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Laplace transform for solving some families of fractional differential equations and its applications

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Abstract

In many recent works, many authors have demonstrated the usefulness of fractional calculus in the derivation of particular solutions of a significantly large number of linear ordinary and partial differential equations of the second and higher orders. The main objective of the present paper is to show how this simple fractional calculus method to the solutions of some families of fractional differential equations would lead naturally to several interesting consequences, which include (for example) a generalization of the classical Frobenius method. The methodology presented here is based chiefly upon some general theorems on (explicit) particular solutions of some families of fractional differential equations with the Laplace transform and the expansion coefficients of binomial series.

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1 Introduction, definitions and preliminaries

In the past two decades, the widely investigated subject of fractional calculus has remarkably gained importance and popularity due to its demonstrated applications in numerous diverse fields of science and engineering. These contributions to the fields of science and engineering are based on the mathematical analysis. It covers the widely known classical fields such as Abel's integral equation and viscoelasticity. Also, including the analysis of feedback amplifiers, capacitor theory, generalized voltage dividers, fractional-order Chua-Hartley systems, electrode-electrolyte interface models, electric conductance of biological systems, fractional-order models of neurons, fitting of experimental data, and the fields of special functions, *etc.* (see, for example, [1–4]).

In this paper, we apply the Laplace of the fractional derivative and the expansion coefficients of binomial series to derive the explicit solutions to homogeneous fractional differential equations.

We present some useful definitions and preliminaries as follows.

Definitions

1. The fractional derivative of a causal function $f(t)$ (*cf.* [3, 4]) is defined by

$$\frac{d^\alpha}{dt^\alpha}f(t)=\begin{cases} f^{(n)}(t) & \text{if } \alpha=n\in\mathbb{N},\\ \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(t)}{(t-x)^{\alpha-n+1}}\,dt & \text{if } n-1<\alpha<n,\end{cases}$$

where the Euler gamma function $\Gamma(\cdot)$ is defined by

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}\,dt \quad \bigl(\mathbb{R}(z)>0\bigr).$$

2. The Laplace transform of a function $f(t)$, $t\in(0,\infty)$ is defined by

$$\mathcal{L}\bigl[f(t)\bigr](s)=F(s)=\int_0^\infty e^{-st}f(t)\,dt \quad (s\in\mathbb{C}).$$

3. The Mittag-Leffler function (*cf.* [5, 6]) is defined by

$$E_{\alpha,\beta}(z)=\sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k+\beta)} \quad \bigl(z,\alpha,\beta\in\mathbb{C},\mathbb{R}(\alpha)>0\bigr).$$

4. The simplest Wright function (*cf.* [7, 8]) is defined by

$$\phi(\alpha,\beta;z)=\sum_{k=0}^\infty \frac{1}{\Gamma(\alpha k+\beta)}\cdot\frac{z^k}{k!} \quad (z,\alpha,\beta\in\mathbb{C}).$$

5. The general Wright function ${}_p\Psi_q(z)$ (*cf.* [7, 8]) is defined for $z\in\mathbb{C}$, complex $a_i,b_j\in\mathbb{C}$, and real $\alpha_i,\beta_j\in\mathbb{R}$ ($i=1,\dots,p$; $j=1,\dots,q$) by the series

$${}_p\Psi_q(z)={}_p\Psi_q\left[\left.\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right|z\right]:=\sum_{k=0}^\infty \frac{\prod_{i=1}^p\Gamma(a_i+\alpha_i k)}{\prod_{j=1}^q\Gamma(b_j+\beta_j k)}\cdot\frac{z^k}{k!},$$

where $z,a_i,b_j\in\mathbb{C}$, $\alpha_i,\beta_j\in\mathbb{R}$, $i=1,2,\dots,p$ and $j=1,2,\dots,q$.

6. The Riemann-Liouville fractional derivatives $D^\alpha_{a+}y$ and $D^\alpha_{b-}y$ of order $\alpha\in\mathbb{C}$ ($\Re(\alpha)\geqq 0$) are defined by

$$\bigl(D^\alpha_{a+}y\bigr)(x)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n\int_a^x\frac{y(t)\,dt}{(x-t)^{\alpha-n+1}} \quad \bigl(n=\bigl[\Re(\alpha)\bigr]+1;x>a\bigr) \tag{1.1}$$

and

$$\bigl(D^\alpha_{b-}y\bigr)(x)=\frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dx}\right)^n\int_x^b\frac{y(t)\,dt}{(t-x)^{\alpha-n+1}} \quad \bigl(n=\bigl[\Re(\alpha)\bigr]+1;x<b\bigr), \tag{1.2}$$

respectively, where $[\Re(\alpha)]$ means the integral part of $\Re(\alpha)$.

7. The Pochhammer symbol (or the shifted factorial, since $(1)_n=n!$ for $n\in N_0=\{0,1,2,\dots\}$) (*cf.* [9]) given by

$$(\lambda)_n=\begin{cases}1 & (n=0),\\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n\in\mathbb{N}_0/\{0\}).\end{cases}$$

8. The binomial coefficients are defined by

$$\binom{\lambda}{n} = \frac{\lambda!}{\lambda!(\lambda-n)!} = \frac{\lambda(\lambda-1)(\lambda-n+1)}{n!},$$

where λ and n are integers. Observe that $0! = 1$, then

$$\binom{\lambda}{0} = 1, \qquad \binom{\lambda}{\lambda} = 1 \quad \text{and} \quad (1-z)^{-\lambda} = \sum_{r=0}^{\infty} \frac{(\lambda)_r}{r!} z^r = \sum_{r=0}^{\infty} \binom{\lambda+r-1}{r} z^r.$$

Preliminaries

1. $\mathcal{L}[\phi(\alpha,\beta;t)](s) = \frac{1}{s} E_{\alpha,\beta}(\frac{1}{s})$ $(\alpha > -1, \beta \in \mathbb{C}; \mathbb{R}(s) > 0)$.
2. The Laplace transform of the generalized Wright function is given by

$$\mathcal{L}\left\{ {}_p\Psi_q \left[\begin{matrix} (a_i,\alpha_i)_{1,p} \\ (b_j,\beta_j)_{1,q} \end{matrix} \middle| -t \right] \right\}(s) = \frac{1}{s} {}_{p+1}\Psi_q \left[\begin{matrix} (1,1),(a_i,\alpha_i)_{1,p} \\ (b_j,\beta_j)_{1,q} \end{matrix} \middle| -\frac{1}{s} \right]$$

$\big(\Re(s) > 0\big), i = 1,2,\ldots,p$ and $j = 1,2,\ldots,q$.

3. $\mathcal{L}[D^\alpha f(t)](s) = s^\alpha[\mathcal{L}f(t)](s) - \sum_{k=1}^{n} s^{\alpha-k} f^{(k-1)}(0)$ (*cf.* [10]), where $\alpha > 0$, $n-1 < \alpha \le n$ $(n \in \mathbb{N})$, $f(t) \in \mathcal{C}^n(0,\infty)$, $f^{(n)}(t) \in L_1(0,b)$ for any $b > 0$.

Remark 1.1 By appropriately appealing to Definition 2, it is not difficult to prove Preliminary 3 by the technique of integral transform as follows

$$\begin{aligned}
\mathcal{L}\big[D^\alpha f(t)\big](s) &= \int_0^\infty e^{-st}\big[D^\alpha f(t)\big]\,dt \\
&= \int_0^\infty e^{-st} \cdot \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\zeta)}{(t-\zeta)^{\alpha-n+1}}\,d\zeta\,dt \\
&= \frac{1}{\Gamma(n-\alpha)} \int_0^\infty \int_\zeta^\infty e^{-st} \cdot \frac{f^{(n)}(\zeta)}{(t-\zeta)^{\alpha-n+1}}\,dt\,d\zeta \\
&= \frac{1}{\Gamma(n-\alpha)} \int_0^\infty f^{(n)}(\zeta) \int_0^\infty e^{-s(u+\zeta)} u^{n-\alpha-1}\,du\,d\zeta \\
&= \frac{1}{\Gamma(n-\alpha)} \int_0^\infty e^{-s\zeta} f^{(n)}(\zeta) \int_0^\infty e^{-su} u^{n-\alpha-1}\,du\,d\zeta \\
&= \frac{1}{\Gamma(n-\alpha)} \int_0^\infty e^{-s\zeta} f^{(n)}(\zeta) \frac{\Gamma(n-\alpha)}{s^{n-\alpha}}\,d\zeta \\
&= s^{\alpha-n} \int_0^\infty e^{-s\zeta} f^{(n)}(\zeta)\,d\zeta = s^{\alpha-n} \mathcal{L}\big[f^{(n)}(t)\big](s) \\
&= s^{\alpha-n}\big(s^n \mathcal{L}\big[f(t)\big] - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)\big) \\
&= s^\alpha \mathcal{L}\big[f(t)\big] - s^{\alpha-1}f(0) - s^{\alpha-2}f'(0) - \cdots - s^{\alpha-n}f^{(n-1)}(0) \\
&= s^\alpha \mathcal{L}\big[f(t)\big] - \sum_{k=1}^{n} s^{\alpha-k} f^{(k-1)}(0).
\end{aligned}$$

The interchange of the order of integration in the above derivation can be justified by applying Fubini's theorem.

2 Solutions of the fractional differential equations

Throughout this section, we let $y(t)$ be such that for some value of the parameter s, the Laplace transform $\mathcal{L}[y]$ converges.

Theorem 2.1 *Let $1 < \alpha < 2$ and $a, b \in \mathbb{R}$. Then the fractional differential equation*

$$y''(t) + ay^{(\alpha)}(t) + by(t) = 0 \tag{2.1}$$

with the initial conditions $y(0) = c_0$ and $y'(0) = c_1$ has its solution given by

$$\begin{aligned} y(t) = {} & c_0 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)(-at^{2-\alpha})^r}{\Gamma[(2-\alpha)r+2k+1]r!} \\ & + c_1 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k+1}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)(-at^{2-\alpha})^r}{\Gamma[(2-\alpha)r+2k+2]r!} \\ & + ac_0 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k-\alpha+2}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)(-at^{2-\alpha})^r}{\Gamma[(2-\alpha)r+2k-\alpha+3]r!} \\ & + ac_1 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k-\alpha+3}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)(-at^{2-\alpha})^r}{\Gamma[(2-\alpha)r+2k-\alpha+4]r!}. \end{aligned} \tag{2.2}$$

Proof Applying the Laplace transform (see Preliminary 3) and taking into account, we have

$$s^2\mathcal{L}[y] - c_0 s - c_1 + as^{\alpha}\mathcal{L}[y] - ac_0 s^{\alpha-1} - ac_1 s^{\alpha-2} + b\mathcal{L}[y] = 0. \tag{2.3}$$

Equation (2.3) yields

$$\begin{aligned} \mathcal{L}[y] &= \frac{c_0 s + c_1 + ac_0 s^{\alpha-1} + ac_1 s^{\alpha-2}}{s^2 + as^{\alpha} + b} \\ &= c_0 \sum_{k=0}^{\infty} (-b)^k \sum_{k=0}^{\infty} \binom{k+r}{r} (-a)^r s^{(\alpha-2)r-2k-1} \\ &\quad + c_1 \sum_{k=0}^{\infty} (-b)^k \sum_{k=0}^{\infty} \binom{k+r}{r} (-a)^r s^{(\alpha-2)r-2k-2} \\ &\quad + ac_0 \sum_{k=0}^{\infty} (-b)^k \sum_{k=0}^{\infty} \binom{k+r}{r} (-a)^r s^{(\alpha-2)r-2k+\alpha-3} \\ &\quad + ac_1 \sum_{k=0}^{\infty} (-b)^k \sum_{k=0}^{\infty} \binom{k+r}{r} (-a)^r s^{(\alpha-2)r-2k+\alpha-4}, \end{aligned} \tag{2.4}$$

since

$$\begin{aligned} \frac{1}{s^2 + as^{\alpha} + b} &= \frac{s^{-\alpha}}{s^{2-\alpha} + a + bs^{-\alpha}} = \frac{s^{-\alpha}}{(s^{2-\alpha} + a)(1 + \frac{bs^{-\alpha}}{s^{2-\alpha}+a})} \\ &= \frac{s^{-\alpha}}{s^{2-\alpha} + a} \sum_{k=0}^{\infty} \left(\frac{-bs^{-\alpha}}{s^{2-\alpha} + a} \right)^k = \sum_{k=0}^{\infty} \frac{(-b)^k s^{-\alpha k - \alpha}}{(s^{2-\alpha} + a)^{k+1}} = \sum_{k=0}^{\infty} \frac{(-b)^k s^{-2k-2}}{(1 + as^{\alpha-2})^{k+1}} \end{aligned}$$

$$= \sum_{k=0}^{\infty}(-b)^k s^{-2k-2} \sum_{r=0}^{\infty}\left(-as^{\alpha-2}\right)^r \binom{k+r}{r}$$
$$= \sum_{k=0}^{\infty}(-b)^k \sum_{r=0}^{\infty} \binom{k+r}{r}(-a)^r s^{(\alpha-2)r-2k-2}. \tag{2.5}$$

Thus, from Equation (2.4), we derive the following solution by the inverse Laplace transform to Equation (2.2):

$$\begin{aligned}
y(t) &= c_0 \sum_{k=0}^{\infty} \frac{(-b)^k}{k!} \sum_{r=0}^{\infty} \frac{(k+r)!(-a)^r}{\Gamma[(2-\alpha)r+2k+1]} \cdot \frac{t^{(2-\alpha)r+2k}}{r!} \\
&\quad + c_1 \sum_{k=0}^{\infty} \frac{(-b)^k}{k!} \sum_{r=0}^{\infty} \frac{(k+r)!(-a)^r}{\Gamma[(2-\alpha)r+2k+2]} \cdot \frac{t^{(2-\alpha)r+2k+1}}{r!} \\
&\quad + ac_0 \sum_{k=0}^{\infty} \frac{(-b)^k}{k!} \sum_{r=0}^{\infty} \frac{(k+r)!(-a)^r}{\Gamma[(2-\alpha)r+2k-\alpha+3]} \cdot \frac{t^{(2-\alpha)r+2k-\alpha+2}}{r!} \\
&\quad + ac_1 \sum_{k=0}^{\infty} \frac{(-b)^k}{k!} \sum_{r=0}^{\infty} \frac{(k+r)!(-a)^r}{\Gamma[(2-\alpha)r+2k-\alpha+4]} \cdot \frac{t^{(2-\alpha)r+2k-\alpha+3}}{r!} \\
&= c_0 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)(-at^{2-\alpha})^r}{\Gamma[(2-\alpha)r+2k+1]r!} \\
&\quad + c_1 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k+1}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)(-at^{2-\alpha})^r}{\Gamma[(2-\alpha)r+2k+2]r!} \\
&\quad + ac_0 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k-\alpha+2}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)(-at^{2-\alpha})^r}{\Gamma[(2-\alpha)r+2k-\alpha+3]r!} \\
&\quad + ac_1 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k-\alpha+3}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)(-at^{2-\alpha})^r}{\Gamma[(2-\alpha)r+2k-\alpha+4]r!}.
\end{aligned}$$

□

Example 2.1 The fractional differential equation of a generalized viscoelastic free damping oscillation (*cf.* [1])

$$y''(t) + ay^{(\frac{3}{2})}(t) + by(t) = 0 \tag{2.6}$$

with the initial conditions $y(0) = c_0$ and $y'(0) = c_1$ has its solution given by

$$\begin{aligned}
y(t) &= c_0 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)(-at^{\frac{1}{2}})^r}{\Gamma[\frac{1}{2}r+2k+1]r!} \\
&\quad + c_1 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k+1}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)(-at^{\frac{1}{2}})^r}{\Gamma[\frac{1}{2}r+2k+2]r!} \\
&\quad + ac_0 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k+\frac{1}{2}}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)(-at^{\frac{1}{2}})^r}{\Gamma[\frac{1}{2}r+2k+\frac{3}{2}]r!} \\
&\quad + ac_1 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k+\frac{3}{2}}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)(-at^{2-\alpha})^r}{\Gamma[(2-\alpha)r+2k+\frac{5}{2}]r!}.
\end{aligned} \tag{2.7}$$

In particular, if $a = \sqrt{3}$ and $b = 8$, then the equation

$$y''(t) + \sqrt{3}y^{(\frac{3}{2})}(t) + 8y(t) = 0 \tag{2.8}$$

with the initial conditions $y(0) = c_0$ and $y'(0) = c_1$ has its solution given by

$$\begin{aligned} y(t) = {} & c_0 \sum_{k=0}^{\infty} \frac{(-8)^k t^{2k}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)(-\sqrt{3}t^{\frac{1}{2}})^r}{\Gamma[\frac{1}{2}r + 2k + 1]r!} \\ & + c_1 \sum_{k=0}^{\infty} \frac{(-8)^k t^{2k+1}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)(-\sqrt{3}t^{\frac{1}{2}})^r}{\Gamma[\frac{1}{2}r + 2k + 2]r!} \\ & + \sqrt{3}c_0 \sum_{k=0}^{\infty} \frac{(-8)^k t^{2k+\frac{1}{2}}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)(-\sqrt{3}t^{\frac{1}{2}})^r}{\Gamma[\frac{1}{2}r + 2k + \frac{3}{2}]r!} \\ & + \sqrt{3}c_1 \sum_{k=0}^{\infty} \frac{(-8)^k t^{2k+\frac{3}{2}}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)(-\sqrt{3}t^{\frac{1}{2}})^r}{\Gamma[\frac{r}{2} + 2k + \frac{5}{2}]r!}. \end{aligned} \tag{2.9}$$

Theorem 2.2 *Let $1 < \alpha < 2$ and $a, b \in \mathbb{R}$. Then the fractional differential equation*

$$y^{(\alpha)}(t) + ay'(t) + by(t) = 0 \tag{2.10}$$

with the initial conditions $y(0) = c_0$ and $y'(0) = c_1$ has its solution given by

$$\begin{aligned} y(t) = {} & c_0 \sum_{k=0}^{\infty} \frac{(-b)^k}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)(-a)^r t^{(\alpha-1)r+\alpha k}}{\Gamma[(\alpha-1)r + \alpha k + 1]r!} \\ & + c_1 \sum_{k=0}^{\infty} \frac{(-b)^k}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)(-a)^r t^{(\alpha-1)r+\alpha k+1}}{\Gamma[(\alpha-1)r + \alpha k + 2]r!} \\ & + ac_0 \sum_{k=0}^{\infty} \frac{(-b)^k}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)(-a)^r t^{(\alpha-1)r+\alpha k+\alpha-1}}{\Gamma[(\alpha-1)r + \alpha k + \alpha]r!}. \end{aligned} \tag{2.11}$$

Proof Applying the Laplace transform (see Preliminary 3) and taking into account, we have

$$s^{\alpha}\mathcal{L}[y] - s^{\alpha-1}y(0) - s^{\alpha-2}y'(0) + as\mathcal{L}[y] - ay'(0) + b\mathcal{L}[y] = 0.$$

That is,

$$\bigl(s^{\alpha} + as + b\bigr)\mathcal{L}[y] = c_0 s^{\alpha-1} + c_1 s^{\alpha-2} + ac_0. \tag{2.12}$$

Equation (2.12) yields

$$\begin{aligned} \mathcal{L}[y] &= \frac{c_0 s^{\alpha-1} + c_1 s^{\alpha-2} + ac_0}{s^{\alpha} + as + b} \\ &= c_0 \sum_{k=0}^{\infty} (-b)^k \sum_{k=0}^{\infty} \binom{k+r}{r} (-a)^r s^{r-\alpha r-\alpha k-1} \end{aligned}$$

$$+ c_1 \sum_{k=0}^{\infty} (-b)^k \sum_{k=0}^{\infty} \binom{k+r}{r} (-a)^r s^{r-\alpha r-\alpha k-2}$$
$$+ ac_0 \sum_{k=0}^{\infty} (-b)^k \sum_{k=0}^{\infty} \binom{k+r}{r} (-a)^r s^{r-\alpha r-\alpha k-\alpha}, \tag{2.13}$$

since

$$\begin{aligned}
\frac{1}{s^\alpha + as + b} &= \frac{s^{-1}}{s^{\alpha-1} + a + bs^{-1}} = \frac{s^{-1}}{(s^{\alpha-1}+a)(1 + \frac{bs^{-1}}{s^{\alpha-1}+a})} \\
&= \frac{s^{-1}}{s^{\alpha-1}+a} \sum_{k=0}^{\infty} (-1)^k \left(\frac{bs^{-1}}{s^{\alpha-1}+a} \right)^k \\
&= \sum_{k=0}^{\infty} \frac{(-b)^k s^{-k-1}}{(s^{\alpha-1}+a)^{k+1}} = \sum_{k=0}^{\infty} \frac{(-b)^k s^{-\alpha k-\alpha}}{(1 + as^{1-\alpha})^{k+1}} \\
&= \sum_{k=0}^{\infty} (-b)^k s^{-\alpha k-\alpha} \sum_{r=0}^{\infty} \binom{k+r}{r} \left(-as^{1-\alpha}\right)^r \\
&= \sum_{k=0}^{\infty} (-b)^k \sum_{r=0}^{\infty} \binom{k+r}{r} (-a)^r s^{r-\alpha r-\alpha k-\alpha}.
\end{aligned} \tag{2.14}$$

Thus, from Equation (2.13), we derive the following solution by the inverse Laplace transform to Equation (2.11):

$$\begin{aligned}
y(t) &= c_0 \sum_{k=0}^{\infty} \frac{(-b)^k}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)(-a)^r}{\Gamma[(\alpha-1)r + \alpha k + 1]} \cdot \frac{t^{(\alpha-1)r+\alpha k}}{r!} \\
&\quad + c_1 \sum_{k=0}^{\infty} \frac{(-b)^k}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)(-a)^r}{\Gamma[(\alpha-1)r + \alpha k + 2]} \cdot \frac{t^{(\alpha-1)r+\alpha k+1}}{r!} \\
&\quad + ac_0 \sum_{k=0}^{\infty} \frac{(-b)^k}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)(-a)^r}{\Gamma[(\alpha-1)r + \alpha k + \alpha]} \cdot \frac{t^{(\alpha-1)r+\alpha k+\alpha-1}}{r!}.
\end{aligned}$$

This solution can be expressed by the Wright function as

$$\begin{aligned}
y(t) &= c_0 \sum_{k=0}^{\infty} \frac{(-b)^k t^{\alpha k}}{k!} {}_1\Psi_1 \left[\begin{matrix} (k+1,1) \\ (\alpha k+1, \alpha-1) \end{matrix} \middle| -at^{\alpha-1} \right] \\
&\quad + c_1 \sum_{k=0}^{\infty} \frac{(-b)^k t^{\alpha k+1}}{k!} {}_1\Psi_1 \left[\begin{matrix} (k+1,1) \\ (\alpha k+2, \alpha-1) \end{matrix} \middle| -at^{\alpha-1} \right] \\
&\quad + ac_0 \sum_{k=0}^{\infty} \frac{(-b)^k t^{\alpha k+\alpha-1}}{k!} {}_1\Psi_1 \left[\begin{matrix} (k+1,1) \\ (\alpha k+\alpha, \alpha-1) \end{matrix} \middle| -at^{\alpha-1} \right].
\end{aligned}$$

□

Example 2.2 If we let $\alpha = \frac{3}{2}$, $a = -1$ and $b = -2$ in Theorem 2.2, then the equation

$$y^{(\frac{3}{2})}(t) - y'(t) - 2y(t) = 0$$

has a solution

$$\begin{aligned} y(t) &= c_0 \sum_{k=0}^{\infty} \frac{2^k}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)t^{\frac{r}{2}+\frac{3}{2}k}}{\Gamma(\frac{r}{2}+\frac{3}{2}k+1)r!} \\ &\quad + c_1 \sum_{k=0}^{\infty} \frac{2^k}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)t^{\frac{r}{2}+\frac{3}{2}k+1}}{\Gamma(\frac{r}{2}+\frac{3}{2}k+2)r!} \\ &\quad - c_0 \sum_{k=0}^{\infty} \frac{2^k}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)t^{\frac{r}{2}+\frac{3}{2}k+\frac{1}{2}}}{\Gamma(\frac{r}{2}+\frac{3}{2}k+\frac{3}{2})r!}. \end{aligned}$$

Theorem 2.3 *Let* $0 < \alpha < 1$ *and* $b \in \mathbb{R}$. *Then the equation*

$$y^{(\alpha)}(t) - by(t) = 0 \tag{2.15}$$

with the initial condition $y(0) = c_0$ *has its solution given by*

$$\begin{aligned} y(t) &= c_0 \sum_{k=0}^{\infty} \frac{(bt^{\alpha})^k}{\Gamma(\alpha k+1)} \\ &= c_0 E_{\alpha,1}\bigl(bt^{\alpha}\bigr). \end{aligned} \tag{2.16}$$

Proof Applying the Laplace transform to Equation (2.15), that is,

$$s^{\alpha}\mathcal{L}[y] - c_0 s^{\alpha-1} - b\mathcal{L}[y] = 0,$$

we have

$$\begin{aligned} \mathcal{L}[y] &= \frac{c_0 s^{\alpha-1}}{s^{\alpha}-b} = \frac{c_0 s^{-1}}{1-bs^{-\alpha}} = c_0 s^{-1} \sum_{k=0}^{\infty} \bigl(bs^{-\alpha}\bigr)^k = c_0 \sum_{k=0}^{\infty} b^k s^{-\alpha k-1}, \\ y(t) &= c_0 \sum_{k=0}^{\infty} \frac{(bt^{\alpha})^k}{\Gamma(\alpha k+1)} = c_0 E_{\alpha,1}\bigl(bt^{\alpha}\bigr). \end{aligned}$$

□

Remark 2.1 If $a = 0$ in Equation (2.10), then the equation

$$y^{\alpha}(t) + by(t) = 0, \quad 1 < \alpha \le 2 \tag{2.17}$$

with the initial conditions $y(0) = c_0$ and $y'(0) = c_1$ has its solution given by

$$\begin{aligned} y(t) &= c_0 \sum_{k=0}^{\infty} \frac{(-bt^{\alpha})^k}{\Gamma(\alpha k+1)} + c_1 t \sum_{k=0}^{\infty} \frac{(-bt^{\alpha})^k}{\Gamma(\alpha k+2)} \\ &= c_0 E_{\alpha,1}\bigl(-bt^{\alpha}\bigr) + c_1 t E_{\alpha,2}\bigl(-bt^{\alpha}\bigr). \end{aligned} \tag{2.18}$$

Theorem 2.4 *A nearly simple harmonic vibration equation* (*cf.* [1])

$$y^{\alpha}(t) + w^2 y(t) = 0, \quad 1 < \alpha \le 2 \tag{2.19}$$

with the initial conditions $y(0) = c_0$ *and* $y'(0) = c_1$ *has its solution given by*

$$y(t) = c_0 E_{\alpha,1}\left(-w^2 t^\alpha\right) + c_1 t E_{\alpha,2}\left(-w^2 t^\alpha\right). \tag{2.20}$$

Proof We complete this proof by putting $b = w^2$ in Equation (2.17). □

In fact, by applying the Laplace transform to a linear fractional differential equation with the initial conditions, we can easily derive its solutions as the previous forms in this paper.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
S-DL carried out the molecular genetic studies, participated in the sequence alignment and drafted the manuscript. C-HL participated in the sequence alignment.

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