lorenz system, Chen system, Lü system, and Chua's circuit can be written in the above form.

At the receiver end, the slave system is written in the following form with an impulsive control scheme:

$$\dot{y}(t) = Ay(t) + \Phi_2\bigl(y(t)\bigr) + u(t), \tag{2.2}$$

where $\Phi_2(x) : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function and $u(t)$ is the modified impulsive hybrid controller designed as

$$u(t) = \sum_{k=1}^{\infty} \delta(t - t_k) \left[B_p e(t_k) + B_l \sum_{i=k-N}^{k-1} e(t_i) \right],$$

where $B_p \in \mathbb{R}^{n\times n}$ and $B_l \in \mathbb{R}^{n\times n}$ are impulsive control gain matrices to be designed and $\delta(\cdot)$ is the Dirac delta function. The impulsive instant sequence $\{t_k\}_{k=1}^{\infty}$ satisfies $0 \le t_1 < t_2 < \cdots < t_k < \cdots$, with $\Delta_{\sup} < \infty$ and $\lim_{k\to\infty} t_k = \infty$. Let $e(t) = y(t) - x(t)$ be the synchronization error between the states of the master system (2.1) and the slave system (2.2).

Remark 1 The proposed modified impulsive control scheme in [18] utilizes feedback from the error at the current time instant and the errors at the previous time instants, which is quite different from the impulsive controllers in [12–17]. By this modification, one can increase the impulsive distance and therefore reduce the control cost effectively. In this paper, we design a more generally modified impulsive control scheme than the one in [18].

Hence, the slave system with the modified impulsive controller can then be described by the following impulsive differential equation:

$$\begin{cases} \dot{y}(t) = Ay(t) + \Phi_2(y(t)), & t \neq t_k, k \in N, t \ge t_0, \\ \Delta y(t) = B_p e(t_k) + B_l \sum_{i=k-N}^{k-1} e(t_i), & t = t_k, \end{cases} \tag{2.3}$$

where $\Delta y(t_k) = y(t_k^+) - y(t_k^-)$ is the 'jump' in the state variable at the time instant t_k, $y(t_k^+) = \lim_{t\to t_k^+} y(t)$ and $y(t_k^-) = \lim_{t\to t_k^-} y(t)$. For simplicity, we assume that $y(t)$ is left continuous at $t = t_k$, *i.e.*, $y(t_k^-) = y(t_k)$.

Subtracting (2.1) from (2.3) yields the following error dynamics:

$$\begin{cases} \dot{e}(t) = Ae(t) + \Psi(x(t), y(t)), & t \neq t_k, k \in N, t \ge t_0, \\ \Delta e(t) = B_p e(t_k) + B_l \sum_{i=k-N}^{k-1} e(t_i), & t = t_k, \end{cases} \tag{2.4}$$

where $\Psi(x(t), y(t)) = \Phi_2(y(t)) - \Phi_1(x(t))$. It is easy to see the master system (2.1) and the slave system (2.2) achieve global exponential synchronization if and only if the trivial solution $e(t) = 0$ is globally exponentially stable in the error system (2.4).

Assumption 1 There exist a positive definite matrix P and constant matrices $D \in \mathbb{R}^{n\times n}$ such that

$$(x - y)^T P\Psi(x, y) \le (x - y)^T PD(x - y).$$

Remark 2 Assumption 1 gives some requirements for the dynamics of the master system and the slave system. If the functions describing the master and slave systems satisfy $\|\Psi(x, y)\| \le l\|y - x\|$, where $x, y \in \mathbb{R}$, one can choose $D = lI_n$ to satisfy Assumption 1. In addition, several groups of chaotic systems such as Lorenz system, Chen system, Lü system, and Chua's circuit also satisfy Assumption 1 with $\Phi_2 = \Phi_1$.

Definition 1 ([19] Average impulsive interval) The average impulsive interval of the impulsive sequence $\zeta = \{t_1, t_2, \ldots\}$ is less than T_a if there exist a positive integer N_0 and a positive number T_a such that

$$N_\zeta(T,t) \geq \frac{T-t}{T_a} - N_0, \quad \forall T \geq t \geq 0,$$

where $N_\zeta(T,t)$ denotes the number of impulsive times of the impulsive sequence ζ in the time interval $[t,T]$.

Definition 2 The error dynamical system (2.4) is said to be globally exponentially synchronized if there exist $\alpha > 0$, $T > 0$, and $K > 0$ such that

$$\|e(t)\| \leq Ke^{-\alpha t}$$

holds for all $t > T$ and any initial value.

We will need the following lemmas.

Lemma 1 (see [20]) *For any vectors $x, y \in \mathbb{R}^n$ and a positive-definite matrix $Q \in \mathbb{R}^{n\times n}$, the following matrix inequality holds*: $2x^T y \leq x^T Qx + y^T Q^{-1} y$.

Lemma 2 (see [21]) *Let $P \in \mathbb{R}^{n\times n}$ be a positive definite matrix, then*

$$\lambda_{\min}(P)x^T x \leq x^T Px \leq \lambda_{\max}(P)x^T x, \quad \forall x \in \mathbb{R}^n.$$

3 Synchronization criteria

In this section, based on the stability analysis for an impulsive delayed system, some sufficient conditions are derived to ensure the global exponential synchronization for the master system and the slave system.

Theorem 1 *Suppose that Assumption* 1 *holds and $\Delta_{\sup} < \infty$. Let λ_1 be the largest eigenvalue of $(I_n + B_p)^T(I_n + B_p)$ and λ_2 be the largest eigenvalue of $B_l^T B_l$. If there exist a positive definite matrix P such that the discrete system*

$$z(k+1) = J_k(N+1)z(k), \quad k \in Z$$

is globally exponentially stable with decay rate $\sigma > 0$, where

$$J_k(N+1) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ \alpha_{k-N} & \alpha_{k-N+1} & \alpha_{k-N+2} & \cdots & \alpha_{k-1} & \tilde{\alpha}_{k-1} \end{bmatrix},$$

where $\tilde{\alpha}_{k-1} = ae^{\alpha\Delta_k}$, $\alpha_{k-N+i-1} = be^{\alpha\Delta_{k-N+i-1}}$, $i = 1, 2, \ldots, N$, $a = 2\lambda_1 \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}$, $b = 2N\lambda_2 \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}$, $\alpha = \frac{\lambda_{\max}(A^T P + PA + D^T P + PD)}{\lambda_{\max}(P)}$, $\lambda_1 = \lambda_{\max}((B_p + I)^T P(B_p + I))$, and $\lambda_2 = \lambda_{\max}(B_l^T B_l)$. Then the error *system* (2.4) *is globally exponentially stable with the convergence rate* $-\frac{\sigma}{2T_a}$, *and hence the slave system* (2.2) *can achieve global exponential synchronization with the master system* (2.1).

Proof Consider a Lyapunov function in the form of

$$V(t) = e^T(t)Pe(t),$$

when $t \in (t_{k-1}, t_k]$. The Dini derivative of $V(t)$ along the trajectory of the error system (2.4) can be obtained as follows:

$$\begin{aligned}
\dot{V}(t) &= \dot{e}^T(t)Pe(t) + e^T(t)P\dot{e}(t) \\
&= e^T(t)\big(A^T P + PA\big)e(t) + e^T(t)P\Psi\big(x(t), y(t)\big) + \Psi^T\big(x(t), y(t)\big)Pe(t) \\
&\le e^T(t)\big(A^T P + PA\big)e(t) + e^T(t)\big(D^T P + PD\big)e(t) \\
&= e^T(t)\big(A^T P + PA + D^T P + PD\big)e(t) \\
&\le \frac{\lambda_{\max}(A^T P + PA + D^T P + PD)}{\lambda_{\min}(P)} V(t) \\
&\triangleq \alpha V(t), \qquad (3.1)
\end{aligned}$$

where the first inequality is obtained by Assumption 1 and $\alpha = \frac{\lambda_{\max}(A^T P+PA+D^T P+PD)}{\lambda_{\min}(P)}$.

Therefore,

$$V(t) \le V\big(t_{k-1}^+\big)\exp\big[\alpha(t - t_{k-1})\big], \quad t \in (t_{k-1}, t_k], k = 1, 2, \ldots. \qquad (3.2)$$

On the other hand, it follows from (2.4) for $t = t_k^+$, $k = 1, 2, \ldots$, that we obtain

$$\begin{aligned}
V\big(e\big(t_k^+\big)\big) &= \left[e^T(t_k)(B_p + I)^T + \sum_{i=k-N}^{k-1} e^T(t_i)B_l^T\right] P \left[(B_p + I)e(t_k) + B_l \sum_{i=k-N}^{k-1} e(t_i)\right] \\
&= e^T(t_k)(B_p + I)^T P(B_p + I)e(t_k) + \sum_{i=k-N}^{k-1} e^T(t_i)B_l^T PB_l \sum_{i=k-N}^{k-1} e(t_i) \\
&\quad + 2e^T(t_k)(B_p + I)^T PB_l \sum_{i=k-N}^{k-1} e(t_i). \qquad (3.3)
\end{aligned}$$

By Lemmas 1 and 2, we can obtain that

$$e^T(t_k)(B_p + I)^T P(B_p + I)e(t_k) \le \lambda_1 \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} e^T(t_k)Pe(t_k) = \frac{a}{2} V\big(e(t_k)\big), \qquad (3.4)$$

$$\sum_{i=k-N}^{k-1} e^T(t_i)B_l^T PB_l \sum_{i=k-N}^{k-1} e(t_i) \le N\lambda_2 \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \sum_{i=k-N}^{k-1} e^T(t_i)Pe(t_i) = \frac{b}{2} \sum_{i=k-N}^{k-1} V\big(e(t_i)\big), \qquad (3.5)$$

$$2e^T(t_k)(B_p + I)^T PB_l \sum_{i=k-N}^{k-1} e(t_i)$$

$$\begin{aligned}
&\le e^T(t_k)(B_p+I)^T P(B_p+I)e(t_k) + \sum_{i=k-N}^{k-1} e^T(t_i)B_l^T PB_l \sum_{i=k-N}^{k-1} e(t_i) \\
&\le \frac{a}{2}V(e(t_k)) + \frac{b}{2}\sum_{i=k-N}^{k-1} V\bigl(e(t_i)\bigr),
\end{aligned} \tag{3.6}$$

where $\lambda_1 = \lambda_{\max}((B_p+I)^T P(B_p+I))$, $\lambda_2 = \lambda_{\max}(B_l^T B_l)$, $a = 2\lambda_1 \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}$ and $b = 2N\lambda_2 \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}$ are utilized.

From (3.4)-(3.6), we have

$$\begin{aligned}
V\bigl(e\bigl(t_k^+\bigr)\bigr) &\le aV\bigl(e(t_k)\bigr) + b\sum_{i=k-N}^{k-1} V\bigl(e(t_i)\bigr) \\
&\le aV\bigl(e\bigl(t_{k-1}^+\bigr)\bigr)e^{\alpha(t_k-t_{k-1})} + b\sum_{i=k-N}^{k-1} V\bigl(e\bigl(t_{i-1}^+\bigr)\bigr)e^{\alpha(t_i-t_{i-1})} \\
&= ae^{\alpha\Delta_k}V\bigl(e\bigl(t_{k-1}^+\bigr)\bigr) + b\sum_{i=k-N}^{k-1} e^{\alpha\Delta_i}V\bigl(e\bigl(t_{i-1}^+\bigr)\bigr) \\
&\triangleq \sum_{i=1}^{N} \alpha_{k-N+i-1}V\bigl(t_{k-N+i-2}^+\bigr) + \tilde{\alpha}_{k-1}V\bigl(t_{k-1}^+\bigr),
\end{aligned} \tag{3.7}$$

where $\tilde{\alpha}_{k-1} = ae^{\alpha\Delta_k}$, $\alpha_{k-N+i-1} = be^{\alpha\Delta_{k-N+i-1}}$, $i = 1, 2, \ldots, N$.

Similar to the proof of Theorem 4.2 in [22], by (3.7), for $k \in Z$, let

$$\begin{cases}
\omega_1(k) = V(e(t_{k+1}^+)), \\
\omega_2(k) = V(e(t_{k+2}^+)), \\
\vdots \\
\omega_{N+1}(k) = V(e(t_{k+N+1}^+))
\end{cases} \tag{3.8}$$

and $\omega(k) = (\omega_1(k), \omega_2(k), \ldots, \omega_{N+1}(k))^T$. Then the system of difference equations obtained above together with (3.7) and (3.8) can be expressed as

$$\omega(k-N) \le J_k(N+1)\omega(k-N-1),$$

where

$$J_k(N+1) = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 \\
\alpha_{k-N} & \alpha_{k-N+1} & \alpha_{k-N+2} & \cdots & \alpha_{k-1} & \tilde{\alpha}_{k-1}
\end{bmatrix}. \tag{3.9}$$

Let the comparison system be

$$\begin{cases}
z(k+1) = J_k(N+1)z(k), \\
z(N) = \omega(-1).
\end{cases} \tag{3.10}$$

Then, by the comparison principle, we can get

$$\omega(k-N-1) \le z(k), \quad k \ge N, k \in Z.$$

Thus, by the condition in the theorem, there exists a constant $K > 0$ such that

$$\left\|\omega(k-N-1)\right\| \le Ke^{-\sigma(k-N)}\left\|\omega(-1)\right\|, \quad k \ge N, k \in Z,$$

where $\|\omega(-1)\| = \sum_{i=0}^{N} V^2(e(t_i^+))$.

From (3.8), for $t = t_k$, $k \in Z$, we can get that

$$V\bigl(e\bigl(t_k^+\bigr)\bigr) = \omega_{N+1}(k-N-1) \le \left\|\omega(k-N-1)\right\| \le K\left\|\omega(-1)\right\|e^{-\sigma(k-N)}. \tag{3.11}$$

Hence, by Lemma 2, (3.2) and (3.11), and for any $t \in (t_{k-1}, t_k]$, $k \in Z$, we get

$$\begin{aligned}\left\|e(t)\right\|^2 &\le \frac{1}{\lambda_{\min}(P)}V\bigl(e(t)\bigr) \le \frac{1}{\lambda_{\min}(P)}V\bigl(t_{k-1}^+\bigr)e^{\alpha(t-t_{k-1})} \\ &\le \frac{K\|\omega(-1)\|e^{\alpha\Delta_{\sup}}}{\lambda_{\min}(P)}e^{\sigma(N+1)}e^{-\sigma k} \triangleq \tilde{K}^2e^{-\sigma k},\end{aligned}$$

where $\tilde{K}^2 = \frac{K\|\omega(-1)\|e^{\alpha\Delta_{\sup}}}{\lambda_{\min}(P)}e^{\sigma(N+1)}$.

Let $N_\zeta(t,t_0)$ be the number of impulsive times of the impulsive sequence ζ in the interval (t_0,t). Hence, we can obtain

$$\left\|e(t)\right\|^2 \le \tilde{K}^2e^{-\sigma N_\zeta(t,t_0)}. \tag{3.12}$$

Since the average impulsive interval of the impulsive sequence $\zeta = \{t_1, t_2, \ldots\}$ is equal to T_a, we have

$$N_\zeta(t,t_0) \ge \frac{t-t_0}{T_a} - N_0, \quad \forall T \ge t \ge 0.$$

Hence, by (3.12), we get

$$\left\|e(t)\right\| \le \tilde{K}e^{\frac{\sigma N_0}{2}}e^{-\frac{\sigma}{2T_a}(t-t_0)}.$$

Thus, the trivial solution $e = 0$ of the error system (2.4) is globally exponentially stable with the convergence rate $-\frac{\sigma}{2T_a}$, and hence the slave system (2.2) can achieve global exponential synchronization with the master system (2.1). □

Remark 3 In this paper, a modified impulsive control system is adopted to provide the basis for developing global exponential synchronization between the master system and the slave system, which can reduce the impulsive times and the control cost effectively. In addition, to stabilize the error system (2.4) more effectively, we can also consider that the error at the current time instant and the previous time instants play different roles in the impulsive control system. For example, we can suppose that $B_p = \frac{\eta b_p}{b_p+b_l}I$ and $B_l = \frac{\eta b_l}{N(b_p+b_l)}I$, where η, b_p and b_l are constants, and $|b_p| \ge |b_l|$. Obviously, it is a special case of Theorem 1.

Remark 4 Note that in the proof of Theorem 1, the concept of an average impulsive interval is employed to prove the global exponential stability for the error system under Assumption 1. By this approach, the requirement on the lower bound and upper bound of impulsive interval is removed in Theorem 1, which is different from the conventional ones in the literature.

Remark 5 If $N = 0$, the modified impulsive control scheme is the normal impulsive one, such as in [15–17]. Hence, by Theorem 1, we only need a positive definite matrix P such that $|\tilde{\alpha}_{k-1}| < 1, \forall k \in Z$, where $\tilde{\alpha}_{k-1} = ae^{\alpha\Delta_k}$, $i = 1, 2, \dots, N$, are the same as in Theorem 1. Then the slave system (2.2) can achieve global exponential synchronization with the master system (2.1). In fact, it can be seen from (3.7) that $\tilde{\alpha}_{k-1}$ is the impulsive strength of the impulsive signal if $N = 0$. If $|\tilde{\alpha}_{k-1}| < 1$, $\forall k \in Z$, the impulse is beneficial for the error system since the difference is reduced. Thus, the error system can be stable easily with the impulsive control system.

In the following, by using Theorem 1, we give some simple corollaries of Theorem 1.

Corollary 1 *Suppose the impulsive interval is a positive constant* Δ, *and the impulsive gain matrix* $B_p = b_p I$, *and* $B_l = b_l I$. *If there exists a positive definite matrix P such that*

$$\rho\bigl(J(N+1)\bigr) < e^{-\sigma},$$

where

$$J(N+1) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ be^{\alpha\Delta} & be^{\alpha\Delta} & be^{\alpha\Delta} & \cdots & be^{\alpha\Delta} & ae^{\alpha\Delta} \end{bmatrix},$$

where $a = 2(1 + b_p)^2 \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}$, $b = 2Nb_l^2 \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}$ *and* α *is the same as in Theorem* 1. *Then the error system* (2.4) *is globally exponentially stable with the convergence rate* $-\frac{\sigma}{2T_a}$, *and hence the slave system* (2.2) *can achieve global exponential synchronization with the master system* (2.1).

Proof The proof is similar to Theorem 1. □

Corollary 2 *If there exists a positive constant* $0 < \gamma < 1$ *such that every root* λ_j $(j = 1, 2, \dots, N+1)$ *of the characteristic polynomial*

$$F_k(\lambda) \triangleq \lambda^{N+1} - \tilde{\alpha}_{k-1}\lambda^N - \alpha_{k-1}\lambda^N - \cdots - \alpha_{k-N+1}\lambda - \alpha_{k-N}$$

satisfies $|\lambda_j| \le \gamma < 1$, $j = 1, 2, \dots, N+1$, *where* $\tilde{\alpha}_{k-1} = ae^{\alpha\Delta_k}$, $\alpha_{k-N+i-1} = be^{\alpha\Delta_{k-N+i-1}}$, $i = 1, 2, \dots, N$, *are the same as in Theorem* 1. *Then the error system* (2.4) *is globally exponentially stable with the convergence rate* $-\frac{\sigma}{2T_a}$, *and hence the slave system* (2.2) *can achieve global exponential synchronization with the master system* (2.1).

Proof In fact, $F_k(\lambda)$ is the characteristic polynomial of $J_k(n+1)$ in Theorem 1. Hence, if every root satisfies $|\lambda_j| \le \gamma < 1$, $j = 1, 2, \ldots, N+1$, there exists a constant $\sigma > 0$ such that $|\lambda_j| \le \gamma \le e^{-\sigma} < 1$, $j = 1, 2, \ldots, N+1$, then the spectral radius of $J_k(n+1)$ satisfies $\rho(J_k(n+1)) \le \gamma < 1$. Thus, we conclude that this corollary is true. □

4 Numerical example

In this section, the chaotic system used in this example and simulation is given by

$$\begin{cases} \dot{x}_1 = \alpha_1(x_2 - x_1 - f(x_1)), \\ \dot{x}_2 = x_1 - x_2 + x_3, \\ \dot{x}_3 = -\beta_1 x_2, \end{cases} \tag{4.1}$$

where α_1, β_1 are parameters and $f(x_1)$ represents the piecewise-linear function of the Chua diode, which is given by $f(x_1) = dx_1 + 0.5(c-d)(|x_1+1| - |x_1-1|)$, where $c < d < 0$ are two constants. When $\alpha_1 = 9.22$, $\beta_1 = 15.99$, $c = -1.25$, and $d = -0.76$, the Chua system is chaotic. We can obtain the double scroll attractor shown in Figure 1 with $x_1(0) = 0.2$, $x_2(0) = 1$, and $x_3(0) = 0.5$.

The Chua oscillator can be written in the form of (2.1), *i.e.*,

$$\dot{x}(t) = Ax(t) + \Phi_1\bigl(x(t)\bigr),$$

where

$$A = \begin{bmatrix} d-\alpha_1 & \alpha_1 & 0 \\ 1 & -1 & 1 \\ 0 & -\beta_1 & 0 \end{bmatrix}, \qquad \Phi_1\bigl(x(t)\bigr) = \begin{bmatrix} 0.5(c-d)(|x_1+1| - |x_1-1|) \\ 0 \\ 0 \end{bmatrix}.$$

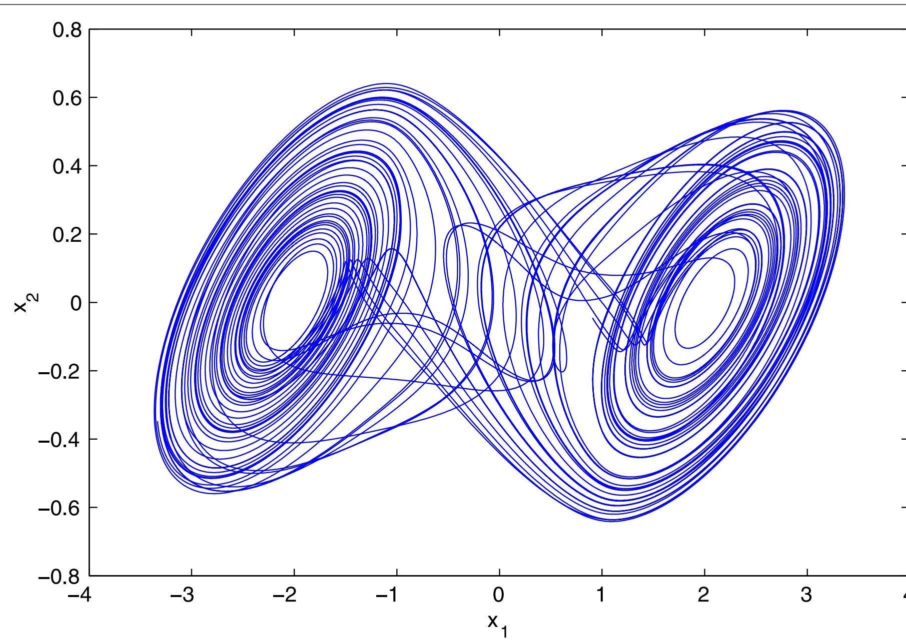

Figure 1 A double scroll attractor in Chua's circuit.

The function Φ_2 is defined as follows:

$$\Phi_2\big(y(t)\big) = \Phi_1\big(y(t)\big).$$

Therefore,

$$(x-y)^T P\Psi(x,y) = (x-y)^T P(\Phi_2\big(y(t)\big) - \Phi_1\big(x(t)\big) \le (x-y)^T P \begin{bmatrix} c-d & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} (x-y)$$

which shows that Assumption 1 holds with $D = \begin{bmatrix} c-d & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $P = I$.

Suppose $B_p = b_p I$, $B_l = b_l I$. We should choose proportional and integral gains (b_p, b_l) to satisfy the conditions in Corollary 1. Set $b_p = -0.8$, $b_l = -0.1$, and $N = 2$, we have $a = 0.08$, $b = 0.08$, $\alpha = 15.5$, and an impulsive interval $\Delta = 0.09$, one obtains $ae^{\alpha\Delta} = be^{\alpha\Delta} = 0.3228$ which results in $\rho(J(N+1)) < e^{-0.01}$. Based on Corollary 1, the error system is globally exponentially stable with the convergence rate $-\frac{1}{18}$, and hence the slave system can achieve global exponential synchronization with the master system. The quantity $e(t) = \sqrt{e_1^2 + e_2^2 + e_3^2}$ is used to measure the quality of synchronization errors of drive-response dynamical systems, which is simulated in Figure 2.

To illustrate the effectiveness of the synchronization scheme with the modified impulsive controller, using the given parameters in the original impulsive method $b_p = -0.8$ and $b_l = 0$, one obtains $ae^{\alpha\Delta} = 0.3228 < 1$ according to Corollary 2, which is simulated in Figure 3.

The effectiveness of the proposed impulsive controller can be observed from the numerical simulations. This implies that by adding the summation term in the error dynamics, one could reduce the synchronization time with the same impulsive distance. In other

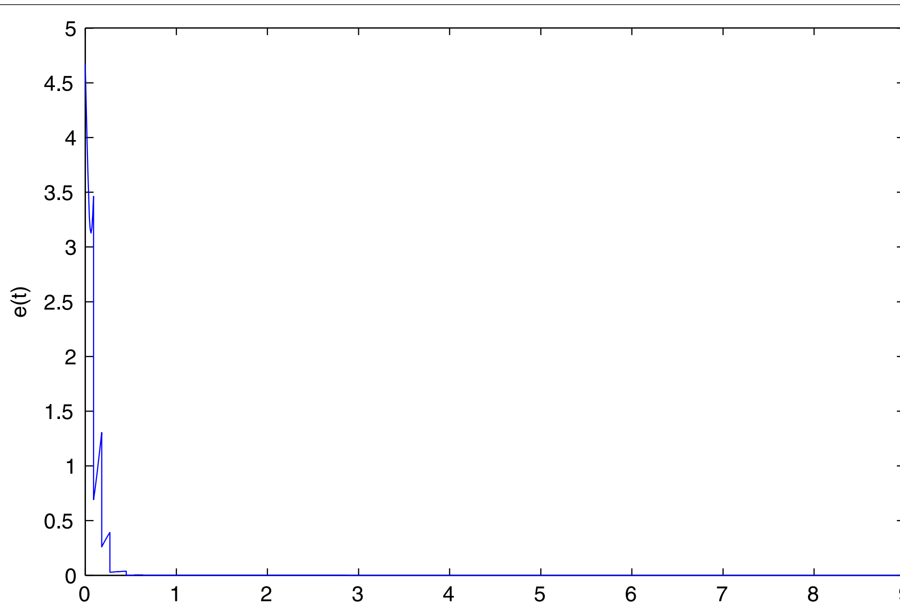

Figure 2 The error system with the impulsive interval $\Delta = 0.09$ and $N = 2$.

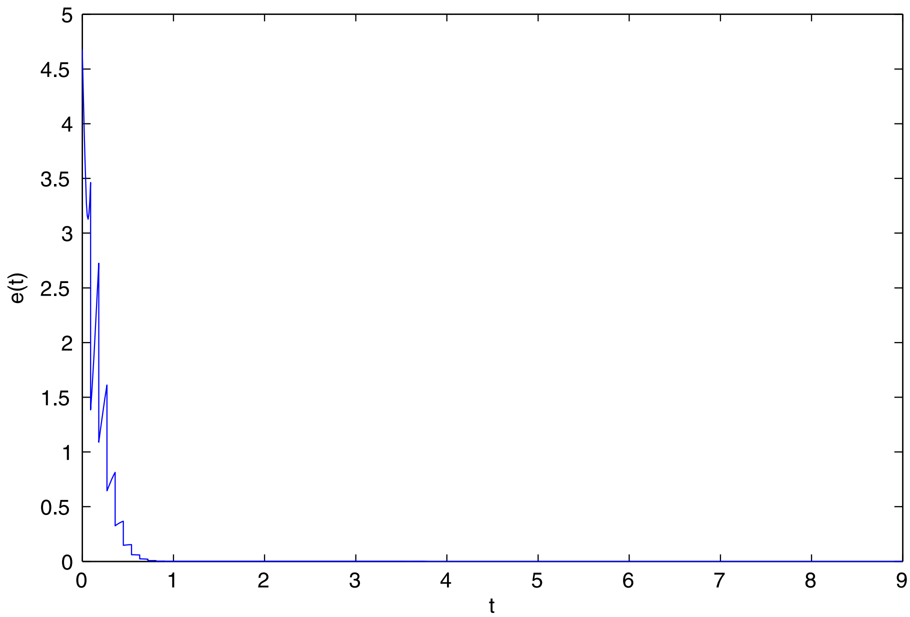

Figure 3 The error system with the impulsive interval $\Delta = 0.09$ and $N = 0$.

words, by adding the summation term in the error dynamics, one could achieve the same effect by increasing the impulse distance and reducing the control cost.

5 Conclusions

This paper is focused on the global exponential synchronization of chaotic systems with an effective modified impulsive controller. Because the modified impulsive controller is a combination of current and past error states, we establish a uniform comparison system for this case and derive a sufficient condition in Theorem 1. At the same time, a numerical example is given to illustrate the effectiveness and feasibility of the proposed methods and results. In other words, by adding the summation term in the error dynamics, one could achieve the same effect by increasing the impulse distance and reducing the control cost.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript. All authors read and approved the final manuscript.

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Acknowledgements
The authors would like to thank the referee and the associate editor for their very helpful suggestions. This work was jointly supported by the National Natural Science Foundation of China under Grants Nos. 61272530 and 11072059, and the Jiangsu Provincial Natural Science, Foundation of China under Grants No. BK2012741.

Received: 4 July 2012 Accepted: 13 January 2013 Published: 29 January 2013

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doi:10.1186/1687-1847-2013-24
Cite this article as: Feng and Cao: **Master-slave synchronization of chaotic systems with a modified impulsive controller.** *Advances in Difference Equations* 2013 **2013**:24.