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Exponential stability of mild solutions to impulsive stochastic neutral partial differential equations with memory

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Abstract
In this paper, we study the exponential stability in the *p*th moment of mild solutions to impulsive stochastic neutral partial differential equations with memory. Sufficient conditions ensuring the stability of the impulsive stochastic system are obtained by establishing a new integral inequality. The results obtained here generalize and improve some well-known results.

1 Introduction

At present, the study of stochastic partial different equations in a separable Hilbert space has become an important area of investigation in the past two decades because of their applications to various problems arising in physics, biology, engineering *etc.* [1, 2]. The existence, uniqueness and stability of solutions of stochastic partial differential equations have been considered by many authors [2–12]. The stability of strong solutions of stochastic differential equations also have been discussed extensively [13–15]. However, there are a number of difficulties encountered in the study of stability by the Lyapunov second method. By the Banach fixed point theory, [16] studied a linear scalar neutral stochastic differential equation with variable delays and gave conditions to ensure that the zero solution is asymptotically mean square stable. Further [17] considered the stability of stochastic partial differential equations with delays by using the Banach fixed point theory.

On the other hand, the impulsive effects exist in many evolution processes, in which states are changed abruptly at certain moments of time, involved in such fields as medicine and biology, economics, mechanics, electronics [18, 19]. In recent years, the investigation of impulsive stochastic differential equations attracts great attention, especially as regards stability. For example, [20] discussed the stability of impulsive stochastic systems. [21, 22] discussed the exponential stability in mean square of impulsive stochastic difference equations by establishing difference inequalities. Jiang and Shen [23] discussed the asymptotic stability of impulsive stochastic neutral partial differential equations with infinite delays.

As known, although the Lyapunov second method is a powerful technique in proving the stability theorems, it is not so suitable in the non-delay case. A difficulty is that mild solutions do not have stochastic differentials, so that one cannot apply the Itô formula to them. Meanwhile, the difficulty of the method of the fixed point theory comes from finding an appropriate fixed point theorem. Therefore, the techniques and the methods for the stability of mild solutions should be developed and explored. In this present work, motivated by [21–23], we study the exponential stability of impulsive stochastic neutral partial differential equations with memory by establishing a new integral inequality. The results obtained here generalize the main results from [3, 6, 17] to cover a class of more general impulsive stochastic neutral systems.

The rest of this paper is organized as follows. In Section 2, we introduce some preliminaries. In Section 3 sufficient conditions ensuring the stability of the impulsive stochastic system are obtained by establishing a new integral inequality.

2 Preliminaries

Throughout this paper, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (*i.e.*, it is increasing and right-continuous while $\mathcal{F}_0$ contains all P-null sets). Moreover, let X, Y be two real separable Hilbert spaces and let $L(Y, X)$ denote the space of all bounded linear operators from Y into X.

For simplicity, we use the notation $|\cdot|$ to denote the norm in X, Y and $\|\cdot\|$ to denote the operator norm in $L(X, X)$ and $L(Y, X)$. Let $\langle\cdot\rangle_X$, $\langle\cdot\rangle_Y$ denote the inner products of X, Y, respectively. Let $\{w(t) : t \geq 0\}$ denote a Y-valued Wiener process defined on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ with a covariance operator Q, that is, $E\langle w(t), x\rangle_Y E\langle w(s), y\rangle_X = (t \wedge s)\langle Qx, y\rangle_Y$, for all $x, y \in Y$, where Q is a positive, self-adjoint, trace class operator on Y. In particular, we denote by $w(t)$ a Y-valued Q-Wiener process with respect to $\{\mathcal{F}_t\}_{t\geq 0}$. We assume that there exists a complete orthonormal system $\{e_i\}$ in Y, a bounded sequence of nonnegative real numbers λ_i such that $Qe_i = \lambda_i e_i$, $i = 1, 2, \ldots$, and a sequence $\{\beta_i\}_{i\geq 1}$ of independent Brownian motions such that $\langle w(t), e\rangle = \sum_{i=1}^{\infty} \sqrt{\lambda_i}\langle e_i, e\rangle\beta_i(t)$, $e \in Y$, and $\mathcal{F}_t = \mathcal{F}_t^w$, where $\mathcal{F}_t^w$ is the σ-algebra generated by $\{w(s) : 0 \leq s \leq t\}$. Let $L_2^0 = L_2(Q^{1/2}Y; X)$ be the space of all Hilbert-Schmidt operators from $Q^{1/2}Y$ to X with the inner product $\langle u, \xi\rangle_{L_2^0} = \mathrm{tr}[uQ\xi]$; see, for example, [2].

Suppose that $\{S(t), t \geq 0\}$ is an analytic semigroup with its infinitesimal generator A; for literature relating to semigroup theory, we suggest Pazy [24]. We suppose $0 \in \rho(A)$, the resolvent set of $-A$. For any $\alpha \in [0, 1]$, it is possible to define the fractional power $(-A)^\alpha$ which is a closed linear operator with its domain $\mathcal{D}((-A)^\alpha)$.

In this paper, we consider the following impulsive stochastic neutral partial differential equations with memory:

$$\begin{cases} d[x(t) - u(t, x(t-\rho(t)))] = [Ax(t) + f(t, x(t-\tau(t)))]\,dt \\ \qquad\qquad\qquad + g(t, x(t-\delta(t)))\,dw(t), \quad t \geq 0, t \neq t_k, \\ \triangle x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad t = t_k, k = 1, 2, \ldots, m, \\ x_0(\cdot) = \varphi \in C_{\mathcal{F}_0}^b([-\tau, 0], X), \end{cases} \tag{1}$$

in a real separable Hilbert space X, where $u : R_+ \times C([-\tau, 0], X) \to X, f : R_+ \times C([-\tau, 0], X) \to X, g : R_+ \times C([-\tau, 0], X) \to L(Y, X)$ are all Borel measurable; $\rho : R_+ \to [0, \tau]$, $\tau : R_+ \to [0, \tau]$, $\delta : R_+ \to [0, \tau]$ are continuous; A is the infinitesimal generator of a semigroup of bounded linear operators $S(t)$, $t \geq 0$, in X; $I_k : X \to X$. Furthermore, the fixed moments of time t_k satisfy $0 < t_1 < \cdots < t_m < \lim_{k\to\infty} t_k = \infty$, $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively. Also, $\triangle x(t_k) = x(t_k^+) - x(t_k^-)$, represents the jump in the state x at time t_k with I_k determining the size of the jump. Let $\tau > 0$ and $C = C([-\tau, 0]; X)$ denote the family of all right continuous functions with left-hand limits η from $[-\tau, 0]$

to X. The space C is assumed to be equipped with the norm $\|\eta\|_C = \sup_{\theta\in[-\tau,0]} |\eta(\theta)|$. Here $C^b_{\mathcal{F}_0}([-\tau,0],X)$ is the family of all almost surely bounded, $\mathcal{F}_0$-measurable, continuous random variables from $[-\tau,0]$ to X.

Definition 2.1 A process $\{x(t), t\in[0,T]\}$, $0\le T<\infty$, is called a mild solution of Eq. (1) if

(i) $x(t)$ is adapted to $\mathcal{F}_t$, $t\ge 0$ with $\int_0^T |x(t)|^p\,dt<\infty$ a.s.;

(ii) $x(t)\in X$ has càdlàg paths on $t\in[0,T]$ a.s. and for each $t\in[0,T]$, $x(t)$ satisfies the integral equation

$$\begin{aligned} x(t) ={}& S(t)\bigl[\varphi(0)-u\bigl(0,x\bigl(-\rho(0)\bigr)\bigr)\bigr]+u\bigl(t,x\bigl(t-\rho(t)\bigr)\bigr)\\ &+\int_0^t AS(t-s)u\bigl(s,x\bigl(s-\rho(s)\bigr)\bigr)\,ds\\ &+\int_0^t S(t-s)f\bigl(s,x\bigl(s-\tau(s)\bigr)\bigr)\,ds+\int_0^t S(t-s)g\bigl(s,x\bigl(s-\delta(s)\bigr)\bigr)\,dw(s)\\ &+\sum_{0<t_k<t} S(t-t_k)I_k\bigl(x\bigl(t_k^-\bigr)\bigr) \end{aligned} \tag{2}$$

and $x_0(\cdot)=\varphi\in C^b_{\mathcal{F}_0}([-\tau,0],X)$.

Definition 2.2 Let $p\ge 2$ be an integer. Equation (1) is said to be exponentially stable in the pth mean if for any initial value φ, there exists a pair of positive constants λ and K_0 such that

$$E\bigl|x(t)\bigr|^p\le K_0\|\varphi\|_C^p e^{-\lambda t}\quad \text{for } t\ge 0. \tag{3}$$

In particular, if $p=2$, then Eq. (1) is said to be mean-square exponentially stable.

To establish the exponential stability of the mild solution of Eq. (1), we employ the following assumptions.

(H1) A is the infinitesimal generator of a semigroup of bounded linear operators $S(t)$, $t\ge 0$, in X satisfying $|S(t)|\le Me^{-at}$, $t\ge 0$, for some constants $M\ge 1$ and $0<a\in R_+$.

(H2) The mappings f and g satisfy the following Lipschitz condition: there exists a constant K for any $x,y\in X$ and $t\ge 0$ such that

$$\bigl|f(t,x)-f(t,y)\bigr|\le K|x-y|,\qquad \bigl\|g(t,x)-g(t,y)\bigr\|\le K|x-y|.$$

(H3) The mapping $u(t,x)$ satisfies that there exists a number $\alpha\in[0,1]$ and a positive constant $\overline{K}$ such that for any $x,y\in X$ and $t\ge 0$, $u(t,x)\in\mathcal{D}((-A)^\alpha)$ and

$$\bigl|(-A)^\alpha u(t,x)-(-A)^\alpha u(t,y)\bigr|\le\overline{K}|x-y|.$$

(H4) There exists a constant q_k such that $|I_k(x)-I_k(y)|\le q_k|x-y|$, $k=1,\dots,m$, for each $x,y\in X$.

Moreover, for the purposes of stability, we always assume that $u(t,0)=0$, $f(t,0)=0$, $g(t,0)=0$, $I_k(0)=0$ $(k=1,2,\dots,m)$. Hence Eq. (1) has a trivial solution when $\varphi=0$.

Lemma 2.1 [24] *If* (H1) *holds, then for any* $\beta \in (0,1]$:

(i) *For each* $x \in \mathcal{D}((-A)^\beta)$, $S(t)(-A)^\beta x = (-A)^\beta S(t)x$;

(ii) *There exist positive constants* $M_\beta > 0$ *and* $a \in R_+$ *such that* $\|(-A)^\beta S(t)\| \le M_\beta t^{-\beta} e^{-at}$, $t > 0$.

3 Stability of mild solutions

In this section, to establish sufficient conditions ensuring the exponential stability in p-moment ($p \ge 2$) for a mild solution to Eq. (1), we firstly establish a new integral inequality to overcome the difficulty when the neutral term and impulsive effects are present.

Lemma 3.1 *For any* $\gamma > 0$, *assume that there exist some positive constants* α_i ($i = 1,2,3$), β_k ($k = 1,2,\ldots,m$) *and a function* $\psi : [-\tau,\infty) \to [0,\infty)$ *such that*

$$\psi(t) \le \alpha_1 e^{-\gamma t} \quad \textit{for } t \in [-\tau,0] \tag{4}$$

and

$$\begin{aligned}\psi(t) \le{}& \alpha_1 e^{-\gamma t} + \alpha_2 \sup_{\theta\in[-\tau,0]} \psi(t+\theta) + \alpha_3 \int_0^t e^{-\gamma(t-s)} \sup_{\theta\in[-\tau,0]} \psi(t+\theta)\,ds \\ &+ \sum_{t_k<t} \beta_k e^{-\gamma(t-t_k)} \psi\left(t_k^-\right)\end{aligned} \tag{5}$$

for each $t \ge 0$. *If*

$$\alpha_2 + \frac{\alpha_3}{\gamma} + \sum_{k=1}^m \beta_k < 1, \tag{6}$$

then

$$\psi(t) \le M_0 e^{-\lambda t} \quad \textit{for } t \ge -\tau, \tag{7}$$

where $\lambda > 0$ *is the unique solution to the equation*: $\alpha_2 e^{\lambda\tau} + \alpha_3 e^{\lambda\tau}/(\gamma - \lambda) + \sum_{k=1}^m \beta_k = 1$ *and* $M_0 = \max\{\alpha_1, \frac{\alpha_1(\gamma-\lambda)}{\alpha_3 e^{\lambda\tau}}\} > 0$.

Proof Let $\Phi(\nu) = \alpha_2 e^{\nu\tau} + \alpha_3 e^{\nu\tau}/(\gamma - \nu) + \sum_{k=1}^m \beta_k - 1$, then by (6) and the existence theorem of the root, there exists a positive constant $\lambda \in (0,\gamma)$ such that $\Phi(\lambda) = 0$.

For any $\varepsilon > 0$, let

$$M_\varepsilon = \max\left\{(\alpha_1 + \varepsilon), \frac{(\alpha_1 + \varepsilon)(\gamma - \lambda)}{\alpha_3 e^{\lambda\tau}}\right\} > 0. \tag{8}$$

To now prove the result, we only claim that (4) and (5) imply

$$\psi(t) \le M_\varepsilon e^{-\lambda t} \quad \text{for } t \ge -\tau. \tag{9}$$

Clearly, for any $t \in [-\tau,0]$, (9) holds. By the contradiction, assume that there is a positive constant t_1 such that

$$\psi(t) \le M_\varepsilon e^{-\lambda t} \quad \text{for } t \in [-\tau,t_1), \qquad \psi(t_1) = M_\varepsilon e^{-\lambda t_1}. \tag{10}$$

This, together with (5), yields (note that $0 < \lambda < \gamma$)

$$\begin{aligned}\psi(t_1) &\le \alpha_1 e^{-\gamma t_1} + \alpha_2 M_\varepsilon \sup_{\theta\in[-\tau,0]} e^{-\lambda(t_1+\theta)} + \alpha_3 M_\varepsilon \int_0^{t_1} e^{-\gamma(t_1-s)} \sup_{\theta\in[-\tau,0]} e^{-\lambda(s+\theta)}\,ds \\ &\quad + M_\varepsilon \sum_{t_k<t_1} \beta_k e^{-\gamma(t_1-t_k)} e^{-\lambda t_k} \\ &\le \alpha_1 e^{-\gamma t_1} + \alpha_2 M_\varepsilon e^{-\lambda(t_1-\tau)} + \alpha_3 M_\varepsilon \int_0^{t_1} e^{-\gamma(t_1-s)} e^{-\lambda(s-\tau)}\,ds + M_\varepsilon \sum_{t_k<t_1} \beta_k e^{-\lambda t_1} \\ &\le \alpha_1 e^{-\gamma t_1} - \frac{\alpha_3 M_\varepsilon e^{\lambda\tau}}{\gamma-\lambda} e^{-\gamma t_1} + \left(\alpha_2 e^{\lambda\tau} + \frac{\alpha_3 e^{\lambda\tau}}{\gamma-\lambda} + \sum_{k=1}^m \beta_k\right) M_\varepsilon e^{-\lambda t_1}. \end{aligned} \tag{11}$$

By (8), we have

$$\alpha_1 e^{-\gamma t_1} - \frac{\alpha_3 M_\varepsilon e^{\lambda\tau}}{\gamma-\lambda} e^{-\gamma t_1} \le \alpha_1 e^{-\gamma t_1} - \frac{\alpha_3 e^{\lambda\tau}}{\gamma-\lambda} e^{-\gamma t_1} \frac{(\alpha_1+\varepsilon)(\gamma-\lambda)}{\alpha_3 e^{\lambda\tau}} < 0. \tag{12}$$

Hence, by (11), we obtain $\psi(t_1) < M_\varepsilon e^{-\lambda t_1}$, which contradicts (10). Therefore (9) holds.

Since ε is arbitrarily small, so (7) holds. This completes the proof. □

We can now state our main result of this paper.

Theorem 3.1 *If* (H1)-(H4) *hold for some* $\alpha \in (1/p, 1]$, $p \ge 2$, *then the mild solution of Eq.* (1) *is exponentially stable in the pth moment, provided*

$$\begin{aligned} &a\kappa(1-\kappa)^{p-1} + 8^{p-1} M_{1-\alpha}^p \overline{K}^p a^{1-p\alpha} \left(\Gamma(1+q\alpha-q)\right)^{\frac{p}{q}} + 4^{p-1} M^p K^p a^{1-p} \\ &\quad + 4^{p-1} c_p M^p K^p \left(\frac{2a(p-1)}{p-2}\right)^{1-\frac{p}{2}} + 4^{p-1} a M^p (1-\kappa)^{p-1} \left(\sum_{k=1}^m q_k\right)^p \\ &< a(1-\kappa)^{p-1}, \end{aligned} \tag{13}$$

where $c_p = (p(p-1)/2)^{p/2}$, $\kappa = \overline{K}|(-A)^{-\alpha}|$ *and* $M_{1-\alpha}$ *is defined in Lemma* 3.1.

Proof From the condition (13), we can always find a number $\epsilon > 0$ small enough such that

$$\begin{aligned} &a\kappa(1-\kappa)^{p-1} + 8^{p-1}(1+\epsilon)^{p-1} M_{1-\alpha}^p \overline{K}^p a^{1-p\alpha} \left(\Gamma(1+q\alpha-q)\right)^{\frac{p}{q}} + 4^{p-1} M^p K^p a^{1-p} \\ &\quad + 4^{p-1} c_p M^p K^p \left(\frac{2a(p-1)}{p-2}\right)^{1-\frac{p}{2}} + 4^{p-1} a M^p (1-\kappa)^{p-1} \left(\sum_{k=1}^m q_k\right)^p < a(1-\kappa)^{p-1}. \end{aligned}$$

On the other hand, recall the inequalities $|u-v|^p \le |u|^p/\epsilon^{p-1} + |v|^p/(1-\epsilon)^{p-1}$ and $|u+v|^p \le (1+\epsilon)^{p-1}|u|^p + (1+1/\epsilon)^{p-1}|v|^p$ for $u, v \in X$, $\epsilon > 0$. Then, for any $x_1, \dots, x_6$,

$$\begin{aligned} |x_1+x_2+x_3+x_4+x_5+x_6|^p &\le 4^{p-1}\left(1+\frac{1}{\epsilon}\right)^{p-1} |x_1|^p + 8^{p-1}(1+\epsilon)^{p-1}\left(|x_2|^p + |x_3|^p\right) \\ &\quad + 4^{p-1}|x_4|^p + 4^{p-1}|x_5|^p + 4^{p-1}|x_6|^p. \end{aligned} \tag{14}$$

From (2) and (14),

$$
\begin{aligned}
E\left|x(t)\right|^p &\le \frac{1}{\kappa^{p-1}}E\left|u\big(t,x\big(t-\rho(t)\big)\big)\right|^p + \frac{1}{(1-\kappa)^{p-1}}E\left|x(t)-u\big(t,x\big(t-\rho(t)\big)\big)\right|^p \\
&\le \frac{1}{\kappa^{p-1}}E\left|u\big(t,x\big(t-\rho(t)\big)\big)\right|^p + \frac{1}{(1-\kappa)^{p-1}}E\Bigg|S(t)\big[\varphi(0)-u\big(0,x\big(-\rho(0)\big)\big)\big] \\
&\quad + \int_0^t AS(t-s)u\big(s,x\big(s-\rho(s)\big)\big)\,ds + \int_0^t S(t-s)f\big(s,x\big(s-\tau(s)\big)\big)\,ds \\
&\quad + \int_0^t S(t-s)g\big(s,x\big(s-\delta(s)\big)\big)\,dw(s) + \sum_{0<t_k<t} S(t-t_k)I_k\big(x\big(t_k^-\big)\big)\Bigg|^p \\
&\le \frac{1}{\kappa^{p-1}}E\left|u\big(t,x\big(t-\rho(t)\big)\big)\right|^p + \frac{1}{(1-\kappa)^{p-1}}\Bigg\{4^{p-1}\left(1+\frac{1}{\epsilon}\right)^{p-1}E\left|S(t)\varphi(0)\right|^p \\
&\quad + 8^{p-1}(1+\epsilon)^{p-1}E\left|S(t)u\big(0,x\big(-\rho(0)\big)\big)\right|^p \\
&\quad + 8^{p-1}(1+\epsilon)^{p-1}E\left|\int_0^t AS(t-s)u\big(s,x\big(s-\rho(s)\big)\big)\,ds\right|^p \\
&\quad + 4^{p-1}E\left|\int_0^t S(t-s)f\big(s,x\big(s-\tau(s)\big)\big)\,ds\right|^p \\
&\quad + 4^{p-1}E\left|\int_0^t S(t-s)g\big(s,x\big(s-\delta(s)\big)\big)\,dw(s)\right|^p \\
&\quad + 4^{p-1}E\left|\sum_{0<t_k<t} S(t-t_k)I_k\big(x\big(t_k^-\big)\big)\right|^p\Bigg\} \\
&=: \frac{1}{\kappa^{p-1}}F_0 + \frac{1}{(1-\kappa)^{p-1}}\sum_{i=1}^{6}F_i.
\end{aligned}
\tag{15}
$$

Now we compute the right-hand terms of (15). Firstly, by (H1) and (H3), we can easily obtain

$$F_0 \le \kappa^p \sup_{\theta\in[-\tau,0]} E\left|x(t+\theta)\right|^p, \tag{16}$$

$$F_1 \le 4^{p-1}\left(1+\frac{1}{\epsilon}\right)^{p-1}M^p e^{-pat}E\|\varphi\|_C^p \tag{17}$$

and

$$F_2 \le 8^{p-1}(1+\epsilon)^{p-1}M^p\left|(-A)^{-\alpha}\right|^p E\|\varphi\|_C^p. \tag{18}$$

By (H4) and the Hölder inequality, for $p \ge 2$, $1 < q \le 2$, $1/p + 1/q = 1$, we have

$$
\begin{aligned}
F_6 &\le 4^{p-1}E\left(\sum_{0<t_k<t}\left|S(t-t_k)\right|\left|I_k\big(x\big(t_k^-\big)\big)\right|\right)^p \\
&\le 4^{p-1}E\left(\sum_{0<t_k<t} Me^{-a(t-t_k)}q_k\left|x\big(t_k^-\big)\right|\right)^p
\end{aligned}
$$

$$\le 4^{p-1}M^pE\left(\sum_{0<t_k<t} q_k^{\frac{1}{q}} q_k^{\frac{1}{p}} e^{-a(t-t_k)}\left|x\left(t_k^-\right)\right|\right)^p$$

$$\le 4^{p-1}M^p\left(\sum_{0<t_k<t} q_k\right)^{\frac{p}{q}} \sum_{0<t_k<t} q_k e^{-pa(t-t_k)}E\left|x\left(t_k^-\right)\right|^p. \tag{19}$$

By (H3), Lemma 3.1 and the Hölder inequality,

$$\begin{aligned}
F_3 &\le 8^{p-1}(1+\epsilon)^{p-1}E\left(\int_0^t \left|(-A)^{-\alpha}S(t-s)(-A)^{\alpha}u\left(s,x\left(s-\rho(s)\right)\right)\right| ds\right)^p \\
&\le 8^{p-1}(1+\epsilon)^{p-1}M_{1-\alpha}^p\overline{K}^p\left(\int_0^t e^{-a(t-s)}(t-s)^{q\alpha-q}\,ds\right)^{\frac{p}{q}} \int_0^t e^{-a(t-s)}E\left|x\left(s-\rho(s)\right)\right|^p ds \\
&\le 8^{p-1}(1+\epsilon)^{p-1}M_{1-\alpha}^p\overline{K}^p a^{1-p\alpha}\left(\Gamma(1+q\alpha-q)\right)^{\frac{p}{q}} \\
&\quad\times \int_0^t e^{-a(t-s)}E\left|x\left(s-\rho(s)\right)\right|^p ds \\
&\le 8^{p-1}(1+\epsilon)^{p-1}M_{1-\alpha}^p\overline{K}^p a^{1-p\alpha}\left(\Gamma(1+q\alpha-q)\right)^{\frac{p}{q}} \\
&\quad\times \int_0^t e^{-a(t-s)}\sup_{\theta\in[-\tau,0]}E\left|x(s+\theta)\right|^p ds.
\end{aligned} \tag{20}$$

Similar to (20), by (H2) and the Hölder inequality, we have

$$F_4 \le 4^{p-1}M^pK^pa^{1-p}\int_0^t e^{-a(t-s)}\sup_{\theta\in[-\tau,0]}E\left|x(s+\theta)\right|^p ds. \tag{21}$$

By Da Prato and Zabczyk [2, Lemma 7.7, p.194], similar to (20), (H2) and the Hölder inequality, we have

$$\begin{aligned}
F_5(t) &\le 4^{p-1}c_pM^p\left(\int_0^t \left(e^{-ap(t-s)}E\left\|g\left(s,x\left(s-\delta(t)\right)\right)\right\|_{L_2^0}^p\right)^{\frac{2}{p}} ds\right)^{\frac{p}{2}} \\
&\le 4^{p-1}c_pM^pK^p\left(\frac{2a(p-1)}{p-2}\right)^{1-\frac{p}{2}} \int_0^t e^{-a(t-s)}\sup_{\theta\in[-\tau,0]}E\left|x(s+\theta)\right|^p ds,
\end{aligned} \tag{22}$$

where $c_p = (p(p-1)/2)^{p/2}$.

Substituting (16)-(22) into (15) yields

$$\begin{aligned}
E\left|x(t)\right|^p \le{}& \kappa\sup_{\theta\in[-\tau,0]}E\left|x(t+\theta)\right|^p + \frac{1}{(1-\kappa)^{p-1}}\left\{4^{p-1}\left(1+\frac{1}{\epsilon}\right)^{p-1}M^pe^{-at}E\|\varphi\|_C^p\right. \\
&+ 8^{p-1}(1+\epsilon)^{p-1}M^p\left|(-A)^{-\alpha}\right|^pE\|\varphi\|_C^p \\
&+ 8^{p-1}(1+\epsilon)^{p-1}M_{1-\alpha}^p\overline{K}^pa^{1-p\alpha}\left(\Gamma(1+q\alpha-q)\right)^{\frac{p}{q}} \\
&\times \int_0^t e^{-a(t-s)}\sup_{\theta\in[-\tau,0]}E\left|x(s+\theta)\right|^p ds \\
&+ 4^{p-1}M^pK^pa^{1-p}\int_0^t e^{-a(t-s)}\sup_{\theta\in[-\tau,0]}E\left|x(s+\theta)\right|^p ds
\end{aligned}$$

$$+ 4^{p-1}c_pM^pK^p\left(\frac{2a(p-1)}{p-2}\right)^{1-\frac{p}{2}}\int_0^t e^{-a(t-s)}\sup_{\theta\in[-\tau,0]}E\left|x(s+\theta)\right|^p\,ds$$
$$+ 4^{p-1}M^p\left(\sum_{0<t_k<t}q_k\right)^{\frac{p}{q}}\sum_{0<t_k<t}q_ke^{-a(t-t_k)}E\left|x\left(t_k^-\right)\right|^p\Bigg\}. \tag{23}$$

This, together with Lemma 3.1 and (13), gives that there exist two positive constants M_0 and $\lambda \in (0,a)$ such that $E|x(t)|^p \le M_0e^{-\lambda t}$ for any $t \ge -\tau$. This completes the proof. □

If $p = 2$, then we get the following corollary from Theorem 3.1.

Corollary 3.1 *If* (H1)-(H4) *hold for some* $\alpha \in (1/2,1]$, *then the mild solution of Eq.* (1) *is mean-square exponentially stable, provided*

$$a\overline{K}\left|(-A)^{-\alpha}\right|\left(1-\overline{K}\left|(-A)^{-\alpha}\right|\right) + 8M_{1-\alpha}^2\overline{K}^2a^{1-2\alpha}\Gamma(2\alpha-1) + 4M^2K^2a^{-1}$$
$$+ 4M^2K^2 + 4aM^2(1-\kappa)\left(\sum_{k=1}^m q_k\right)^2 < a\left(1-\overline{K}\left|(-A)^{-\alpha}\right|\right). \tag{24}$$

Remark 3.1 Unlike earlier studies, ours does not make use of general methods such as Lyapunov methods, fixed point theory and so forth. As we know, in general, it is impossible to construct a suitable Lyapunov function (functional) and to find an appropriate fixed point theorem for stochastic partial differential equations with memory, even for constant delays, to deal with stability. In this work, we use the new impulsive integral inequality to derive the sufficient conditions for stability.

Remark 3.2 Without delay and impulsive effect, Eq. (1) becomes stochastic neutral partial differential equations, which is investigated in [3]. Without the neutral term and impulsive effect, Eq. (1) reduces to stochastic partial differential delay equations, which is studied in [6, 17]. Therefore, we generalize by the integral inequality the results to cover a class of more general impulsive stochastic neutral partial differential equations with memory. Moreover, unlike [6], we need not require the functions $\rho(t)$, $\tau(t)$, $\delta(t)$ to be differentiable.

Remark 3.3 In Eq. (1), provided $\triangle x(t_k) = 0$, Eq. (1) becomes stochastic neutral partial differential equations without impulsive effects, that is to say, our Theorem 3.1 is effective for it.

4 Conclusion

In this paper, we discuss the exponential stability in the pth moment of mild solutions to impulsive stochastic neutral partial differential equations with memory. By establishing a new integral inequality, we obtain sufficient conditions ensuring the stability of the impulsive stochastic system. The results generalize and improve earlier publications.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
HY gave the proof of the main result and drafted the manuscript. FJ established the new integral inequality and participated in the study of the main result of the paper. All authors read and approved the final manuscript.

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References

1. Chow, PL: Stochastic Partial Differential Equations. Chapman & Hall/CRC Press, Boca Raton (2007)
2. Da Prato, G, Zabczyk, J: Stochastic Equations in Infinite Dimensions. Cambridge University Press, Cambridge (1992)
3. Mahmudov, NI: Existence and uniqueness results for neutral SDEs in Hilbert spaces. Stoch. Anal. Appl. **24**, 79-95 (2006)
4. Bao, J, Hou, Z: Existence of mild solutions to stochastic neutral partial functional differential equations with non-Lipschitz coefficients. Comput. Math. Appl. **59**, 207-214 (2010)
5. Chen, H: Integral inequality and exponential stability for neutral stochastic partial differential equations with delays. J. Inequal. Appl. **2009**, Article ID 297478 (2009)
6. Caraballo, T, Liu, K: Exponential stability of mild solutions of stochastic partial differential equations with delays. Stoch. Anal. Appl. **17**, 743-763 (1999)
7. Wan, L, Duan, J: Exponential stability of non-autonomous stochastic partial differential equations with finite memory. Stat. Probab. Lett. **78**, 490-498 (2008)
8. Bao, J, Truman, A, Yuan, C: Stability in distribution of mild solutions to stochastic partial differential delay equations with jumps. Proc. R. Soc. Lond. A **465**, 2111-2134 (2009)
9. Chen, H: Impulsive-integral inequality and exponential stability for stochastic partial differential equations with delays. Stat. Probab. Lett. **80**, 50-56 (2010)
10. Chen, H: The asymptotic behavior for second-order neutral stochastic partial differential equations with infinite delay. Discrete Dyn. Nat. Soc. **2011**, 1-15 (2011)
11. Chen, H: Asymptotic behavior of stochastic two-dimensional Navier-Stokes equations with delays. Proc. Indian Acad. Sci. Math. Sci. **122**, 283-295 (2012)
12. Jiang, J, Shen, Y: A note on the existence and uniqueness of mild solutions to neutral stochastic partial functional differential equations with non-Lipschitz coefficients. Comput. Math. Appl. **61**, 1590-1594 (2011)
13. Mao, X: Stochastic Differential Equations and Applications. Horwood, Chichestic (1997)
14. Liu, L: Stability of Infinite Dimensional Stochastic Differential Equations with Applications. Chapman & Hall/CRC Press, London (2004)
15. Li, C, Sun, J: Stability analysis of nonlinear stochastic differential delay systems under impulsive control. Phys. Lett. A **374**, 1154-1158 (2010)
16. Luo, J: Fixed points and stability of neutral stochastic delay differential equations. J. Math. Anal. Appl. **334**, 431-440 (2007)
17. Luo, J: Fixed points and exponential stability of mild solutions of stochastic partial differential equations with delays. J. Math. Anal. Appl. **342**, 753-760 (2008)
18. Lakshmikantham, V, Bainov, D, Simeonov, P: Theory of Impulsive Differential Equations. World Scientific, Singapore (1989)
19. Li, X: New results on global exponential stabilization of impulsive functional differential equations with infinite delays or finite delays. Nonlinear Anal., Real World Appl. **11**, 4194-4201 (2010)
20. Cheng, P, Deng, F, Peng, Y: Robust exponential stability and delayed-state-feedback stabilization of uncertain impulsive stochastic systems with time-varying delay. Commun. Nonlinear Sci. Numer. Simul. **17**, 4740-4752 (2012)
21. Yang, Z, Xu, D: Mean square exponential stability of impulsive stochastic difference equations. Appl. Math. Lett. **20**, 938-945 (2007)
22. Bao, J, Hou, Z, Wang, F: Exponential stability in mean square of impulsive stochastic difference equations with continuous time. Appl. Math. Lett. **22**, 749-753 (2009)
23. Jiang, F, Shen, Y: Stability of impulsive stochastic neutral partial differential equations with infinite delays. Asian J. Control **14**, 1706-1709 (2012)
24. Pazy, A: Semigroups of Linear Operators and Applications to Partial Differential Equations. Springer, New York (1992)

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