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On the Ulam stability of mixed type QA mappings in IFN-spaces

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Abstract
We give Ulam-type stability results concerning the quadratic-additive functional equation in intuitionistic fuzzy normed spaces.

Keywords: *t*-norm; *t*-conorm; quadratic-additive functional equation; intuitionistic fuzzy normed space; Hyers-Ulam stability

1 Introduction

In 1940, Ulam [1] proposed the following stability problem: 'When is it true that a function which satisfies some functional equation approximately must be close to one satisfying the equation exactly?'. Hyers [2] gave the first affirmative partial answer to the question of Ulam for Banach spaces. Aoki [3] presented a generalization of Hyers results by considering additive mappings, and later on Rassias [4] did for linear mappings by considering an unbounded Cauchy difference. The paper of Rassias has significantly influenced the development of what we now call the Hyers-Ulam-Rassias stability of functional equations. Various extensions, generalizations and applications of the stability problems have been given by several authors so far; see, for example, [5–24] and references therein.

The notion of intuitionistic fuzzy set introduced by Atanassov [25] has been used extensively in many areas of mathematics and sciences. Using the idea of intuitionistic fuzzy set, Saadati and Park [26] presented the notion of intuitionistic fuzzy normed space which is a generalization of the concept of a fuzzy metric space due to Bag and Samanta [27]. The authors of [28–34] defined and studied some summability problems in the setting of an intuitionistic fuzzy normed space.

In the recent past, several Hyers-Ulam stability results concerning the various functional equations were determined in [35–46], respectively, in the fuzzy and intuitionistic fuzzy normed spaces. Quite recently, Alotaibi and Mohiuddine [47] established the stability of a cubic functional equation in random 2-normed spaces, while the notion of random 2-normed spaces was introduced by Goleţ [48] and further studied in [49–51].

The Hyers-Ulam stability problems of quadratic-additive functional equation

$$f(x+y+z)+f(x)+f(y)+f(z)=f(x+y)+f(y+z)+f(x+z)$$

under the approximately even (or odd) condition were established by Jung [52] and the solution of the above functional equation where the range is a field of characteristic 0 was determined by Kannappan [53]. In this paper we determine the stability results concerning the above functional equation in the setting of intuitionistic fuzzy normed spaces. This work indeed presents a relationship between two various disciplines: the theory of fuzzy spaces and the theory of functional equations.

2 Definitions and preliminaries

We shall assume throughout this paper that the symbol $\mathbb{N}$ denotes the set of all natural numbers.

A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is said to be a *continuous t-norm* if it satisfies the following conditions:

(a) $*$ is associative and commutative, (b) $*$ is continuous, (c) $a * 1 = a$ for all $a \in [0,1]$, (d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0,1]$.

A binary operation $\diamondsuit : [0,1] \times [0,1] \to [0,1]$ is said to be a *continuous t-conorm* if it satisfies the following conditions:

(a′) $\diamondsuit$ is associative and commutative, (b′) $\diamondsuit$ is continuous, (c′) $a \diamondsuit 0 = a$ for all $a \in [0,1]$, (d′) $a \diamondsuit b \leq c \diamondsuit d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0,1]$.

The five-tuple $(X, \mu, \nu, *, \diamondsuit)$ is said to be *intuitionistic fuzzy normed spaces* (for short, IFN-spaces) [26] if X is a vector space, $*$ is a continuous t-norm, $\diamondsuit$ is a continuous t-conorm, and μ, ν are fuzzy sets on $X \times (0, \infty)$ satisfying the following conditions. For every $x, y \in X$ and $s, t > 0$,

(i) $\mu(x,t) + \nu(x,t) \leq 1$,
(ii) $\mu(x,t) > 0$,
(iii) $\mu(x,t) = 1$ if and only if $x = 0$,
(iv) $\mu(\alpha x, t) = \mu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$,
(v) $\mu(x,t) * \mu(y,s) \leq \mu(x+y, t+s)$,
(vi) $\mu(x, \cdot) : (0,\infty) \to [0,1]$ is continuous,
(vii) $\lim_{t\to\infty} \mu(x,t) = 1$ and $\lim_{t\to 0} \mu(x,t) = 0$,
(viii) $\nu(x,t) < 1$,
(ix) $\nu(x,t) = 0$ if and only if $x = 0$,
(x) $\nu(\alpha x, t) = \nu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$,
(xi) $\nu(x,t) \diamondsuit \nu(y,s) \geq \nu(x+y, t+s)$,
(xii) $\nu(x, \cdot) : (0,\infty) \to [0,1]$ is continuous,
(xiii) $\lim_{t\to\infty} \nu(x,t) = 0$ and $\lim_{t\to 0} \nu(x,t) = 1$.

In this case (μ, ν) is called an *intuitionistic fuzzy norm*. For simplicity in notation, we denote the intuitionistic fuzzy normed spaces by (X, μ, ν) instead of $(X, \mu, \nu, *, \diamondsuit)$. For example, let $(X, \|\cdot\|)$ be a normed space, and let $a * b = ab$ and $a \diamondsuit b = \min\{a+b, 1\}$ for all $a, b \in [0,1]$. For all $x \in X$ and every $t > 0$, consider

$$\mu(x,t) := \frac{t}{t + \|x\|} \quad \text{and} \quad \nu(x,t) := \frac{\|x\|}{t + \|x\|}.$$

Then (X, μ, ν) is an intuitionistic fuzzy normed space.

The notions of convergence and Cauchy sequence in the setting of IFN-spaces were introduced by Saadati and Park [26] and further studied by Mursaleen and Mohiuddine [30].

Let (X, μ, ν) be an intuitionistic fuzzy normed space. Then the sequence $x = (x_k)$ is said to be:

(i) *Convergent* to $L \in X$ with respect to the intuitionistic fuzzy norm (μ,ν) if, for every $\epsilon > 0$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\mu(x_k - L, t) > 1 - \epsilon$ and $\nu(x_k - L, t) < \epsilon$ for all $k \geq k_0$. In this case, we write $(\mu,\nu)\text{-}\lim x_k = L$ or $x_k \xrightarrow{(\mu,\nu)} L$ as $k \to \infty$.

(ii) *Cauchy sequence* with respect to the intuitionistic fuzzy norm (μ,ν) if, for every $\epsilon > 0$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\mu(x_k - x_\ell, t) > 1 - \epsilon$ and $\nu(x_k - x_\ell, t) < \epsilon$ for all $k, \ell \geq k_0$. An IFN-space (X,μ,ν) is said to be *complete* if every Cauchy sequence in (X,μ,ν) is convergent in the IFN-space. In this case, (X,μ,ν) is called an *intuitionistic fuzzy Banach space*.

3 Stability of a quadratic-additive functional equation in the IFN-space

We shall assume the following abbreviation throughout this paper:

$$Df(x,y,z) = f(x+y+z) - f(x+y) - f(y+z) - f(x+z) + f(x) + f(y) + f(z).$$

Theorem 3.1 *Let X be a linear space and (X,μ,ν) be an IFN-space. Suppose that f is an intuitionistic fuzzy q-almost quadratic-additive mapping from (X,μ,ν) to an intuitionistic fuzzy Banach space (Y,μ',ν') such that*

$$\left.\begin{array}{l} \mu'(Df(x,y,z), s+t+u) \geq \mu(x,s^q) * \mu(y,t^q) * \mu(z,u^q) \quad \textit{and} \\ \nu'(Df(x,y,z), s+t+u) \leq \nu(x,s^q) \diamond \nu(y,t^q) \diamond \nu(z,u^q) \end{array}\right\} \tag{3.1}$$

for all $x,y,z \in X$ and $s,t,u > 0$, where q is a positive real number with $q \neq \frac{1}{2}, 1$. Then there exists a unique quadratic-additive mapping $T : X \to Y$ such that

$$\left.\begin{array}{l} \mu'(T(x) - f(x), t) \geq \begin{cases} \sup_{t'<t} \mu(x, (\frac{2-2^p}{3})^q t'^q) & \textit{if } q > 1, \\ \sup_{t'<t} \mu(x, (\frac{(4-2^p)(2-2^p)}{6})^q t'^q) & \textit{if } \frac{1}{2} < q < 1, \\ \sup_{t'<t} \mu(x, (\frac{2^p-4}{3})^q t'^q) & \textit{if } 0 < q < \frac{1}{2} \end{cases} \\ \textit{and} \\ \nu'(T(x) - f(x), t) \leq \begin{cases} \sup_{t'<t} \nu(x, (\frac{2-2^p}{3})^q t'^q) & \textit{if } q > 1, \\ \sup_{t'<t} \nu(x, (\frac{(4-2^p)(2-2^p)}{6})^q t'^q) & \textit{if } \frac{1}{2} < q < 1, \\ \sup_{t'<t} \nu(x, (\frac{2^p-4}{3})^q t'^q) & \textit{if } 0 < q < \frac{1}{2}, \end{cases} \end{array}\right\} \tag{3.2}$$

for all $x \in X$ and all $t > 0$ with $t' \in (0,t)$, where $p = 1/q$.

Proof Putting $x = 0 = y = z$ in (3.1), it follows that

$$\mu'\bigl(f(0), t\bigr) \geq \mu\bigl(0, (t/3)^q\bigr) * \mu\bigl(0, (t/3)^q\bigr) * \mu\bigl(0, (t/3)^q\bigr) = 1$$

and

$$\nu'\bigl(f(0), t\bigr) \leq \nu\bigl(0, (t/3)^q\bigr) \diamond \nu\bigl(0, (t/3)^q\bigr) \diamond \nu\bigl(0, (t/3)^q\bigr) = 0$$

for all $t > 0$. Using the definition of IFN-space, we have $f(0) = 0$. Now we are ready to prove our theorem for three cases. We consider the cases as $q > 1$, $\frac{1}{2} < q < 1$ and $0 < q < \frac{1}{2}$.

Case 1. Let $q > 1$. Consider a mapping $J_n f : X \to Y$ to be such that

$$J_n f(x) = \frac{1}{2}\bigl(4^{-n}\bigl(f(2^n x) + f(-2^n x)\bigr) + 2^{-n}\bigl(f(2^n x) - f(-2^n x)\bigr)\bigr)$$

for all $x \in X$. Notice that $J_0 f(x) = f(x)$ and

$$\begin{aligned} J_j f(x) - J_{j+1} f(x) &= \frac{Df(2^j x, 2^j x, -2^j x)}{2 \cdot 4^{j+1}} + \frac{Df(-2^j x, -2^j x, 2^j x)}{2 \cdot 4^{j+1}} \\ &\quad + \frac{Df(2^j x, 2^j x, -2^j x)}{2^{j+2}} - \frac{Df(-2^j x, -2^j x, 2^j x)}{2^{j+2}} \end{aligned} \tag{3.3}$$

for all $x \in X$ and $j \geq 0$. Using the definition of IFN-space and (3.1), this equation implies that if $n + m > m \geq 0$, then

$$\begin{aligned} &\mu'\left(J_m f(x) - J_{n+m} f(x), \sum_{j=m}^{n+m-1} \frac{3}{2}\left(\frac{2^p}{2}\right)^j t^p\right) \\ &= \mu'\left(\sum_{j=m}^{n+m-1} \bigl(J_j f(x) - J_{j+1} f(x)\bigr), \sum_{j=m}^{n+m-1} \frac{3 \cdot 2^{jp}}{2^{j+1}} t^p\right) \\ &\geq \prod_{j=m}^{n+m-1} \mu'\left(J_j\bigl(f(x) - J_{j+1} f(x)\bigr), \frac{3 \cdot 2^{jp}}{2^{j+1}}\right) \\ &\geq \prod_{j=m}^{n+m-1} \left\{\mu'\left(\frac{(2^{j+1}+1) Df(2^j x, 2^j x, -2^j x)}{2 \cdot 4^{j+1}}, \frac{3(2^{j+1}+1) 2^{jp} t^p}{2 \cdot 4^{j+1}}\right)\right. \\ &\quad \left. * \mu'\left(\frac{1 - (2^{j+1}) Df(-2^j x, -2^j x, 2^j x)}{2 \cdot 4^{j+1}}, \frac{3(2^{j+1}-1) 2^{jp} t^p}{2 \cdot 4^{j+1}}\right)\right\} \\ &\geq \prod_{j=m}^{n+m-1} \mu\bigl(2^j x, 2^j t\bigr) = \mu(x, t) \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} &\nu'\left(J_m f(x) - J_{n+m} f(x), \sum_{j=m}^{n+m-1} \frac{3}{2}\left(\frac{2^p}{2}\right)^j t^p\right) \\ &= \nu'\left(\sum_{j=m}^{n+m-1} \bigl(J_j f(x) - J_{j+1} f(x)\bigr), \sum_{j=m}^{n+m-1} \frac{3 \cdot 2^{jp}}{2^{j+1}} t^p\right) \\ &\leq \coprod_{j=m}^{n+m-1} \nu'\left(J_j\bigl(f(x) - J_{j+1} f(x)\bigr), \frac{3 \cdot 2^{jp}}{2^{j+1}}\right) \\ &\leq \coprod_{j=m}^{n+m-1} \left\{\nu'\left(\frac{(2^{j+1}+1) Df(2^j x, 2^j x, -2^j x)}{2 \cdot 4^{j+1}}, \frac{3(2^{j+1}+1) 2^{jp} t^p}{2 \cdot 4^{j+1}}\right)\right. \\ &\quad \left. \diamond \nu'\left(\frac{1 - (2^{j+1}) Df(-2^j x, -2^j x, 2^j x)}{2 \cdot 4^{j+1}}, \frac{3(2^{j+1}-1) 2^{jp} t^p}{2 \cdot 4^{j+1}}\right)\right\} \\ &\leq \coprod_{j=m}^{n+m-1} \nu\bigl(2^j x, 2^j t\bigr) = \nu(x, t) \end{aligned} \tag{3.5}$$

for all $x \in X$ and $t > 0$, where $\prod_{j=1}^{n} a_j = a_1 * a_2 * \cdots * a_n$, $\coprod_{j=1}^{n} a_j = a_1 \diamondsuit a_2 \diamondsuit \cdots \diamondsuit a_n$. Let $\epsilon > 0$ and $\delta > 0$ be given. Since $\lim_{t\to\infty} \mu(x,t) = 1$ and $\lim_{t\to\infty} \nu(x,t) = 0$, there exists $t_0 > 0$ such that $\mu(x,t_0) \geq 1-\epsilon$ and $\nu(x,t_0) \leq \epsilon$ for all $x \in X$. We observe that for some $\tilde{t} > t_0$, the series $\sum_{j=0}^{\infty} \frac{3\cdot 2^{jp}}{2^{j+1}} \tilde{t}^p$ converges for $p = \frac{1}{q} < 1$, there exists some $n_0 \geq 0$ such that $\sum_{j=m}^{n+m-1} \frac{3\cdot 2^{jp}}{2^{j+1}} \tilde{t}^p < \delta$ for each $m \geq n_0$ and $n > 0$. Using (3.4) and (3.5), we have

$$\begin{aligned} \mu'\big(J_m f(x) - J_{n+m} f(x), \delta\big) &\geq \mu'\left(J_m f(x) - J_{n+m} f(x), \sum_{j=m}^{n+m-1} \frac{3\cdot 2^{jp}}{2^{j+1}} \tilde{t}^p\right) \\ &\geq \mu(x,\tilde{t}) \geq \mu(x,t_0) \geq 1-\epsilon \end{aligned}$$

and

$$\nu'\big(J_m f(x) - J_{n+m} f(x), \delta\big) \leq \nu'\left(J_m f(x) - J_{n+m} f(x), \sum_{j=m}^{n+m-1} \frac{3\cdot 2^{jp}}{2^{j+1}} \tilde{t}^p\right) \leq \nu(x,\tilde{t}) \leq \nu(x,t_0) \leq \epsilon$$

for all $x \in X$ and $\delta > 0$. Hence $\{J_n f(x)\}$ is a Cauchy sequence in the fuzzy Banach space (Y,μ',ν'). Thus, we define a mapping $T : X \to Y$ such that $T(x) := (\mu',\nu') - \lim_{n\to\infty} J_n f(x)$ for all $x \in X$. Moreover, if we put $m = 0$ in (3.4) and (3.5), we get

$$\left.\begin{aligned} &\mu'(f(x) - J_n f(x), t) \geq \mu\Big(x, \frac{t^q}{(\sum_{j=0}^{n-1} \frac{3\cdot 2^{jp}}{2^{j+1}})^q}\Big) \quad \text{and} \\ &\nu'(f(x) - J_n f(x), t) \leq \nu\Big(x, \frac{t^q}{(\sum_{j=0}^{n-1} \frac{3\cdot 2^{jp}}{2^{j+1}})^q}\Big) \end{aligned}\right\} \tag{3.6}$$

for all $x \in X$ and $t > 0$. Now we have to show that T is quadratic additive. Let $x, y, z \in X$. Then

$$\begin{aligned} \mu'\big(DT(x,y,z), t\big) \geq{}& \mu'\left((T - J_n f)(x+y+z), \frac{t}{28}\right) * \mu'\left((T - J_n f)(x), \frac{t}{28}\right) \\ &* \mu'\left((T - J_n f)(y), \frac{t}{28}\right) * \mu'\left((T - J_n f)(z), \frac{t}{28}\right) \\ &* \mu'\left((J_n f - T)(x+y), \frac{t}{28}\right) * \mu'\left((J_n f - T)(x+z), \frac{t}{28}\right) \\ &* \mu'\left((J_n f - T)(y+z), \frac{t}{28}\right) * \mu'\left(DJ_n f(x,y,z), \frac{3t}{4}\right) \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \nu'\big(DT(x,y,z), t\big) \leq{}& \nu'\left((T - J_n f)(x+y+z), \frac{t}{28}\right) \diamondsuit \nu'\left((T - J_n f)(x), \frac{t}{28}\right) \\ &\diamondsuit \nu'\left((T - J_n f)(y), \frac{t}{28}\right) \diamondsuit \nu'\left((T - J_n f)(z), \frac{t}{28}\right) \\ &\diamondsuit \nu'\left((J_n f - T)(x+y), \frac{t}{28}\right) \diamondsuit \nu'\left((J_n f - T)(x+z), \frac{t}{28}\right) \\ &\diamondsuit \nu'\left((J_n f - T)(y+z), \frac{t}{28}\right) \diamondsuit \nu'\left(DJ_n f(x,y,z), \frac{3t}{4}\right) \end{aligned} \tag{3.8}$$

for all $t > 0$ and $n \in \mathbb{N}$. Taking the limit as $n \to \infty$ in the inequalities (3.7) and (3.8), we can see that first seven terms on the right-hand side of (3.7) and (3.8) tend to 1 and 0, respectively, by using the definition of T. It is left to find the value of the last term on the right-hand side of (3.7) and (3.8). By using the definition of $J_n f(x)$, write

$$\begin{aligned}
&\mu'\left(DJ_n f(x,y,z), \frac{3t}{4}\right) \\
&\quad \geq \mu'\left(\frac{Df(2^n x, 2^n y, 2^n z)}{2\cdot 4^n}, \frac{3t}{16}\right) * \mu'\left(\frac{Df(-2^n x, -2^n y, -2^n z)}{2\cdot 4^n}, \frac{3t}{16}\right) \\
&\qquad * \mu'\left(\frac{Df(2^n x, 2^n y, 2^n z)}{2\cdot 2^n}, \frac{3t}{16}\right) * \mu'\left(\frac{Df(-2^n x, -2^n y, -2^n z)}{2\cdot 2^n}, \frac{3t}{16}\right)
\end{aligned} \tag{3.9}$$

and, similarly,

$$\begin{aligned}
&\nu'\left(DJ_n f(x,y,z), \frac{3t}{4}\right) \\
&\quad \leq \nu'\left(\frac{Df(2^n x, 2^n y, 2^n z)}{2\cdot 4^n}, \frac{3t}{16}\right) \diamond \nu'\left(\frac{Df(-2^n x, -2^n y, -2^n z)}{2\cdot 4^n}, \frac{3t}{16}\right) \\
&\qquad \diamond \nu'\left(\frac{Df(2^n x, 2^n y, 2^n z)}{2\cdot 2^n}, \frac{3t}{16}\right) \diamond \nu'\left(\frac{Df(-2^n x, -2^n y, -2^n z)}{2\cdot 2^n}, \frac{3t}{16}\right)
\end{aligned} \tag{3.10}$$

for all $x, y, z \in X$, $t > 0$ and $n \in \mathbb{N}$. Also, from (3.1), we have

$$\begin{aligned}
&\mu'\left(\frac{Df(\pm 2^n x, \pm 2^n y, \pm 2^n z)}{2\cdot 4^n}, \frac{3t}{16}\right) \\
&\quad = \mu'\left(Df\left(\pm 2^n x, \pm 2^n y, \pm 2^n z\right), \frac{3\cdot 4^n t}{8}\right) \\
&\quad \geq \mu\left(2^n x, \left(\frac{4^n t}{8}\right)^q\right) * \mu\left(2^n y, \left(\frac{4^n t}{8}\right)^q\right) * \mu\left(2^n z, \left(\frac{4^n t}{8}\right)^q\right) \\
&\quad \geq \mu\left(x, 2^{(2q-1)n-3q} t^q\right) * \mu\left(y, 2^{(2q-1)n-3q} t^q\right) * \mu\left(z, 2^{(2q-1)n-3q} t^q\right)
\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}
&\mu'\left(\frac{Df(\pm 2^n x, \pm 2^n y, \pm 2^n z)}{2\cdot 2^n}, \frac{3t}{16}\right) \\
&\quad \geq \mu\left(x, 2^{(2q-1)n-3q} t^q\right) * \mu\left(y, 2^{(2q-1)n-3q} t^q\right) * \mu\left(z, 2^{(2q-1)n-3q} t^q\right)
\end{aligned} \tag{3.12}$$

for all $x, y, z \in X$, $t > 0$ and $n \in \mathbb{N}$. Since $q > 1$, therefore (3.9) tends to 1 as $n \to \infty$ with the help of (3.11) and (3.12). Similarly, by proceeding along the same lines as in (3.11) and (3.12), we can show that (3.10) tends to 0 as $n \to \infty$. Thus, inequalities (3.7) and (3.8) become

$$\mu'\left(DT(x,y,z), t\right) = 1 \quad \text{and} \quad \nu'\left(DT(x,y,z), t\right) = 0$$

for all $x, y, z \in X$ and $t > 0$. Accordingly, $DT(x,y,z) = 0$ for all $x, y, z \in X$. Now we approximate the difference between f and T in a fuzzy sense. Choose $\epsilon \in (0,1)$ and $0 < t' < t$. Since

T is the intuitionistic fuzzy limit of $\{J_nf(x)\}$ such that

$$\mu'\big(T(x)-J_nf(x),t-t'\big)\geq 1-\epsilon \quad \text{and} \quad \nu'\big(T(x)-J_nf(x),t-t'\big)\leq \epsilon$$

for all $x\in X$, $t>0$ and $n\in\mathbb{N}$. From (3.6), we have

$$\begin{aligned}\mu'\big(T(x)-f(x),t\big) &\geq \mu'\big(T(x)-J_nf(x),t-t'\big)*\mu'\big(J_nf(x)-f(x),t'\big)\\ &\geq (1-\epsilon)*\mu\left(x,\frac{t'^q}{(\sum_{j=0}^{n-1}\frac{3\cdot 2^{jp}}{2^{j+1}})^q}\right)\geq (1-\epsilon)*\mu\left(x,\left(\frac{(2-2^p)t'}{3}\right)^q\right)\end{aligned}$$

and

$$\begin{aligned}\nu'\big(T(x)-f(x),t\big) &\leq \nu'\big(T(x)-J_nf(x),t-t'\big)\diamondsuit\nu'\big(J_nf(x)-f(x),t'\big)\\ &\leq (1-\epsilon)\diamondsuit\nu\left(x,\left(\frac{(2-2^p)t'}{3}\right)^q\right).\end{aligned}$$

Since $\epsilon\in(0,1)$ is arbitrary, we get the inequality (3.2) in this case.

To prove the uniqueness of T, assume that T' is another quadratic-additive mapping from X into Y, which satisfies the required inequality, *i.e.*, (3.2). Then, by (3.3), for all $x\in X$ and $n\in\mathbb{N}$,

$$\left.\begin{aligned}T(x)-J_nT(x)&=\textstyle\sum_{j=0}^{n-1}(J_jT(x)-J_{j+1}T(x))=0,\\ T'(x)-J_nT'(x)&=\textstyle\sum_{j=0}^{n-1}(J_jT'(x)-J_{j+1}T'(x))=0.\end{aligned}\right\} \tag{3.13}$$

Therefore

$$\begin{aligned}\mu'\big(T(x)-T'(x),t\big) &= \mu'\big(J_nT(x)-J_nT'(x),t\big)\\ &\geq \mu'\left(J_nT(x)-J_nf(x),\frac{t}{2}\right)*\mu'\left(J_nf(x)-J_nT'(x),\frac{t}{2}\right)\\ &\geq \mu'\left(\frac{(T-f)(2^nx)}{2\cdot 4^n},\frac{t}{8}\right)*\mu'\left(\frac{(f-T')(2^nx)}{2\cdot 4^n},\frac{t}{8}\right)\\ &\quad *\mu'\left(\frac{(T-f)(-2^nx)}{2\cdot 4^n},\frac{t}{8}\right)*\mu'\left(\frac{(f-T')(-2^nx)}{2\cdot 4^n},\frac{t}{8}\right)\\ &\quad *\mu'\left(\frac{(T-f)(2^nx)}{2\cdot 2^n},\frac{t}{8}\right)*\mu'\left(\frac{(f-T')(2^nx)}{2\cdot 2^n},\frac{t}{8}\right)\\ &\quad *\mu'\left(\frac{(T-f)(-2^nx)}{2\cdot 2^n},\frac{t}{8}\right)*\mu'\left(\frac{(f-T')(-2^nx)}{2\cdot 2^n},\frac{t}{8}\right)\\ &\geq \sup_{t'<t}\mu\left(x,2^{(q-1)n-2q}\left(\frac{2-2^p}{3}\right)^q t'^q\right)\end{aligned}$$

and

$$\begin{aligned}\nu'\big(T(x)-T'(x),t\big) &= \nu'\big(J_nT(x)-J_nT'(x),t\big)\\ &\leq \nu'\left(J_nT(x)-J_nf(x),\frac{t}{2}\right)\diamondsuit\nu'\left(J_nf(x)-J_nT'(x),\frac{t}{2}\right)\end{aligned}$$

$$\begin{aligned}
&\leq \nu'\left(\frac{(T-f)(2^n x)}{2\cdot 4^n}, \frac{t}{8}\right) \diamond \nu'\left(\frac{(f-T')(2^n x)}{2\cdot 4^n}, \frac{t}{8}\right) \\
&\quad \diamond \nu'\left(\frac{(T-f)(-2^n x)}{2\cdot 4^n}, \frac{t}{8}\right) \diamond \nu'\left(\frac{(f-T')(-2^n x)}{2\cdot 4^n}, \frac{t}{8}\right) \\
&\quad \diamond \nu'\left(\frac{(T-f)(2^n x)}{2\cdot 2^n}, \frac{t}{8}\right) \diamond \nu'\left(\frac{(f-T')(2^n x)}{2\cdot 2^n}, \frac{t}{8}\right) \\
&\quad \diamond \nu'\left(\frac{(T-f)(-2^n x)}{2\cdot 2^n}, \frac{t}{8}\right) \diamond \nu'\left(\frac{(f-T')(-2^n x)}{2\cdot 2^n}, \frac{t}{8}\right) \\
&\leq \sup_{t'<t} \nu\left(x, 2^{(q-1)n-2q}\left(\frac{2-2^p}{3}\right)^q t'^q\right)
\end{aligned}$$

for all $x \in X$, $t > 0$ and $n \in \mathbb{N}$. Since $q = 1/p > 1$ and taking limit as $n \to \infty$ in the last two inequalities, we get $\mu'(T(x) - T'(x), t) = 1$ and $\nu'(T(x) - T'(x), t) = 0$ for all $x \in X$ and $t > 0$. Hence $T(x) = T'(x)$ for all $x \in X$.

Case 2. Let $\frac{1}{2} < q < 1$. Consider a mapping $J_n f : X \to Y$ to be such that

$$J_n f(x) = \frac{1}{2}\left(4^{-n}\left(f\left(2^n x\right) + f\left(-2^n x\right)\right) + 2^n\left(f\left(\frac{x}{2^n}\right) - f\left(-\frac{x}{2^n}\right)\right)\right)$$

for all $x \in X$. Then $J_0 f(x) = f(x)$ and

$$\begin{aligned}
J_j f(x) - J_{j+1} f(x) &= \frac{Df(-2^j x, -2^j x, 2^j x)}{2\cdot 4^{j+1}} + \frac{Df(2^j x, 2^j x, -2^j x)}{2\cdot 4^{j+1}} \\
&\quad - 2^{j-1}\left(Df\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, \frac{-x}{2^{j+1}}\right) - Df\left(\frac{-x}{2^{j+1}}, \frac{-x}{2^{j+1}}, \frac{x}{2^{j+1}}\right)\right)
\end{aligned}$$

for all $x \in X$ and $j \geq 0$. Thus, for each $n + m > m \geq 0$, we have

$$\begin{aligned}
&\mu'\left(J_m f(x) - J_{n+m} f(x), \sum_{j=m}^{n+m-1}\left(\frac{3}{4}\left(\frac{2^p}{4}\right)^j + \frac{3}{2^p}\left(\frac{2}{2^p}\right)^j\right) t^p\right) \\
&\quad \geq \prod_{j=m}^{n+m-1}\left\{\mu'\left(\frac{Df(2^j x, 2^j x, -2^j x)}{2\cdot 4^{j+1}}, \frac{3\cdot 2^{jp} t^p}{2\cdot 4^{j+1}}\right) * \mu'\left(\frac{Df(-2^j x, -2^j x, 2^j x)}{2\cdot 4^{j+1}}, \frac{3\cdot 2^{jp} t^p}{2\cdot 4^{(j+1)}}\right)\right. \\
&\qquad * \mu'\left(-2^{j-1} Df\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, \frac{-x}{2^{j+1}}\right), \frac{3\cdot 2^{j-1} t^p}{2^{(j+1)p}}\right) \\
&\qquad \left.* \mu'\left(2^{j-1} Df\left(\frac{-x}{2^{j+1}}, \frac{-x}{2^{j+1}}, \frac{x}{2^{j+1}}\right), \frac{3\cdot 2^{j-1} t^p}{2^{(j+1)p}}\right)\right\} \\
&\quad \geq \prod_{j=m}^{n+m-1}\left\{\mu\left(2^j x, 2^j t\right) * \mu\left(\frac{x}{2^{j+1}}, \frac{t}{2^{j+1}}\right)\right\} = \mu(x, t) \quad \text{and} \\
&\nu'\left(J_m f(x) - J_{n+m} f(x), \sum_{j=m}^{n+m-1}\left(\frac{3}{4}\left(\frac{2^p}{4}\right)^j + \frac{3}{2^p}\left(\frac{2}{2^p}\right)^j\right) t^p\right) \\
&\quad \leq \coprod_{j=m}^{n+m-1}\left\{\nu'\left(\frac{Df(2^j x, 2^j x, -2^j x)}{2\cdot 4^{j+1}}, \frac{3\cdot 2^{jp} t^p}{2\cdot 4^{j+1}}\right) \diamond \nu'\left(\frac{Df(-2^j x, -2^j x, 2^j x)}{2\cdot 4^{j+1}}, \frac{3\cdot 2^{jp} t^p}{2\cdot 4^{(j+1)}}\right)\right. \\
&\qquad \diamond \nu'\left(-2^{j-1} Df\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, \frac{-x}{2^{j+1}}\right), \frac{3\cdot 2^{j-1} t^p}{2^{(j+1)p}}\right)
\end{aligned}$$

$$
\begin{aligned}
&\diamond \nu'\left(2^{j-1}Df\left(\frac{-x}{2^{j+1}}, \frac{-x}{2^{j+1}}, \frac{x}{2^{j+1}}\right), \frac{3 \cdot 2^{j-1}t^p}{2^{(j+1)^p}}\right)\Bigg\} \\
&\leq \coprod_{j=m}^{n+m-1}\left\{\nu\left(2^j x, 2^j t\right) \diamond \nu\left(\frac{x}{2^{j+1}}, \frac{t}{2^{j+1}}\right)\right\} = \nu(x,t),
\end{aligned}
$$

where $\prod$ and $\coprod$ are the same as in Case 1. Proceeding along a similar argument as in Case 1, we see that $\{J_n f(x)\}$ is a Cauchy sequence in (Y, μ', ν'). Thus, we define $T(x) := (\mu', \nu')\text{-}\lim_{n\to\infty} J_n f(x)$ for all $x \in X$. Putting $m = 0$ in the last two inequalities, we get

$$
\left.
\begin{aligned}
&\mu'(f(x) - J_n f(x), t) \geq \mu\Big(x, \frac{t^p}{(\sum_{j=0}^{n-1}(\frac{3}{4}(\frac{2^p}{4})^j + \frac{3}{2^p}(\frac{2}{2^p})^j))^q}\Big) \quad \text{and} \\
&\nu'(f(x) - J_n f(x), t) \leq \nu\Big(x, \frac{t^p}{(\sum_{j=0}^{n-1}(\frac{3}{4}(\frac{2^p}{4})^j + \frac{3}{2^p}(\frac{2}{2^p})^j))^q}\Big)
\end{aligned}
\right\}
\tag{3.14}
$$

for all $x \in X$ and $t > 0$. To prove that t is a quadratic-additive function, it is enough to show that the last term on the right-hand side of (3.7) and (3.8) tends to 1 and 0, respectively, as $n \to \infty$. Using the definition of $J_n f(x)$ and (3.1), we obtain

$$
\begin{aligned}
&\mu'\left(DJ_n f(x,y,z), \frac{3t}{4}\right) \\
&\quad\geq \mu'\left(\frac{Df(2^n x, 2^n y, 2^n z)}{2 \cdot 4^n}, \frac{3t}{16}\right) * \mu'\left(\frac{Df(-2^n x, -2^n y, -2^n z)}{2 \cdot 4^n}, \frac{3t}{16}\right) \\
&\qquad * \mu'\left(2^{n-1}Df\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right), \frac{3t}{16}\right) * \mu'\left(2^{n-1}Df\left(\frac{-x}{2^n}, \frac{-y}{2^n}, \frac{-z}{2^n}\right), \frac{3t}{16}\right) \\
&\quad\geq \mu\left(x, 2^{(2q-1)n-3q}t^q\right) * \mu\left(y, 2^{(2q-1)n-3q}t^q\right) * \mu\left(z, 2^{(2q-1)n-3q}t^q\right) \\
&\qquad * \mu\left(x, 2^{(1-q)n-3q}t^q\right) * \mu\left(y, 2^{(1-q)n-3q}t^q\right) * \mu\left(z, 2^{(1-q)n-3q}t^q\right)
\end{aligned}
\tag{3.15}
$$

and

$$
\begin{aligned}
&\nu'\left(DJ_n f(x,y,z), \frac{3t}{4}\right) \\
&\quad\leq \nu'\left(\frac{Df(2^n x, 2^n y, 2^n z)}{2 \cdot 4^n}, \frac{3t}{16}\right) \diamond \nu'\left(\frac{Df(-2^n x, -2^n y, -2^n z)}{2 \cdot 4^n}, \frac{3t}{16}\right) \\
&\qquad \diamond \nu'\left(2^{n-1}Df\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right), \frac{3t}{16}\right) \diamond \nu'\left(2^{n-1}Df\left(\frac{-x}{2^n}, \frac{-y}{2^n}, \frac{-z}{2^n}\right), \frac{3t}{16}\right) \\
&\quad\leq \nu\left(x, 2^{(2q-1)n-3q}t^q\right) \diamond \nu\left(y, 2^{(2q-1)n-3q}t^q\right) \diamond \nu\left(z, 2^{(2q-1)n-3q}t^q\right) \\
&\qquad \diamond \nu\left(x, 2^{(1-q)n-3q}t^q\right) \diamond \nu\left(y, 2^{(1-q)n-3q}t^q\right) \diamond \nu\left(z, 2^{(1-q)n-3q}t^q\right)
\end{aligned}
\tag{3.16}
$$

for each $x, y, z \in X$, $t > 0$ and $n \in \mathbb{N}$. Since $1/2 < q < 1$ and taking the limit as $n \to \infty$, we see that (3.15) and (3.16) tend to 1 and 0, respectively. As in Case 1, we have $DT(x,y,z) = 0$ for all $x, y, z \in X$. Using the same argument as in Case 1, we see that (3.2) follows from (3.14). To prove the uniqueness of T, assume that T' is another quadratic-additive mapping from X into Y satisfying (3.2). Using (3.2) and (3.13), we have

$$
\begin{aligned}
\mu'\left(T(x) - T'(x), t\right) &= \mu'\left(J_n T(x) - J_n T'(x), t\right) \\
&\geq \mu'\left(J_n T(x) - J_n f(x), \frac{t}{2}\right) * \mu'\left(J_n f(x) - J_n T'(x), \frac{t}{2}\right)
\end{aligned}
$$

$$\begin{aligned}
&\geq \mu'\left(\frac{(T-f)(2^n x)}{2\cdot 4^n}, \frac{t}{8}\right) * \mu'\left(\frac{(f-T')(2^n x)}{2\cdot 4^n}, \frac{t}{8}\right) \\
&\quad * \mu'\left(\frac{(T-f)(-2^n x)}{2\cdot 4^n}, \frac{t}{8}\right) * \mu'\left(\frac{(f-T')(-2^n x)}{2\cdot 4^n}, \frac{t}{8}\right) \\
&\quad * \mu'\left(2^{n-1}\left((T-f)\left(\frac{x}{2^n}\right)\right), \frac{t}{8}\right) * \mu'\left(2^{n-1}\left((f-T')\left(\frac{x}{2^n}\right)\right), \frac{t}{8}\right) \\
&\quad * \mu'\left(2^{n-1}\left((T-f)\left(\frac{-x}{2^n}\right)\right), \frac{t}{8}\right) * \mu'\left(2^{n-1}\left((f-T')\left(\frac{-x}{2^n}\right)\right), \frac{t}{8}\right) \\
&\geq \sup_{t'<t} \mu\left(x, 2^{(2q-1)n-2q}\left(\frac{(4-2^p)(2^p-2)}{6}\right)^q t'^q\right) \\
&\quad * \sup_{t'<t} \mu\left(x, 2^{2(1-q)n-2q}\left(\frac{(4-2^p)(2^p-2)}{6}\right)^q t'^q\right) \qquad (3.17)
\end{aligned}$$

and

$$\begin{aligned}
\nu'\big(T(x)-T'(x), t\big) &\leq \nu'\left(J_n T(x) - J_n f(x), \frac{t}{2}\right) \diamond \nu'\left(J_n f(x) - J_n T'(x), \frac{t}{2}\right) \\
&\leq \nu'\left(\frac{(T-f)(2^n x)}{2\cdot 4^n}, \frac{t}{8}\right) \diamond \nu'\left(\frac{(f-T')(2^n x)}{2\cdot 4^n}, \frac{t}{8}\right) \\
&\quad \diamond \nu'\left(\frac{(T-f)(-2^n x)}{2\cdot 4^n}, \frac{t}{8}\right) \diamond \nu'\left(\frac{(f-T')(-2^n x)}{2\cdot 4^n}, \frac{t}{8}\right) \\
&\quad \diamond \nu'\left(2^{n-1}\left((T-f)\left(\frac{x}{2^n}\right)\right), \frac{t}{8}\right) \diamond \nu'\left(2^{n-1}\left((f-T')\left(\frac{x}{2^n}\right)\right), \frac{t}{8}\right) \\
&\quad \diamond \nu'\left(2^{n-1}\left((T-f)\left(\frac{-x}{2^n}\right)\right), \frac{t}{8}\right) \diamond \nu'\left(2^{n-1}\left((f-T')\left(\frac{-x}{2^n}\right)\right), \frac{t}{8}\right) \\
&\leq \sup_{t'<t} \mu\left(x, 2^{(2q-1)n-2q}\left(\frac{(4-2^p)(2^p-2)}{6}\right)^q t'^q\right) \\
&\quad \diamond \sup_{t'<t} \mu\left(x, 2^{2(1-q)n-2q}\left(\frac{(4-2^p)(2^p-2)}{6}\right)^q t'^q\right) \qquad (3.18)
\end{aligned}$$

for all $x \in X$, $t > 0$ and $n \in \mathbb{N}$. Letting $n \to \infty$ in (3.17) and (3.18), and using the fact that $\lim_{n\to\infty} 2^{(2q-1)n-2q} = \lim_{n\to\infty} 2^{(1-q)n-2q} = \infty$ together with the definition of IFN-space, we get $\mu'(T(x)-T'(x),t) = 1$ and $\nu'(T(x)-T'(x),t) = 0$ for all $x \in X$ and $t > 0$. Hence $T(x) = T'(x)$ for all $x \in X$.

Case 3. Let $0 < q < \frac{1}{2}$. Define a mapping $J_n f : X \to Y$ by

$$J_n f(x) = \frac{1}{2}\left(4^n\big(f(2^{-n}x) + f(-2^{-n}x)\big) + 2^n\left(f\left(\frac{x}{2^n}\right) - f\left(-\frac{x}{2^n}\right)\right)\right)$$

for all $x \in X$. In this case, $J_0 f(x) = f(x)$ and

$$\begin{aligned}
J_j f(x) - J_{j+1} f(x) = &-\frac{4^j}{2}\left(Df\left(\frac{-x}{2^{j+1}}, \frac{-x}{2^{j+1}}, \frac{x}{2^{j+1}}\right) + Df\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, \frac{-x}{2^{j+1}}\right)\right) \\
&- 2^{j-1}\left(Df\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, \frac{-x}{2^{j+1}}\right) - Df\left(\frac{-x}{2^{j+1}}, \frac{-x}{2^{j+1}}, \frac{x}{2^{j+1}}\right)\right)
\end{aligned}$$

for all $x \in X$ and $j \geq 0$. Thus, for each $n + m > m \geq 0$, we have

$$\begin{aligned}
&\mu'\left(J_m f(x) - J_{n+m} f(x) \sum_{j=m}^{n+m-1} \frac{3}{2^p}\left(\frac{4}{2^p}\right)^j t^p\right)\\
&\quad \geq \prod_{j=m}^{n+m-1} \left\{\mu'\left(-\frac{(4^j + 2^j)Df(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, \frac{-x}{2^{j+1}})}{2}, \frac{3(4^j + 2^j)t^p}{2 \cdot 2^{(j+1)p}}\right)\right.\\
&\qquad \left. * \mu'\left(-\frac{(4^j - 2^j)Df(\frac{-x}{2^{j+1}}, \frac{-x}{2^{j+1}}, \frac{x}{2^{j+1}})}{2}, \frac{3(4^j - 2^j)t^p}{2 \cdot 2^{(j+1)p}}\right)\right\}\\
&\quad \geq \prod_{j=m}^{n+m-1} \mu\left(\frac{x}{2^{j+1}}, \frac{t}{2^{j+1}}\right) = \mu(x,t) \quad \text{and}\\
&\nu'\left(J_m f(x) - J_{n+m} f(x) \sum_{j=m}^{n+m-1} \frac{3}{2^p}\left(\frac{4}{2^p}\right)^j t^p\right)\\
&\quad \leq \coprod_{j=m}^{n+m-1} \left\{\nu'\left(-\frac{(4^j + 2^j)Df(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, \frac{-x}{2^{j+1}})}{2}, \frac{3(4^j + 2^j)t^p}{2 \cdot 2^{(j+1)p}}\right)\right.\\
&\qquad \left. \diamond \nu'\left(-\frac{(4^j - 2^j)Df(\frac{-x}{2^{j+1}}, \frac{-x}{2^{j+1}}, \frac{x}{2^{j+1}})}{2}, \frac{3(4^j - 2^j)t^p}{2 \cdot 2^{(j+1)p}}\right)\right\}\\
&\quad \leq \coprod_{j=m}^{n+m-1} \nu\left(\frac{x}{2^{j+1}}, \frac{t}{2^{j+1}}\right) = \nu(x,t)
\end{aligned}$$

for all $x \in X$ and $t > 0$. Proceeding along a similar argument as in the previous cases, we see that $\{J_n f(x)\}$ is a Cauchy sequence in (Y, μ', ν'). Thus, we define $T(x) := (\mu', \nu') - \lim_{n\to\infty} J_n f(x)$ for all $x \in X$. Putting $m = 0$ in the last two inequalities, we get

$$\left.\begin{aligned}
&\mu'(f(x) - J_n f(x), t) \geq \mu(x, \tfrac{t^q}{(\sum_{j=0}^{n-1} \frac{3}{2^p}(\frac{4}{2^p})^j)^q}) \quad \text{and}\\
&\nu'(f(x) - J_n f(x), t) \leq \nu(x, \tfrac{t^q}{(\sum_{j=0}^{n-1} \frac{3}{2^p}(\frac{4}{2^p})^j)^q})
\end{aligned}\right\} \tag{3.19}$$

for all $x \in X$ and $t > 0$. Write

$$\begin{aligned}
&\mu'\left(DJ_n f(x,y,z), \frac{3t}{4}\right)\\
&\quad \geq \mu'\left(\frac{4^n}{2} Df\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right), \frac{3t}{16}\right)\\
&\qquad * \mu'\left(\frac{4^n}{2} Df\left(\frac{-x}{2^n}, \frac{-y}{2^n}, \frac{-z}{2^n}\right), \frac{3t}{16}\right)\\
&\qquad * \mu'\left(2^{n-1} Df\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right), \frac{3t}{16}\right)\\
&\qquad * \mu'\left(2^{n-1} Df\left(\frac{-x}{2^n}, \frac{-y}{2^n}, \frac{-z}{2^n}\right), \frac{3t}{16}\right)\\
&\quad \geq \mu\left(x, 2^{(1-2q)n-3q} t^q\right) * \mu\left(y, 2^{(1-2q)n-3q} t^q\right) * \mu\left(z, 2^{(1-2q)n-3q} t^q\right)\\
&\qquad * \mu\left(x, 2^{(1-q)n-3q} t^q\right) * \mu\left(y, 2^{(1-q)n-3q} t^q\right) * \mu\left(z, 2^{(1-q)n-3q} t^q\right)
\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}
&\nu'\left(DJ_nf(x,y,z),\frac{3t}{4}\right)\\
&\quad\le \nu'\left(\frac{4^n}{2}Df\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\right),\frac{3t}{16}\right)\diamond\nu'\left(\frac{4^n}{2}Df\left(\frac{-x}{2^n},\frac{-y}{2^n},\frac{-z}{2^n}\right),\frac{3t}{16}\right)\\
&\qquad\diamond\nu'\left(2^{n-1}Df\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\right),\frac{3t}{16}\right)\diamond\nu'\left(2^{n-1}Df\left(\frac{-x}{2^n},\frac{-y}{2^n},\frac{-z}{2^n}\right),\frac{3t}{16}\right)\\
&\quad\le \nu\left(x,2^{(1-2q)n-3q}t^q\right)\diamond\nu\left(y,2^{(1-2q)n-3q}t^q\right)\diamond\nu\left(z,2^{(1-2q)n-3q}t^q\right)\\
&\qquad\diamond\nu\left(x,2^{(1-q)n-3q}t^q\right)\diamond\nu\left(y,2^{(1-q)n-3q}t^q\right)\diamond\nu\left(z,2^{(1-q)n-3q}t^q\right)
\end{aligned}\tag{3.21}$$

for all $x,y,z\in X$, $t>0$ and $n\in\mathbb{N}$. Since $1/2<q<1$ and taking the limit as $n\to\infty$, we see that (3.20) and (3.21) tend to 1 and 0, respectively. As in the previous cases, we have that $DT(x,y,z)=0$ for all $x,y,z\in X$. By the same argument as in previous cases, we can see that (3.2) follows from (3.19). To prove the uniqueness of T, assume that T' is another quadratic-additive mapping from X into Y satisfying (3.2). From (3.2) and (3.13), for all $x\in X$ and $t>0$, write

$$\begin{aligned}
\mu'\left(T(x)-T'(x),t\right) &= \nu'\left(J_nT(x)-J_nT'(x),t\right)\\
&\ge \mu'\left(J_nT(x)-J_nf(x),\frac{t}{2}\right)*\mu'\left(J_nf(x)-J_nT'(x),\frac{t}{2}\right)\\
&\ge \mu'\left(\frac{4^n}{2}\left((T-f)\left(\frac{x}{2^n}\right)\right),\frac{t}{8}\right)*\mu\left(\frac{4^n}{2}\left((f-T')\left(\frac{x}{2^n}\right)\right),\frac{t}{8}\right)\\
&\quad*\mu'\left(\frac{4^n}{2}\left((T-f)\left(-\frac{x}{2^n}\right)\right),\frac{t}{8}\right)*\mu'\left(\frac{4^n}{2}\left((f-T')\left(-\frac{x}{2^n}\right)\right),\frac{t}{8}\right)\\
&\quad*\mu'\left(2^{n-1}\left((T-f)\left(\frac{x}{2^n}\right)\right),\frac{t}{8}\right)*\mu'\left(2^{n-1}\left((f-T')\left(\frac{x}{2^n}\right)\right),\frac{t}{8}\right)\\
&\quad*\mu'\left(2^{n-1}\left((T-f)\left(\frac{-x}{2^n}\right)\right),\frac{t}{8}\right)*\mu'\left(2^{n-1}\left((f-T')\left(\frac{-x}{2^n}\right)\right),\frac{t}{8}\right)\\
&\ge \sup_{t'<t}\mu\left(x,2^{(1-2q)n-2q}\left(\frac{2^p-4}{3}\right)^q t^q\right)
\end{aligned}$$

and, similarly,

$$\begin{aligned}
\nu'\left(T(x)-T'(x),t\right) &\le \nu'\left(\frac{4^n}{2}\left((T-f)\left(\frac{x}{2^n}\right)\right),\frac{t}{8}\right)\diamond\nu\left(\frac{4^n}{2}\left((f-T')\left(\frac{x}{2^n}\right)\right),\frac{t}{8}\right)\\
&\quad\diamond\nu'\left(\frac{4^n}{2}\left((T-f)\left(-\frac{x}{2^n}\right)\right),\frac{t}{8}\right)\diamond\nu'\left(\frac{4^n}{2}\left((f-T')\left(-\frac{x}{2^n}\right)\right),\frac{t}{8}\right)\\
&\quad\diamond\nu'\left(2^{n-1}\left((T-f)\left(\frac{x}{2^n}\right)\right),\frac{t}{8}\right)\diamond\nu'\left(2^{n-1}\left((f-T')\left(\frac{x}{2^n}\right)\right),\frac{t}{8}\right)\\
&\quad\diamond\nu'\left(2^{n-1}\left((T-f)\left(\frac{-x}{2^n}\right)\right),\frac{t}{8}\right)\diamond\nu'\left(2^{n-1}\left((f-T')\left(\frac{-x}{2^n}\right)\right),\frac{t}{8}\right)\\
&\le \sup_{t'<t}\nu\left(x,2^{(1-2q)n-2q}\left(\frac{2^p-4}{3}\right)^q t^q\right)
\end{aligned}$$

for $n \in \mathbb{N}$. Letting $n \to \infty$ in (3.17) and (3.18), and using the fact that $\lim_{n\to\infty} 2^{(2q-1)n-2q} = \lim_{n\to\infty} 2^{(1-q)n-2q} = \infty$ together with the definition of IFN-space, we get $\mu'(T(x) - T'(x), t) = 1$ and $\nu'(T(x) - T'(x), t) = 0$ for all $x \in X$ and $t > 0$. Hence $T(x) = T'(x)$ for all $x \in X$. □

Remark 3.2 Let (X, μ, ν) be an IFN-space and (X, μ, ν) be an intuitionistic fuzzy Banach space (Y, μ', ν'). Let $f : X \to Y$ be a mapping satisfying (3.1) with a real number $q < 0$ and for all $t > 0$. If we choose a real number α with $0 < 3\alpha < t$, then

$$\mu'\bigl(Df(x,y,z),t\bigr) \geq \mu'\bigl(Df(x,y,z),3\alpha\bigr) \geq \mu\bigl(x,\alpha^q\bigr) * \mu\bigl(y,\alpha^q\bigr) * \mu\bigl(z,\alpha^q\bigr) \quad \text{and}$$
$$\nu'\bigl(Df(x,y,z),t\bigr) \leq \nu'\bigl(Df(x,y,z),3\alpha\bigr) \leq \nu\bigl(x,\alpha^q\bigr) \diamond \nu\bigl(y,\alpha^q\bigr) \diamond \nu\bigl(z,\alpha^q\bigr)$$

for all $x, y, z \in X$, $t > 0$ and $q < 0$. Since $q < 0$, we have $\lim_{\alpha\to 0^+} \alpha^q = \infty$. This implies that

$$\lim_{\alpha\to 0^+} \mu\bigl(x,\alpha^q\bigr) = 1 = \lim_{\alpha\to 0^+} \mu\bigl(y,\alpha^q\bigr) = \lim_{\alpha\to 0^+} \mu\bigl(z,\alpha^q\bigr) \quad \text{and}$$
$$\lim_{\alpha\to 0^+} \nu\bigl(x,\alpha^q\bigr) = 0 = \lim_{\alpha\to 0^+} \nu\bigl(y,\alpha^q\bigr) = \lim_{\alpha\to 0^+} \nu\bigl(z,\alpha^q\bigr).$$

Thus, we have $\mu'(Df(x,y,z),t) = 1$ and $\nu'(Df(x,y,z),t) = 0$ for all $x, y, z \in X$ and $t > 0$. Hence $Df(x,y,z) = 0$ for all $x, y, z \in X$. In other words, if f is an intuitionistic fuzzy q-almost quadratic-additive mapping for the case $q < 0$, then f is itself a quadratic-additive mapping.

Corollary 3.3 *Suppose that f is an even mapping satisfying the conditions of Theorem* 3.1. *Then there exists a unique quadratic mapping $T : X \to Y$ such that*

$$\left.\begin{aligned} &\mu'(T(x) - f(x), t) \geq \sup_{t'<t} \mu\bigl(x, \bigl(\tfrac{|4-2^p|t'}{3}\bigr)^q\bigr) \quad \textit{and} \\ &\nu'(T(x) - f(x), t) \leq \sup_{t'<t} \nu\bigl(x, \bigl(\tfrac{|4-2^p|t'}{3}\bigr)^q\bigr) \end{aligned}\right\} \tag{3.22}$$

for all $x \in X$ and $t > 0$, where $p = 1/q$.

Proof Since f is an even mapping, we get

$$J_nf(x) = \begin{cases} \frac{f(2^nx)+f(-2^nx)}{2\cdot 4^n} & \text{if } q > \frac{1}{2}, \\ \frac{1}{2}(4^n(f(2^{-n}x) + f(-2^{-n}x))) & \text{if } 0 < q < \frac{1}{2}, \end{cases}$$

for all $x \in X$, where J_nf is defined as in Theorem 3.1. In this case, $J_0f(x) = f(x)$. For all $x \in X$ and $j \in \mathbb{N} \cup \{0\}$, we have

$$J_jf(x) - J_{j+1}f(x) = \begin{cases} \frac{Df(2^jx,2^jx,-2^jx)}{2\cdot 4^{j+1}} + \frac{Df(-2^jx,-2^jx,2^jx)}{2\cdot 4^{j+1}} & \text{if } q > \frac{1}{2}, \\ -\frac{4^j}{2}(Df(\frac{-x}{2^{j+1}}, \frac{-x}{2^{j+1}}, \frac{x}{2^{j+1}}) + Df(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, \frac{-x}{2^{j+1}})) & \text{if } 0 < q < \frac{1}{2}. \end{cases}$$

Proceeding along the same lines as in Theorem 3.1, we obtain that T is a quadratic-additive function satisfying (3.22). Notice that $T(x) := (\mu', \nu') - \lim_{n\to\infty} J_nf(x)$, T is even and $DT(x,y,z) = 0$ for all $x, y, z \in X$. Hence, we get

$$T(x + y) + T(x - y) - 2T(x) - 2T(y) = -DT(x, y, -x) = 0$$

for all $x, y \in X$. It follows that T is a quadratic mapping. □

Corollary 3.4 *Suppose that f is an even mapping satisfying the conditions of Theorem* 3.1. *Then there exists a unique additive mapping $T : X \to Y$ such that*

$$\left.\begin{array}{l} \mu'(T(x) - f(x), t) \geq \sup_{t'<t} \mu(x, (\frac{|2-2^p|t'}{3})^q) \quad \textit{and} \\ \nu'(T(x) - f(x), t) \leq \sup_{t'<t} \nu(x, (\frac{|2-2^p|t'}{3})^q) \end{array}\right\} \tag{3.23}$$

for all $x \in X$ and $t > 0$, where $p = 1/q$.

Proof Since f is an odd mapping, we get

$$J_n f(x) = \begin{cases} \frac{f(2^n x)+f(-2^n x)}{2^{n+1}} & \text{if } q > 1, \\ 2^{n-1}(f(2^{-n}x) + f(-2^{-n}x)) & \text{if } 0 < q < 1, \end{cases}$$

for all $x \in X$, where $J_n f$ is defined as in Theorem 3.1. Here $J_0 f(x) = f(x)$. For all $x \in X$ and $j \in \mathbb{N} \cup \{0\}$, we have

$$J_j f(x) - J_{j+1} f(x) = \begin{cases} \frac{Df(2^j x, 2^j x, -2^j x)}{2^{j+2}} - \frac{Df(-2^j x, -2^j x, 2^j x)}{2^{j+2}} & \text{if } q > 1, \\ -2^{j-1}(Df(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, \frac{-x}{2^{j+1}}) - Df(\frac{-x}{2^{j+1}}, \frac{-x}{2^{j+1}}, \frac{x}{2^{j+1}})) & \text{if } 0 < q < 1. \end{cases}$$

Proceeding along the same lines as in Theorem 3.1, we obtain that T is a quadratic-additive function satisfying (3.23). Here $T(x) := (\mu', \nu') - \lim_{n\to\infty} J_n f(x)$, T is odd and $DT(x, y, z) = 0$ for all $x, y, z \in X$. Hence, we obtain

$$T(x + y) - T(x) - T(y) = Df\left(\frac{x - y}{2}, \frac{x + y}{2}, \frac{-x + y}{2}\right) = 0$$

for all $x, y \in X$. It follows that T is an additive mapping. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally and significantly in writing this paper. Both the authors read and approved the final manuscript.

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