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On $\mathcal{I}$-asymptotically lacunary statistical equivalent sequences

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Abstract

This paper presents the following definition, which is a natural combination of the definitions for asymptotically equivalent, $\mathcal{I}$-statistically limit and $\mathcal{I}$-lacunary statistical convergence. Let θ be a lacunary sequence; the two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be $\mathcal{I}$-asymptotically lacunary statistical equivalent of multiple L provided that for every $\epsilon > 0$, and $\delta > 0$,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\left\{k \in I_r : \left|\frac{x_k}{y_k} - L\right| \geq \varepsilon\right\}\right| \geq \delta\right\} \in \mathcal{I}$$

(denoted by $x \overset{S_\theta^L(\mathcal{I})}{\sim} y$) and simply $\mathcal{I}$-asymptotically lacunary statistical equivalent if $L = 1$.

MSC: Primary 40A99; secondary 40A05

Keywords: asymptotical equivalent; ideal convergence; lacunary sequence; statistical convergence

1 Introduction

In 1993, Marouf [1] presented definitions for asymptotically equivalent sequences and asymptotic regular matrices. In 2003, Patterson [2] extended these concepts by presenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices.

In [3], asymptotically lacunary statistical equivalent, which is a natural combination of the definitions for asymptotically equivalent, statistical convergence and lacunary sequences. Later on, the extension asymptotically lacunary statistical equivalent sequences is presented (see [4]).

Recently, Das, Savaş and Ghosal [5] introduced new notions, namely $\mathcal{I}$-statistical convergence and $\mathcal{I}$-lacunary statistical convergence by using ideal.

In this short paper, we shall use asymptotical equivalent and lacunary sequence to introduce the concepts $\mathcal{I}$-asymptotically statistical equivalent and $\mathcal{I}$-asymptotically lacunary statistical equivalent. In addition to these definitions, natural inclusion theorems shall also be presented.

First, we introduce some definitions.

2 Definitions and notations

Definition 2.1 (Marouf [1]) Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be *asymptotically equivalent* if

$$\lim_k \frac{x_k}{y_k} = 1$$

(denoted by $x \sim y$).

Definition 2.2 (Fridy [6]) The sequence $x = (x_k)$ has *statistic limit* L, denoted by $st\text{-}\lim s = L$ provided that for every $\epsilon > 0$,

$$\lim_n \frac{1}{n}\bigl\{\text{the number of } k \le n : |x_k - L| \ge \epsilon\bigr\} = 0.$$

The next definition is natural combination of Definitions 2.1 and 2.2.

Definition 2.3 (Patterson [2]) Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically statistical equivalent of multiple L provided that for every $\epsilon > 0$,

$$\lim_n \frac{1}{n}\left\{\text{the number of } k < n : \left|\frac{x_k}{y_k} - L\right| \ge \epsilon\right\} = 0$$

(denoted by $x \overset{S_L}{\sim} y$) and simply asymptotically statistical equivalent if $L = 1$.

By a lacunary $\theta = (k_r)$; $r = 0, 1, 2, \dots$, where $k_0 = 0$, we shall mean an increasing sequence of nonnegative integers with $k_r - k_{r-1}$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and $h_r = k_r - k_{r-1}$. The ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r.

Definition 2.4 ([3]) Let θ be a lacunary sequence; the two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically lacunary statistical equivalent of multiple L provided that for every $\epsilon > 0$

$$\lim_r \frac{1}{h_r}\left|\left\{k \in I_r : \left|\frac{x_k}{y_k} - L\right| \ge \epsilon\right\}\right| = 0$$

(denoted by $x \overset{S_\theta^L}{\sim} y$) and simply asymptotically lacunary statistical equivalent if $L = 1$.

More investigations in this direction and more applications of asymptotically statistical equivalent can be found in [7, 8] where many important references can be found.

The following definitions and notions will be needed.

Definition 2.5 ([9]) A nonempty family $\mathcal{I} \subset 2^Y$ of subsets a nonempty set Y is said to be an ideal in Y if the following conditions hold:

(i) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$;
(ii) $A \in \mathcal{I}$, $B \subset A$ imply $B \in \mathcal{I}$.

Definition 2.6 ([10]) A nonempty family $\mathcal{F} \subset 2^{\mathbb{N}}$ is said to be a filter of $\mathbb{N}$ if the following conditions hold:

(i) $\emptyset \notin \mathcal{F}$;
(ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$;
(iii) $A \in \mathcal{F}$, $B \subset A$ imply $B \in \mathcal{F}$.

If $\mathcal{I}$ is proper ideal of $\mathbb{N}$ (*i.e.*, $\mathbb{N} \notin \mathcal{I}$), then the family of sets $\mathcal{F}(\mathcal{I}) = \{M \subset \mathbb{N} : \exists A \in \mathcal{I} : M = \mathbb{N} \backslash A\}$ is a filter of $\mathbb{N}$. It is called the filter associated with the ideal.

Definition 2.7 ([9, 10]) A proper ideal $\mathcal{I}$ is said to be admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

Throughout $\mathcal{I}$ will stand for a proper admissible ideal of $\mathbb{N}$, and by sequence we always mean sequences of real numbers.

Definition 2.8 ([9]) Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be a proper admissible ideal in $\mathbb{N}$.

The sequence (x_k) of elements of $\mathbb{R}$ is said to be $\mathcal{I}$-convergent to $L \in \mathbb{R}$ if for each $\epsilon > 0$ the set $A(\epsilon) = \{n \in \mathbb{N} : |x_n - L| \geq \epsilon\} \in \mathcal{I}$.

Following these results, we introduce two new notions $\mathcal{I}$-asymptotically lacunary statistical equivalent of multiple L and strong $\mathcal{I}$-asymptotically lacunary equivalent of multiple L.

The following definitions are given in [5].

Definition 2.9 A sequence $x = (x_k)$ is said to be $\mathcal{I}$-statistically convergent to L or $S(\mathcal{I})$-convergent to L if, for any $\varepsilon > 0$ and $\delta > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : |x_k - L| \geq \varepsilon\right\}\right| \geq \delta\right\} \in \mathcal{I}.$$

In this case, we write $x_k \to L(S(\mathcal{I}))$. The class of all $\mathcal{I}$-statistically convergent sequences will be denoted by $S(\mathcal{I})$.

Definition 2.10 Let θ be a lacunary sequence. A sequence $x = (x_k)$ is said to be $\mathcal{I}$-lacunary statistically convergent to L or $S_\theta(\mathcal{I})$-convergent to L if, for any $\varepsilon > 0$ and $\delta > 0$,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\left\{k \in I_r : |x_k - L| \geq \varepsilon\right\}\right| \geq \delta\right\} \in \mathcal{I}.$$

In this case, we write $x_k \to L(S_\theta(\mathcal{I}))$. The class of all $\mathcal{I}$-lacunary statistically convergent sequences will be denoted by $S_\theta(\mathcal{I})$.

Definition 2.11 Let θ be a lacunary sequence. A sequence $x = (x_k)$ is said to be strong $\mathcal{I}$-lacunary convergent to L or $N_\theta(\mathcal{I})$-convergent to L if, for any $\varepsilon > 0$

$$\left\{r \in N : \frac{1}{h_r}\sum_{k \in I_r} |x_k - L| \geq \varepsilon\right\} \in \mathcal{I}.$$

In this case, we write $x_k \to L(N_\theta(\mathcal{I}))$. The class of all strong $\mathcal{I}$-lacunary statistically convergent sequences will be denoted by $N_\theta(\mathcal{I})$.

3 New definitions

The next definitions are combination of Definitions 2.1, 2.9, 2.10 and 2.11.

Definition 3.1 Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be $\mathcal{I}$-asymptotically statistical equivalent of multiple L provided that for every $\epsilon > 0$ and $\delta > 0$,

$$\left\{n \in N : \frac{1}{n}\left|\left\{k \le n : \left|\frac{x_k}{y_k} - L\right| \ge \varepsilon\right\}\right| \ge \delta\right\} \in \mathcal{I}$$

(denoted by $x \overset{S^L(\mathcal{I})}{\sim} y$) and simply $\mathcal{I}$-asymptotically statistical equivalent if $L = 1$.

For $\mathcal{I} = \mathcal{I}_{\text{fin}}$, $\mathcal{I}$-asymptotically statistical equivalent of multiple L coincides with asymptotically statistical equivalent of multiple L, which is defined in [3].

Definition 3.2 Let θ be a lacunary sequence; the two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be $\mathcal{I}$-asymptotically lacunary statistical equivalent of multiple L provided that for every $\epsilon > 0$ and $\delta > 0$,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\left\{k \in I_r : \left|\frac{x_k}{y_k} - L\right| \ge \varepsilon\right\}\right| \ge \delta\right\} \in \mathcal{I}$$

(denoted by $x \overset{S_\theta^L(\mathcal{I})}{\sim} y$) and simply $\mathcal{I}$-asymptotically lacunary statistical equivalent if $L = 1$.

For $\mathcal{I} = \mathcal{I}_{\text{fin}}$, $\mathcal{I}$-asymptotically lacunary statistical equivalent of multiple L coincides with asymptotically lacunary statistical equivalent of multiple L, which is defined in [3].

Definition 3.3 Let θ be a lacunary sequence; two number sequences $x = (x_k)$ and $y = (y_k)$ are strong $\mathcal{I}$-asymptotically lacunary equivalent of multiple L provided that

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k \in I_r}\left|\frac{x_k}{y_k} - L\right| \ge \varepsilon\right\} \in \mathcal{I}$$

(denoted by $x \overset{N_\theta^L(\mathcal{I})}{\sim} y$) and strong simply $\mathcal{I}$-asymptotically lacunary equivalent if $L = 1$.

4 Main result

In this section, we state and prove the results of this article.

Theorem 4.1 *Let $\theta = \{k_r\}$ be a lacunary sequence then*

(1) (a) *If $x \overset{N_\theta^L(\mathcal{I})}{\sim} y$ then $x \overset{S_\theta^L(\mathcal{I})}{\sim} y$,*

(b) *$x \overset{N_\theta^L(\mathcal{I})}{\sim} y$ is a proper subset of $x \overset{S_\theta^L(\mathcal{I})}{\sim} y$;*

(2) *If $x, y \in l_\infty$ and $x \overset{S_\theta^L(\mathcal{I})}{\sim} y$ then $x \overset{N_\theta^L(\mathcal{I})}{\sim} y$;*

(3) *$x \overset{S_\theta^L(\mathcal{I})}{\sim} y \cap l_\infty = x \overset{N_\theta^L(\mathcal{I})}{\sim} y \cap l_\infty$,*

where l_∞ denote the set of bounded sequences.

Proof Part (1a): If $\epsilon > 0$ and $x \overset{N_\theta^L(\mathcal{I})}{\sim} y$ then

$$\begin{aligned}\sum_{k\in I_r}\left|\frac{x_k}{y_k}-L\right| &\geq \sum_{k\in I_r \& |\frac{x_k}{y_k}-L|\geq\epsilon}\left|\frac{x_k}{y_k}-L\right| \\ &\geq \epsilon\left|\left\{k\in I_r:\left|\frac{x_k}{y_k}-L\right|\geq\epsilon\right\}\right|\end{aligned}$$

and so

$$\frac{1}{\varepsilon h_r}\sum_{k\in I_r}\left|\frac{x_k}{y_k}-L\right| \geq \frac{1}{h_r}\left|\left\{k\in I_r:\left|\frac{x_k}{y_k}-L\right|\geq\epsilon\right\}\right|.$$

Then, for any $\delta > 0$,

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\left|\left\{k\in I_r:\left|\frac{x_k}{y_k}-L\right|\geq\epsilon\right\}\right|\geq\delta\right\}\subseteq\left\{r\in N:\frac{1}{h_r}\sum_{k\in I_r}\left|\frac{x_k}{y_k}-L\right|\geq\epsilon\cdot\delta\right\}\in\mathcal{I}.$$

Hence, we have $x \overset{S_\theta^L(\mathcal{I})}{\sim} y$.

Part (1b): $x \overset{N_\theta^L(\mathcal{I})}{\sim} y \subset x \overset{S_\theta^L(\mathcal{I})}{\sim} y$, let $x = (x_k)$ be defined as follows: x_k to be $1, 2, \ldots, [\sqrt{h_r}]$ at the first $[\sqrt{h_r}]$ integers in I_r and zero otherwise. $y_k = 1$ for all k. These two satisfy the following $x \overset{S_\theta^L(\mathcal{I})}{\sim} y$, but the following fails $x \overset{N_\theta^L(\mathcal{I})}{\sim} y$.

Part (2): Suppose $x = (x_k)$ and $y = (y_k)$ are in l_∞ and $x \overset{S_\theta^L(\mathcal{I})}{\sim} y$. Then we can assume that

$$\left|\frac{x_k}{y_k}-L\right| \leq M \quad \text{for all } k.$$

Given $\epsilon > 0$, we have

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r}\left|\frac{x_k}{y_k}-L\right| &= \frac{1}{h_r}\sum_{k\in I_r \& |\frac{x_k}{y_k}-L|\geq\epsilon}\left|\frac{x_k}{y_k}-L\right| + \frac{1}{h_r}\sum_{k\in I_r \& |\frac{x_k}{y_k}-L|<\epsilon}\left|\frac{x_k}{y_k}-L\right| \\ &\leq \frac{M}{h_r}\left|\left\{k\in I_r:\left|\frac{x_k}{y_k}-L\right|\geq\frac{\epsilon}{2}\right\}\right| + \frac{\epsilon}{2}.\end{aligned}$$

Consequently, we have

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in I_r}\left|\frac{x_k}{y_k}-L\right|\geq\varepsilon\right\}\subseteq\left\{r\in N:\frac{1}{h_r}\left|\left\{k\in I_r:\left|\frac{x_k}{y_k}-L\right|\geq\frac{\epsilon}{2}\right\}\right|\geq\frac{\varepsilon}{2M}\right\}\in\mathcal{I}.$$

Therefore, $x \overset{N_\theta^L(\mathcal{I})}{\sim} y$.

Part (3): Follows from (1) and (2). □

Theorem 4.2 *Let $\mathcal{I}$ is an ideal and $\theta = \{k_r\}$ is a lacunary sequence with* $\liminf q_r > 1$, *then*

$$x \overset{S^L(\mathcal{I})}{\sim} y \quad \textit{implies} \quad x \overset{S_\theta^L(\mathcal{I})}{\sim} y.$$

Proof Suppose first that $\liminf q_r > 1$, then there exists a $\delta > 0$ such that $q_r \geq 1 + \delta$ for sufficiently large r, which implies

$$\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta}.$$

If $x \overset{S_\theta^L(\mathcal{I})}{\sim} y$, then for every $\varepsilon > 0$ and for sufficiently large r, we have

$$\begin{aligned}\frac{1}{k_r}\left|\left\{k \leq k_r : \left|\frac{x_k}{y_k} - L\right| \geq \epsilon\right\}\right| &\geq \frac{1}{k_r}\left|\left\{k \in I_r : \left|\frac{x_k}{y_k} - L\right| \geq \epsilon\right\}\right| \\ &\geq \frac{\delta}{1+\delta}\frac{1}{h_r}\left|\left\{k \in I_r : \left|\frac{x_k}{y_k} - L\right| \geq \epsilon\right\}\right|.\end{aligned}$$

Then, for any $\delta > 0$, we get

$$\begin{aligned}&\left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\left\{k \in I_r : \left|\frac{x_k}{y_k} - L\right| \geq \varepsilon\right\}\right| \geq \delta\right\} \\ &\quad\subseteq \left\{r \in N : \frac{1}{k_r}\left|\left\{k \leq k_r : \left|\frac{x_k}{y_k} - L\right| \geq \epsilon\right\}\right| \geq \frac{\delta\alpha}{(1+\alpha)}\right\} \in \mathcal{I}.\end{aligned}$$

This completes the proof. □

For the next result we assume that the lacunary sequence θ satisfies the condition that for any set $C \in F(\mathcal{I})$, $\bigcup\{n : k_{r-1} < n < k_r, r \in C\} \in F(\mathcal{I})$.

Theorem 4.3 *Let $\mathcal{I}$ is an ideal and $\theta = (k_r)$ is a lacunary sequence with $\sup q_r < \infty$, then*

$$x \overset{S_\theta^L(\mathcal{I})}{\sim} y \quad \textit{implies} \quad x \overset{S^L(\mathcal{I})}{\sim} y.$$

Proof If $\limsup_r q_r < \infty$, then without any loss of generality, we can assume that there exists a $0 < B < \infty$ such that $q_r < B$ for all $r \geq 1$. Suppose that $x \overset{S_\theta^L}{\sim} y$ and for $\epsilon, \delta, \delta_1 > 0$ define the sets

$$C = \left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\left\{k \in I_r : \left|\frac{x_k}{y_k} - L\right| \geq \epsilon\right\}\right| < \delta\right\}$$

and

$$T = \left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \left|\frac{x_k}{y_k} - L\right| \geq \epsilon\right\}\right| < \delta_1\right\}.$$

It is obvious from our assumption that $C \in F(\mathcal{I})$, the filter associated with the ideal $\mathcal{I}$. Further observe that

$$A_j = \frac{1}{h_j}\left|\left\{k \in I_j : \left|\frac{x_k}{y_k} - L\right| \geq \epsilon\right\}\right| < \delta$$

for all $j \in C$. Let $n \in \mathbb{N}$ be such that $k_{r-1} < n < k_r$ for some $r \in C$. Now

$$\begin{aligned}
&\frac{1}{n}\left|\left\{k \leq n : \left|\frac{x_k}{y_k} - L\right| \geq \epsilon\right\}\right| \\
&\quad \leq \frac{1}{k_{r-1}}\left|\left\{k \leq k_r : \left|\frac{x_k}{y_k} - L\right| \geq \epsilon\right\}\right| \\
&\quad = \frac{1}{k_{r-1}}\left|\left\{k \in I_1 : \left|\frac{x_k}{y_k} - L\right| \geq \epsilon\right\}\right| + \cdots + \frac{1}{k_{r-1}}\left|\left\{k \in I_r : \left|\frac{x_k}{y_k} - L\right| \geq \epsilon\right\}\right| \\
&\quad = \frac{k_1}{k_{r-1}}\frac{1}{h_1}\left|\left\{k \in I_1 : \left|\frac{x_k}{y_k} - L\right| \geq \epsilon\right\}\right| + \frac{k_2 - k_1}{k_{r-1}}\frac{1}{h_2}\left|\left\{k \in I_2 : \left|\frac{x_k}{y_k} - L\right| \geq \epsilon\right\}\right| + \cdots \\
&\qquad + \frac{k_r - k_{r-1}}{k_{r-1}}\frac{1}{h_r}\left|\left\{k \in I_r : \left|\frac{x_k}{y_k} - L\right| \geq \epsilon\right\}\right| \\
&\quad = \frac{k_1}{k_{r-1}}A_1 + \frac{k_2 - k_1}{k_{r-1}}A_2 + \cdots + \frac{k_r - k_{r-1}}{k_{r-1}}A_r \\
&\quad \leq \sup_{j \in C} A_j \cdot \frac{k_r}{k_{r-1}} < B\delta.
\end{aligned}$$

Choosing $\delta_1 = \frac{\delta}{B}$ and in view of the fact that $\bigcup\{n : k_{r-1} < n < k_r, r \in C\} \subset T$ where $C \in F(\mathcal{I})$, it follows from our assumption on θ that the set T also belongs to $F(\mathcal{I})$ and this completes the proof of the theorem. □

Competing interests
The author declares that they have no competing interests.

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