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*Research Article*

# Transformations of Difference Equations I

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We consider a general weighted second-order difference equation. Two transformations are studied which transform the given equation into another weighted second order difference equation of the same type, these are based on the Crum transformation. We also show how Dirichlet and non-Dirichlet boundary conditions transform as well as how the spectra and norming constants are affected.

## 1. Introduction

Our interest in this topic arose from the work done on transformations and factorisations of continuous (as opposed to discrete) Sturm-Liouville boundary value problems by, amongst others, Binding et al., notably [1, 2]. We make use of similar ideas to those discussed in [3–5] to study the transformations of difference equations.

In this paper, we consider a weighted second-order difference equation of the form

$$ly := -c(n)y(n+1) + b(n)y(n) - c(n-1)y(n-1) = c(n)\lambda y(n), \tag{1.1}$$

where $c(n) > 0$ represents a weight function and $b(n)$ a potential function.

Two factorisations of the formal difference operator, $l$, associated with (1.1), are given. Although there may be many alternative factorisations of this operator (see e.g., [2, 6]), the factorisations given in Theorems 2.1 and 3.1 are of particular interest to us as they are analogous to those used in the continuous Sturm-Liouville case. Moreover, if the original operator is factorised by $SQ$, as in Theorem 2.1, or by $PR$, as in Theorem 3.1, then the Darboux-Crum type transformation that we wish to study is given by the mapping $Q$ or $R$, respectively. This results in eigenfunctions of the difference boundary value problem being transformed to eigenfunctions of another, so-called, transformed boundary value problem given by permuting the factors $S$ and $Q$ or the factors $P$ and $R$, that is, by $QS$ or $RP$, respectively, as in the continuous case. Applying this transformation must then result in a transformed equation of exactly the same type as the original equation. In order to ensure this, we require that the original difference equation which we consider has the form given in (1.1). In particular the weight, $c(n)$, also determines the dependence on the off-diagonal elements. We note that the more general equation

$$c(n)y(n+1) - b(n)y(n) + c(n-1)y(n-1) = -a(n)\lambda y(n), \tag{1.2}$$

can be factorised as $SQ$, however, reversing the factors that is, finding $QS$ does not necessarily result in a transformed equation of the same type as (1.2). The importance of obtaining a transformed equation of exactly the same form as the original equation, is that ultimately we will (in a sequel to the current paper) use these transformations to establish a hierarchy of boundary value problems with (1.1) and various boundary conditions; see [4] for the differential equations case. Initially we transform, in this paper, non-Dirichlet boundary conditions to Dirichlet boundary conditions and back again. In the sequel to this paper, amongst other things, non-Dirichlet boundary conditions are transformed to boundary conditions which depend affinely on the eigenparameter $\lambda$ and vice versa. At all times, it is possible to keep track of how the eigenvalues of the various transformed boundary value problems relate to the eigenvalues of the original boundary value problem.

The transformations given in Theorems 2.1 and 3.1 are almost isospectral. In particular, depending on which transformation is applied at a specific point in the hierarchy, we either lose the least eigenvalue or gain an eigenvalue below the least eigenvalue. It should be noted that if we apply the two transformations of Sections 2 and 3 successively the resulting boundary value problem has precisely the same spectrum as the boundary value problem we began with. In fact, for a suitable choice of the solution $z(n)$ of (1.1), with $\lambda$ less than the least eigenvalue of the boundary value problem fixed, Corollary 3.3 gives that applying the transformation given in Theorem 2.1 followed by the transformation given in Theorem 3.1 yields a boundary value problem which is exactly the same as the original boundary value problem, that is, the same difference equation, boundary conditions, and hence spectrum.

It should be noted that the work [6, Chapter 11] of Teschl, on spectral and inverse spectral theory of Jacobi operators, provides a factorisation of a second-order difference equation, where the factors are adjoints of each another. It is easy to show that the factors given in this paper are not adjoints of each other, making our work distinct from that of Teschl's.

Difference equations, difference operators, and results concerning the existence and construction of their solutions have been discussed in [7, 8]. Difference equations occur in a variety of settings, especially where there are recursive computations. As such they have applications in electrical circuit analysis, dynamical systems, statistics, and many other fields.

More specifically, from Atkinson [9], we obtained the following three physical applications of the difference equation (1.1). Firstly, we have the vibrating string. The string is taken to be weightless and bears $m$ particles $p_0, \ldots, p_{m-1}$ at the points say $x_0, \ldots, x_{m-1}$ with masses $c(0), \ldots, c(m-1)$ and distances between them given by $x_{r+1} - x_r = 1/c(r)$, $r = 0, \ldots, m-2$. Beyond $c(m-1)$ the string extends to a length $1/c(m-1)$ and beyond $c(0)$ to a length $1/c(-1)$. The string is stretched to unit tension. If $s(n)$ is the displacement of the particle $p_n$ at time $t$, the restoring forces on it due to the tension of the string are $c(n-1)(s(n) - s(n-1))$ and $-c(n)(s(n+1) - s(n))$ considering small oscillations only. Hence, we can find the second-order differential equation of motion for the particles. We require solutions to be of the form $s(n) = y(n)\cos(\omega t)$, where $y(n)$ is the amplitude of oscillation of the particle $p_n$. Solving for $y(n)$ then reduces to solving a difference equation of the form (1.1). Imposing various boundary conditions forces the string to be pinned down at one end, both ends, or at a particular particle, see Atkinson [9] for details. Secondly, there is an equivalent scenario in electrical network theory. In this case, the $c(n)$ are inductances, $1/c(n)$ capacitances, and the $s(n)$ are loop currents in successive meshes. The third application of the three-term difference equation (1.1) is in Markov processes, in particular, birth and death processes and random walks. Although the above three applications are somewhat restricted due to the imposed relationship between the weight and the off-diagonal elements, they are nonetheless interesting.

There is also an obvious connection between the three-term difference equation and orthogonal polynomials; see [10]. Although, not the focus of this paper, one can investigate which orthogonal polynomials satisfy the three-term recurrence relation given by (1.1) and establish the properties of those polynomials. In Atkinson [9], the link between the norming constants and the orthogonality of polynomials obeying a three-term recurrence relation is given. Hence the necessity for showing how the norming constants are transformed under the transformations given in Theorems 2.1 and 3.1. As expected, from the continuous case, we find that the $n$th new norming constant is just $\lambda_n - \lambda_0$ multiplied by the original $n$th norming constant or $1/(\lambda_n - \lambda_0)$ multiplied by the original $n$th norming constant depending on which transformation is used.

The paper is set out as follows.

In Section 2, we transform (1.1) with non-Dirichlet boundary conditions at both ends to an equation of the same form but with Dirichlet boundary conditions at both ends. We prove that the spectrum of the new boundary value problem is the same as that of the original boundary value problem but with one eigenvalue less, namely, the least eigenvalue.

In Section 3, we again consider an equation of the form (1.1), but with Dirichlet boundary conditions at both ends. We assume that we have a strictly positive solution, $z(n)$, to (1.1) for $\lambda = \lambda_0$ with $\lambda_0$ less than the least eigenvalue of the given boundary value problem. We can then transform the given boundary value problem to one consisting of an equation of the same type but with specified non-Dirichlet boundary conditions at the ends. The spectrum of the transformed boundary value problem has one extra eigenvalue, in particular $\lambda_0$.

The transformation in Section 2 followed by the transformation in Section 3, gives in general, an isospectral transformation of the weighted second-order difference equation of the form (1.1) with non-Dirichlet boundary conditions. However, for a particular choice of $z(n)$ this results in the original boundary value problem being recovered.

In the final section, we show that the process outlined in Sections 2 and 3 can be reversed.

## 2. Transformation 1

### *2.1. Transformation of the Equation*

Consider the second-order difference equation (1.1), which may be rewritten as

$$c(n)y(n+1) - (b(n) - \lambda_0 c(n))y(n) + c(n-1)y(n-1) = -c(n)(\lambda - \lambda_0)y(n), \tag{2.1}$$

where $n = 0, \ldots, m-1$. Denote by $\lambda_0$ the least eigenvalue of (1.1) with boundary conditions

$$hy(-1) + y(0) = 0, \qquad Hy(m-1) + y(m) = 0, \tag{2.2}$$

where $h$ and $H$ are constants; see [9]. We wish to find a factorisation of the formal operator,

$$ly(n) := -y(n+1) + \left(\frac{b(n)}{c(n)} - \lambda_0\right)y(n) - \frac{c(n-1)}{c(n)}y(n-1) = (\lambda - \lambda_0)y(n), \tag{2.3}$$

for $n = 0, \ldots, m-1$, such that $l = SQ$, where $S$ and $Q$ are both first order formal difference operators.

**Theorem 2.1.** *Let $u_0(n)$ be a solution of* (1.1) *corresponding to $\lambda = \lambda_0$ and define the formal difference operators*

$$\begin{aligned} Sy(n) &:= y(n) - y(n-1)\frac{u_0(n-1)c(n-1)}{u_0(n)c(n)}, \quad n = 0, \ldots, m, \\ Qy(n) &:= y(n+1) - y(n)\frac{u_0(n+1)}{u_0(n)}, \quad n = -1, \ldots, m-1. \end{aligned} \tag{2.4}$$

*Then formally $ly(n) = SQy(n)$, $n = 0, \ldots, m-1$ and the so-called transformed operator is given by $\widetilde{l}\widetilde{y}(n) = QS\widetilde{y}(n)$, $n = 0, \ldots, m-1$. Hence the transformed equation is*

$$\widetilde{c}(n)\widetilde{y}(n+1) - \widetilde{b}(n)\widetilde{y}(n) + \widetilde{c}(n-1)\widetilde{y}(n-1) = -\widetilde{c}(n)\lambda\widetilde{y}(n), \quad n = 0, \ldots, m-2, \tag{2.5}$$

*where*

$$\widetilde{c}(n) = \frac{u_0(n)c(n)}{u_0(n+1)} > 0, \quad n = -1, \ldots, m-1, \tag{2.6}$$

$$\widetilde{b}(n) = \left[\frac{u_0(n)c(n)}{u_0(n+1)c(n+1)} - \frac{c(n-1)u_0(n-1)}{c(n)u_0(n)} + \frac{b(n)}{c(n)} - \lambda_0\right]\frac{u_0(n)c(n)}{u_0(n+1)}, \quad n = 0, \ldots, m-1. \tag{2.7}$$

*Proof.* By the definition of $S$ and $Q$, we have that

$$\begin{aligned} SQy(n) &= S\left(y(n+1) - \frac{u_0(n+1)}{u_0(n)}y(n)\right) \\ &= y(n+1) - \frac{u_0(n+1)y(n)}{u_0(n)} - \left[y(n) - \frac{u_0(n)}{u_0(n-1)}y(n-1)\right]\frac{u_0(n-1)c(n-1)}{u_0(n)c(n)}. \end{aligned} \tag{2.8}$$

Using (2.3), substituting in for $u_0(n+1)$ and cancelling terms, gives

$$\begin{aligned}
SQy(n) &= y(n+1) - \frac{1}{u_0(n)}\left[-\lambda_0 u_0(n) - \frac{c(n-1)}{c(n)}u_0(n-1) + \frac{b(n)}{c(n)}u_0(n)\right]y(n) \\
&\quad - y(n)\frac{u_0(n-1)c(n-1)}{u_0(n)c(n)} + y(n-1)\frac{c(n-1)}{c(n)} \\
&= y(n+1) - \left(\frac{b(n)}{c(n)} - \lambda_0\right)y(n) + \frac{c(n-1)}{c(n)}y(n-1) \\
&= -(\lambda - \lambda_0)y(n), \quad n = 0, \dots, m-1.
\end{aligned} \tag{2.9}$$

Hence $l = SQ$.

Now, setting $Qy(n) = \widetilde{y}(n)$, $n = -1, \dots, m-1$, gives

$$\begin{aligned}
QS\widetilde{y}(n) &= QSQy(n) \\
&= -Q(\lambda - \lambda_0)y(n) \\
&= -(\lambda - \lambda_0)\widetilde{y}(n), \quad n = 0, \dots, m-1,
\end{aligned} \tag{2.10}$$

which is the required transformed equation.

To find $\widetilde{l}$, we need to determine $QS\widetilde{y}(n)$.

Firstly,

$$S\widetilde{y}(n) = \widetilde{y}(n) - \widetilde{y}(n-1)\frac{u_0(n-1)c(n-1)}{u_0(n)c(n)}, \quad n = 0, \dots, m-1, \tag{2.11}$$

thus for $n = 0, \dots, m-2$,

$$\begin{aligned}
Q\big(S\widetilde{y}(n)\big) &= \widetilde{y}(n+1) \\
&\quad - \widetilde{y}(n)\frac{u_0(n)c(n)}{u_0(n+1)c(n+1)} - \left(\widetilde{y}(n) - \widetilde{y}(n-1)\frac{u_0(n-1)c(n-1)}{u_0(n)c(n)}\right)\frac{u_0(n+1)}{u_0(n)} \\
&= \widetilde{y}(n+1) - \widetilde{y}(n)\left[\frac{u_0(n)c(n)}{u_0(n+1)c(n+1)} + \frac{u_0(n+1)}{u_0(n)}\right] \\
&\quad + \widetilde{y}(n-1)\left[\frac{u_0(n-1)c(n-1)u_0(n+1)}{u_0(n)c(n)u_0(n)}\right].
\end{aligned} \tag{2.12}$$

By multiplying by $u_0(n)c(n)/u_0(n+1)$, this may be rewritten as

$$\begin{aligned}
&\frac{u_0(n)c(n)}{u_0(n+1)}\tilde{y}(n+1)-\left[\frac{u_0(n)c(n)}{u_0(n+1)c(n+1)}-\frac{c(n-1)u_0(n-1)}{c(n)u_0(n)}+\frac{b(n)}{c(n)}-\lambda_0\right]\frac{u_0(n)c(n)}{u_0(n+1)}\tilde{y}(n)\\
&\quad+\frac{u_0(n-1)c(n-1)}{u_0(n)}\tilde{y}(n-1)=-(\lambda-\lambda_0)\tilde{y}(n)\frac{u_0(n)c(n)}{u_0(n+1)}.
\end{aligned}\tag{2.13}$$

Thus we obtain (2.5). □

## 2.2. Transformation of the Boundary Conditions

We now show how the non-Dirichlet boundary conditions (2.2) are transformed under $Q$.

By the boundary conditions (2.2) $y$ is defined for $n=-1,\dots,m$.

**Theorem 2.2.** *The mapping $y\mapsto\tilde{y}$ given by $\tilde{y}(n)=y(n+1)-y(n)u_0(n+1)/u_0(n)$, $n=-1,\dots,m-1$, where $u_0$ is an eigenfunction to the least eigenvalue $\lambda_0$ of* (1.1), (2.2), *transforms $y$ obeying boundary conditions* (2.2) *to $\tilde{y}$ obeying Dirichlet boundary conditions of the form*

$$\tilde{y}(-1)=0,\qquad \tilde{y}(m-1)=0.\tag{2.14}$$

*Proof.* Since $\tilde{y}(n)=y(n+1)-y(n)u_0(n+1)/u_0(n)$, we get that

$$\begin{aligned}
\tilde{y}(-1)&=y(0)-y(-1)\frac{u_0(0)}{u_0(-1)}\\
&=-hy(-1)-y(-1)(-h)\\
&=0.
\end{aligned}\tag{2.15}$$

Hence as $y$ obeys the non-Dirichlet boundary condition $hy(-1)+y(0)=0$, $\tilde{y}$ obeys the Dirichlet boundary condition, $\tilde{y}(-1)=0$.

Similarly, for the second boundary condition,

$$\begin{aligned}
\tilde{y}(m-1)&=y(m)-y(m-1)\frac{u_0(m)}{u_0(m-1)}\\
&=-Hy(m-1)-y(m-1)(-H)\\
&=0.
\end{aligned}\tag{2.16}$$

□

We call (2.14) the transformed boundary conditions.

Combining the above results we obtain the following corollary.

**Corollary 2.3.** *The transformation $y\mapsto\tilde{y}$, given in Theorem 2.2, takes eigenfunctions of the boundary value problem* (1.1), (2.2) *to eigenfunctions of the boundary value problem* (2.5), (2.14).

*The spectrum of the transformed boundary value problem* (2.5), (2.14) *is the same as that of* (1.1), (2.2), *except for the least eigenvalue,* $\lambda_0$, *which has been removed.*

*Proof.* Theorems 2.1 and 2.2 prove that the mapping $y \mapsto \tilde{y}$ transforms eigenfunctions of (1.1), (2.2) to eigenfunctions (or possibly the zero solution) of (2.5), (2.14). The boundary value problem (1.1), (2.2) has $m$ eigenvalues which are real and distinct and the corresponding eigenfunctions $u_0(n), \ldots, u_{m-1}(n)$ are linearly independent when considered for $n = 0, \ldots, m-1$; see [11] for the case of vector difference equations of which the above is a special case. In particular, if $\lambda_0 < \lambda_1 < \cdots < \lambda_{m-1}$ are the eigenvalues of (1.1), (2.2) with eigenfunctions $u_0, \ldots, u_{m-1}$, then $\tilde{u}_0 \equiv 0$ and $\tilde{u}_1, \ldots, \tilde{u}_{m-1}$ are eigenfunctions of (2.5), (2.14) with eigenvalues $\lambda_1, \ldots, \lambda_{m-1}$. By a simple computation it can be shown that $\tilde{u}_1, \ldots, \tilde{u}_{m-1} \not\equiv 0$. Since the interval of the transformed boundary value problem is precisely one shorter than the original interval, (2.5), (2.14) has one less eigenvalue. Hence $\lambda_1, \ldots, \lambda_{m-1}$ constitute all the eigenvalues of (2.5), (2.14). □

### 2.3. *Transformation of the Norming Constants*

Let $\lambda_0 < \cdots < \lambda_{m-1}$ be the eigenvalues of (1.1) with boundary conditions (2.2) and $y_0, \ldots, y_{m-1}$ be associated eigenfunctions normalised by $y_n(0) = 1$. We prove, in this subsection, that under the mapping given in Theorem 2.2, the new norming constant is $1/(\lambda_n - \lambda_0)$ times the original norming constant.

**Lemma 2.4.** *Let* $\rho_n$ *denote the norming constants of* (1.1) *and be defined by*

$$\rho_n := \sum_{j=0}^{m-1} (-c(j)) \left( \frac{y_n(j)}{y_n(0)} \right)^2 = \sum_{j=0}^{m-1} (-c(j)) y_n(j)^2. \tag{2.17}$$

*If* $\tilde{\tau}_n$ *is defined by*

$$\tilde{\tau}_n := \sum_{j=0}^{m-2} (-\tilde{c}(j)) \tilde{y}_n(j)^2, \tag{2.18}$$

*then, for* $u_0$ *an eigenfunction for* $\lambda = \lambda_0$ *normalised by* $u_0(0) = 1$,

$$\begin{aligned}(\lambda_n - \lambda_0)\rho_n = \tilde{\tau}_n - c(-1)\frac{u_0(-1)}{u_0(0)} y_n(0)^2 + c(-1) y_n(-1) y_n(0) - y_n(m-1)^2 c(m-1) \frac{u_0(m)}{u_0(m-1)} \\ + y_n(m-1) y_n(m) c(m-1).\end{aligned} \tag{2.19}$$

*Proof.* Substituting in for $\tilde{y}_n(j)$ and $\tilde{c}(j)$, $n = 1, \dots, m-1$, we have that

$$\begin{aligned}
\tilde{\tau}_n &= \sum_{j=0}^{m-2}\left(\frac{-u_0(j)c(j)}{u_0(j+1)}y_n(j+1)^2 + 2y_n(j)y_n(j+1)c(j) - \frac{u_0(j+1)}{u_0(j)}c(j)y_n(j)^2\right) \\
&= \sum_{j=0}^{m-2}\left(\frac{-u_0(j)c(j)}{u_0(j+1)}y_n(j+1)^2 + 2y_n(j)y_n(j+1)c(j) - (b(j) - c(j)\lambda_0)y_n(j)^2\right. \\
&\qquad \left. + c(j-1)\frac{u_0(j-1)}{u_0(j)}y_n(j)^2\right) \\
&= \sum_{j=0}^{m-2}\left(2y_n(j)y_n(j+1)c(j) - [b(j) - c(j)\lambda_0]y_n(j)^2\right) + c(-1)\frac{u_0(-1)}{u_0(0)}y_n(0)^2 \\
&\quad - c(m-2)\frac{u_0(m-2)}{u_0(m-1)}y_n(m-1)^2.
\end{aligned} \tag{2.20}$$

Then, using the definition of $\rho_n$, we obtain that

$$\begin{aligned}
(\lambda_n - \lambda_0)\rho_n &= \sum_{j=0}^{m-1}(\lambda_n - \lambda_0)(-c(j))y_n(j)y_n(j) \\
&= \sum_{j=0}^{m-1}(c(j)y_n(j+1) - (b(j) - \lambda_0 c(j))y_n(j) + c(j-1)y_n(j-1))y_n(j) \\
&= 2\sum_{j=0}^{m-1}[c(j)y_n(j+1)y_n(j)] - \sum_{j=0}^{m-1}\left[(b(j) - \lambda_0 c(j))y_n(j)^2\right] \\
&\quad + c(-1)y_n(-1)y_n(0) - c(m-1)y_n(m-1)y_n(m) \\
&= \sum_{j=0}^{m-2}\left[2(c(j)y_n(j+1)y_n(j)) - [b(j) - \lambda_0 c(j)]y_n(j)^2\right] \\
&\quad + c(m-1)y_n(m)y_n(m-1) - [b(m-1) - \lambda_0 c(m-1)]y_n(m-1)^2 \\
&\quad + c(-1)y_n(-1)y_n(0) \\
&= \tilde{\tau}_n + c(m-2)\frac{u_0(m-2)}{u_0(m-1)}y_n(m-1)^2 + c(m-1)y_n(m)y_n(m-1) \\
&\quad - (b(m-1) - \lambda_0 c(m-1))y_n(m-1)^2 + c(-1)y_n(-1)y_n(0) \\
&\quad - c(-1)\frac{u_0(-1)}{u_0(0)}y_n(0)^2.
\end{aligned} \tag{2.21}$$

Now,

$$\begin{aligned}
&-(b(m-1)-\lambda_0 c(m-1))y_n(m-1)^2 \\
&\quad = -c(m-1)y_n(m)y_n(m-1)-c(m-2)y_n(m-2)y_n(m-1) \\
&\qquad - c(m-1)(\lambda_n-\lambda_0)y_n(m-1)^2.
\end{aligned} \tag{2.22}$$

Therefore,

$$\begin{aligned}
(\lambda_n-\lambda_0)\rho_n = \widetilde{\tau}_n - c(-1)\frac{u_0(-1)}{u_0(0)}y_n(0)^2 + c(-1)y_n(-1)y_n(0) + c(m-2)\frac{u_0(m-2)}{u_0(m-1)}y_n(m-1)^2 \\
- c(m-2)y_n(m-2)y_n(m-1) - c(m-1)(\lambda_n-\lambda_0)y_n(m-1)^2.
\end{aligned} \tag{2.23}$$

Using (1.1) to substitute in for $c(m-2)u_0(m-2)$ and $c(m-2)y_n(m-2)$ gives

$$\begin{aligned}
(\lambda_n-\lambda_0)\rho_n &= \widetilde{\tau}_n - c(-1)\frac{u_0(-1)}{u_0(0)}y_n(0)^2 + c(-1)y_n(-1)y_n(0) \\
&\quad + y_n(m-1)^2\left[-c(m-1)\lambda_0 + b(m-1) - c(m-1)\frac{u_0(m)}{u_0(m-1)}\right] \\
&\quad - y_n(m-1)\left[-c(m-1)\lambda_n y_n(m-1) + b(m-1)y_n(m-1) - c(m-1)y_n(m)\right] \\
&\quad - c(m-1)(\lambda_n-\lambda_0)y_n(m-1)^2 \\
&= \widetilde{\tau}_n - c(-1)\frac{u_0(-1)}{u_0(0)}y_n(0)^2 + c(-1)y_n(-1)y_n(0) - y_n(m-1)^2c(m-1)\frac{u_0(m)}{u_0(m-1)} \\
&\quad + y_n(m-1)y_n(m)c(m-1).
\end{aligned} \tag{2.24}$$

□

**Theorem 2.5.** *If $\rho_n$, as defined in Lemma 2.4, are the norming constants of* (1.1) *with boundary conditions* (2.2) *and*

$$\widetilde{\rho}_n := \sum_{j=0}^{m-2}(-\widetilde{c}(j))\left(\frac{\widetilde{y}_n(j)}{\widetilde{y}_n(0)}\right)^2 \tag{2.25}$$

*are the norming constants of* (2.5) *with boundary conditions* (2.14)*, then*

$$\rho_n = (\lambda_n-\lambda_0)\widetilde{\rho}_n. \tag{2.26}$$

*Proof.* The boundary conditions (2.2) together with Lemma 2.4 give

$$(\lambda_n-\lambda_0)\rho_n = \widetilde{\tau}_n. \tag{2.27}$$

Now by (2.14), $\tilde{y}(-1) = 0$, and thus

$$\tilde{y}(0) = y(1) - \frac{u_0(1)}{u_0(0)} y(0) = y(1) - u_0(1) = -(\lambda - \lambda_0). \tag{2.28}$$

Therefore,

$$(\lambda_n - \lambda_0)\rho_n = \left(\tilde{y}_n(0)\right)^2 \sum_{j=0}^{m-2} \left(-\tilde{c}(j)\right) \left(\frac{\tilde{y}_n(j)}{\tilde{y}_n(0)}\right)^2 = (\lambda_n - \lambda_0)^2 \sum_{j=0}^{m-2} \left(-\tilde{c}(j)\right) \left(\frac{\tilde{y}_n(j)}{\tilde{y}_n(0)}\right)^2. \tag{2.29}$$

Thus we have that

$$\rho_n = (\lambda_n - \lambda_0)\tilde{\rho}_n. \tag{2.30}$$

□

# 3. Transformation 2

## 3.1. Transformation of the Equation

Consider (2.5), where $n = 0, \ldots, m-2$ and $\tilde{y}(n)$, $n = -1, \ldots, m-1$, obeys the boundary conditions (2.14).

Let $z(n)$ be a solution of (2.5) with $\lambda = \lambda_0$ such that $z(n) > 0$ for all $n = -1, \ldots, m-1$, where $\lambda_0$ is less than the least eigenvalue of (2.5), (2.14).

We want to factorise the operator $l_z$, where

$$l_z \tilde{y}(n) = -\tilde{y}(n+1) + \left(\frac{\tilde{b}(n)}{\tilde{c}(n)} - \lambda_0\right)\tilde{y}(n) - \frac{\tilde{c}(n-1)}{\tilde{c}(n)}\tilde{y}(n-1) = (\lambda - \lambda_0)\tilde{y}(n), \tag{3.1}$$

for $n = 0, \ldots, m-2$, such that $l_z = PR$, where $P$ and $R$ are both formal first order difference operators.

**Theorem 3.1.** *Let*

$$\begin{aligned} P\tilde{y}(n) &:= \tilde{y}(n+1) - \tilde{y}(n)\frac{z(n-1)\tilde{c}(n-1)}{z(n)\tilde{c}(n)}, \quad n = 0, \ldots, m-2, \\ R\tilde{y}(n) &:= \tilde{y}(n) - \tilde{y}(n-1)\frac{z(n)}{z(n-1)}, \quad n = 0, \ldots, m-1. \end{aligned} \tag{3.2}$$

*Then $l_z = PR$ and $\hat{y}(n) = R\tilde{y}(n)$ is a solution of the transformed equation $RP\hat{y} = -(\lambda - \lambda_0)\hat{y}$ giving, for $n = 1, \ldots, m-2$,*

$$\hat{l}\hat{y}(n) := -\hat{c}(n)\hat{y}(n+1) + \hat{b}(n)\hat{y}(n) - \hat{c}(n-1)\hat{y}(n-1) = \hat{c}(n)\lambda\hat{y}(n), \tag{3.3}$$

*where, for* $n = 0, \ldots, m-1$,

$$\begin{aligned} \widehat{c}(n) &= \frac{z(n-1)\widetilde{c}(n-1)}{z(n)}, \\ \widehat{b}(n) &= \left[\frac{z(n-1)\widetilde{c}(n-1)}{z(n)\widetilde{c}(n)} + \frac{z(n)}{z(n-1)}\right]\frac{z(n-1)\widetilde{c}(n-1)}{z(n)}. \end{aligned} \tag{3.4}$$

*Proof.* By the definition of $P$ and $R$, we get

$$\begin{aligned} PR\widetilde{y}(n) &= \widetilde{y}(n+1) - \widetilde{y}(n)\frac{z(n+1)}{z(n)} - \left(\widetilde{y}(n) - \widetilde{y}(n-1)\frac{z(n)}{z(n-1)}\right)\frac{z(n-1)\widetilde{c}(n-1)}{z(n)\widetilde{c}(n)} \\ &= \widetilde{y}(n+1) - \widetilde{y}(n)\left[-\lambda_0 - \frac{\widetilde{c}(n-1)z(n-1)}{\widetilde{c}(n)z(n)} + \frac{\widetilde{b}(n)}{\widetilde{c}(n)}\right] - \widetilde{y}(n)\frac{z(n-1)\widetilde{c}(n-1)}{z(n)\widetilde{c}(n)} \\ &\quad + \widetilde{y}(n-1)\frac{\widetilde{c}(n-1)}{\widetilde{c}(n)} \\ &= \widetilde{y}(n+1) - \widetilde{y}(n)\left(\frac{\widetilde{b}(n)}{\widetilde{c}(n)} - \lambda_0\right) + \widetilde{y}(n-1)\frac{\widetilde{c}(n-1)}{\widetilde{c}(n)} \\ &= -(\lambda - \lambda_0)\widetilde{y}(n). \end{aligned} \tag{3.5}$$

Hence $l_z = PR$.

Setting $\widehat{y}(n) = R\widetilde{y}(n)$ gives

$$RP\widehat{y}(n) = R\big(PR\widetilde{y}(n)\big) = -R(\lambda_0 - \lambda)\widetilde{y}(n) = -(\lambda - \lambda_0)\widehat{y}(n) \tag{3.6}$$

giving that $\widehat{y}$ is a solution of the transformed equation.

We now explicitly obtain the transformed equation. From the definitions of $R$ and $P$, we get

$$\begin{aligned} RP\widehat{y}(n) &= R\left(\widehat{y}(n+1) - \widehat{y}(n)\frac{z(n-1)\widetilde{c}(n-1)}{z(n)\widetilde{c}(n)}\right) \\ &= \widehat{y}(n+1) - \widehat{y}(n)\frac{z(n-1)\widetilde{c}(n-1)}{z(n)\widetilde{c}(n)} - \left[\widehat{y}(n) - \widehat{y}(n-1)\frac{z(n-2)\widetilde{c}(n-2)}{z(n-1)\widetilde{c}(n-1)}\right]\frac{z(n)}{z(n-1)} \\ &= \widehat{y}(n+1) - \widehat{y}(n)\left[\frac{z(n-1)\widetilde{c}(n-1)}{z(n)\widetilde{c}(n)} + \frac{z(n)}{z(n-1)}\right] + \widehat{y}(n-1)\frac{z(n-2)\widetilde{c}(n-2)z(n)}{z(n-1)\widetilde{c}(n-1)z(n-1)}. \end{aligned} \tag{3.7}$$

This implies that

$$\begin{aligned}&\frac{z(n-1)\tilde{c}(n-1)}{z(n)}\hat{y}(n+1)-\left[\frac{z(n-1)\tilde{c}(n-1)}{z(n)\tilde{c}(n)}+\frac{z(n)}{z(n-1)}\right]\frac{z(n-1)\tilde{c}(n-1)}{z(n)}\hat{y}(n)\\&\quad+\frac{z(n-2)\tilde{c}(n-2)}{z(n-1)}\hat{y}(n-1)=-(\lambda-\lambda_0)\hat{y}(n)\frac{z(n-1)\tilde{c}(n-1)}{z(n)}.\end{aligned}\tag{3.8}$$

□

### *3.2. Transformation of the Boundary Conditions*

At present, $\hat{y}(n)$ is defined for $n = 0,\dots,m-1$. We extend the definition of $\hat{y}(n)$ to $n = -1,\dots,m$ by forcing the boundary conditions

$$\hat{h}\hat{y}(-1)+\hat{y}(0)=0,\qquad \widehat{H}\hat{y}(m-1)+\hat{y}(m)=0,\tag{3.9}$$

where

$$\begin{aligned}\hat{h}&:=\left[\frac{\hat{c}(0)}{\hat{c}(-1)}\left(\frac{\tilde{b}(0)}{\tilde{c}(0)}-\frac{z(1)}{z(0)}-\frac{\hat{b}(0)}{\hat{c}(0)}\right)\right]^{-1},\\ \widehat{H}&:=\frac{\tilde{b}(m-2)}{\tilde{c}(m-2)}-\frac{\hat{b}(m-1)}{\hat{c}(m-1)}-\frac{z(m-2)\hat{c}(m-2)}{z(m-1)\hat{c}(m-1)}.\end{aligned}\tag{3.10}$$

Here we take $\hat{c}(-1)=c(-1)$.

**Theorem 3.2.** *The mapping $\tilde{y}\mapsto\hat{y}$ given by $\hat{y}(n)=\tilde{y}(n)-\tilde{y}(n-1)(z(n)/z(n-1))$, $n=0,\dots,m-1$, where $z(n)$ is as previously defined (in the beginning of the section), transforms $\tilde{y}$ which obeys boundary conditions* (2.14) *to $\hat{y}$ which obeys the non-Dirichlet boundary conditions* (3.9) *and $\hat{y}$ is a solution of $\hat{l}\hat{y}(n)=\lambda\hat{c}(n)\hat{y}(n)$ for $n=0,\dots,m-1$.*

*Proof.* By the construction of $\hat{h}$ and $\widehat{H}$ it follows that the boundary conditions (3.9) are obeyed by $\hat{y}$.

We now show that $\hat{y}$ is a solution to the extended problem. From Theorem 3.1 we need only prove that $\hat{l}\hat{y}(n)=\lambda\hat{c}(n)\hat{y}(n)$ for $n=0$ and $n=m-1$. For $n=0$, from (3.3) with (3.9), we have that

$$\hat{c}(0)\hat{y}(1)+\hat{c}(-1)\left(\frac{-\hat{y}(0)}{\hat{h}}\right)=\left(\hat{b}(0)-\hat{c}(0)\lambda\right)\hat{y}(0).\tag{3.11}$$

Also the mapping, for $n=0$, gives

$$\hat{y}(0)=\tilde{y}(0)-\tilde{y}(-1)\frac{z(0)}{z(-1)}.\tag{3.12}$$

Thus using (2.14), we obtain that $\widehat{y}(0) = \widetilde{y}(0)$. So we now have

$$\widehat{c}(0)\widehat{y}(1) + \widehat{c}(-1)\left(\frac{-\widetilde{y}(0)}{\widehat{h}}\right) = \left(\widehat{b}(0) - \widehat{c}(0)\lambda\right)\widetilde{y}(0). \tag{3.13}$$

Next, using the mapping at $n = 1$, we obtain that

$$\widehat{c}(0)\left(\widetilde{y}(1) - \widetilde{y}(0)\frac{z(1)}{z(0)}\right) + \widehat{c}(-1)\left(\frac{-\widetilde{y}(0)}{\widehat{h}}\right) = \left(\widehat{b}(0) - \widehat{c}(0)\lambda\right)\widetilde{y}(0). \tag{3.14}$$

Rearranging the terms above results in

$$\widetilde{y}(1) - \left(\frac{\widehat{b}(0)}{\widehat{c}(0)} + \frac{\widehat{c}(-1)}{\widehat{c}(0)\widehat{h}} + \frac{z(1)}{z(0)}\right)\widetilde{y}(0) = -\lambda\widetilde{y}(0). \tag{3.15}$$

Also, (2.5), for $n = 0$, together with (2.14) gives

$$\widetilde{y}(1) - \frac{\widetilde{b}(0)}{\widetilde{c}(0)}\widetilde{y}(0) = -\lambda\widetilde{y}(0). \tag{3.16}$$

Subtracting (3.15) from (3.16) yields

$$\widehat{c}(0)\left(\frac{\widetilde{b}(0)}{\widetilde{c}(0)} - \frac{z(1)}{z(0)} - \frac{\widehat{b}(0)}{\widehat{c}(0)}\right) = \frac{\widehat{c}(-1)}{\widehat{h}}. \tag{3.17}$$

In a similar manner, we can show that (3.3) also holds for $n = m - 1$. Hence $\widehat{y}$ is a solution of $\widehat{l}\widehat{y}(n) = \lambda\widehat{c}(n)\widehat{y}(n)$ for $n = 0, \ldots, m - 1$. □

Combining Theorems 3.1 and 3.2 we obtain the corollary below.

**Corollary 3.3.** *Let $z(n)$ be a solution of* (2.5) *for $\lambda = \lambda_0$, where $\lambda_0$ is less than the least eigenvalue of* (2.5), (2.14), *such that $z(n) > 0$ for $n = -1, \ldots, m - 1$. Then we can transform the given equation,* (2.5), *to an equation of the same type,* (3.3) *with a specified non-Dirichlet boundary condition,* (3.9), *at either the initial or end point. The spectrum of the transformed boundary value problem* (3.3), (3.9) *is the same as that of* (2.5), (2.14) *except for one additional eigenvalue, namely, $\lambda_0$.*

*Proof.* Theorems 3.1 and 3.2 prove that the mapping $\widetilde{y} \mapsto \widehat{y}$, transforms eigenfunctions of (2.5), (2.14) to eigenfunctions of (3.3), (3.9). In particular if $\lambda_1 < \cdots < \lambda_{m-1}$ are the eigenvalues of (2.5), (2.14), $n = -1, \ldots, m - 1$, with eigenfunctions $\widetilde{u}_1, \ldots, \widetilde{u}_{m-1}$, then $z, \widetilde{u}_1, \ldots, \widetilde{u}_{m-1}$ are eigenfunctions of (3.3), (3.9), $n = -1, \ldots, m$, with eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_{m-1}$. Since the index set of the transformed boundary value problem is precisely one larger than the original, (3.3), (3.9) has one more eigenvalue. Hence $\lambda_0, \lambda_1, \ldots, \lambda_{m-1}$ constitute all the eigenvalues of (3.3), (3.9). □

Thus we have proved the following.

**Corollary 3.4.** *The transformation of* (1.1), (2.2) *to* (2.5), (2.14) *and then to* (3.3), (3.9) *is an isospectral transformation. That is, the spectrum of* (1.1), (2.2) *is the same as the spectrum of* (3.3), (3.9).

We now show that for a suitable choice of $z(n)$ the transformation of (1.1), (2.2) to (2.5), (2.14) and then to (3.3), (3.9) results in the original boundary value problem.

Without loss of generality, by a shift of the spectrum, it may be assumed that the least eigenvalue, $\lambda_0$, of (1.1), (2.2) is $\lambda_0 = 0$. Furthermore, let $u_0(n)$ be an eigenfunction to (1.1), (2.2) for the eigenvalue $\lambda_0 = 0$.

**Theorem 3.5.** *If* $z(n) := 1/u_0(n)c(n)$, *then* $z(n)$ *is a solution of* (2.5), *for* $\lambda = \lambda_0 = 0$. *Here* $\lambda_0 = 0$ *is less than the least eigenvalue of* (2.5), (2.14) *and* $z(n)$ *has no zeros in the interval* $n = -1, \ldots, m-1$. *In addition,* $\widehat{h} = h$, $\widehat{H} = H$, $\widehat{c}(n) = c(n)$ *for* $n = -1, \ldots, m-1$ *and* $\widehat{b}(n) = b(n)$ *for* $n = 0, \ldots, m-1$.

*Proof.* The left hand-side of (2.5), with $\widetilde{y} = z$, becomes

$$\widetilde{c}(n)z(n+1) - \widetilde{b}(n)z(n) + \widetilde{c}(n-1)z(n-1), \quad n = 0, \ldots, m-1, \tag{3.18}$$

which, when we substitute in for $z$, $\widetilde{c}$, and $\widetilde{b}$, simplifies to zero. Obviously the right-hand side of (2.5) is equal to 0 for $\lambda = \lambda_0 = 0$. Thus $z(n)$ is a solution of (2.5) for $\lambda = \lambda_0 = 0$, where $\lambda_0 = 0$ is less than the least eigenvalue of (2.5), (2.14).

Substituting for $z(n)$, $z(n-1)$ and $\widetilde{c}(n-1)$, in the equation for $\widehat{c}(n)$, we obtain immediately that $\widehat{c}(n) = c(n)$ for $n = 0, \ldots, m-1$ and by assumption $\widehat{c}(-1) = c(-1)$.

Next, a similar substitution into the equation for $\widehat{b}(n)$ yields

$$\begin{aligned}
\widehat{b}(n) &= \left[\frac{u_0(n)c(n)}{u_0(n-1)c(n-1)} \frac{u_0(n-1)c(n-1)u_0(n+1)}{u_0(n)u_0(n)c(n)} + \frac{u_0(n-1)c(n-1)}{u_0(n)c(n)}\right]c(n) \\
&= \frac{c(n)u_0(n+1) + u_0(n-1)c(n-1)}{u_0(n)}.
\end{aligned} \tag{3.19}$$

But $u_0(n)$ is an eigenfunction of (1.1), (2.2) corresponding to the eigenvalue $\lambda_0 = 0$, thus

$$\widehat{b}(n) = \frac{-c(n)\lambda_0 u_0(n) + b(n)u_0(n)}{u_0(n)} = b(n), \quad n = 0, \ldots, m-1. \tag{3.20}$$

Lastly, by definition

$$\widehat{h} = \left[\frac{\widehat{c}(0)}{\widehat{c}(-1)}\left(\frac{\widetilde{b}(0)}{\widetilde{c}(0)} - \frac{z(1)}{z(0)} - \frac{\widehat{b}(0)}{\widehat{c}(0)}\right)\right]^{-1}. \tag{3.21}$$

Substituting in for $\widehat{b}(0)$, $\widehat{c}(0)$, and using that $\widehat{c}(-1) = c(-1)$, we get

$$\widehat{h} = \left[\frac{z(-1)\widetilde{c}(-1)}{z(0)c(-1)}\left(\frac{\widetilde{b}(0)}{\widetilde{c}(0)} - \frac{z(1)}{z(0)} - \left(\frac{z(-1)\widetilde{c}(-1)}{z(0)\widetilde{c}(0)} + \frac{z(0)}{z(-1)}\right)\right)\right]^{-1}. \tag{3.22}$$

Since $z$ is a solution of (2.5) for $\lambda_0 = 0$, we have, for $n = 0$,

$$\frac{z(1)}{z(0)} - \frac{\widetilde{b}(0)}{\widetilde{c}(0)} + \frac{z(-1)\widetilde{c}(-1)}{z(0)\widetilde{c}(0)} = 0. \tag{3.23}$$

Putting this into the equation for $\widehat{h}$ above yields

$$\widehat{h} = -\frac{c(-1)}{\widetilde{c}(-1)}. \tag{3.24}$$

Substituting in for $\widetilde{c}(-1)$ gives

$$\widehat{h} = \frac{-u_0(0)}{u_0(-1)}, \tag{3.25}$$

so by (2.2) $\widehat{h} = h$.

Using precisely the same method, it can be easily shown that $\widehat{H} = H$. □

Hence, as claimed, we have proved the following result.

**Corollary 3.6.** *The transformation of* (1.1), (2.2) *to* (2.5), (2.14) *and then to* (3.3), (3.9) *with* $z(n) := 1/u_0(n)c(n)$ *results in the original boundary value problem.*

### *3.3. Transformation of the Norming Constants*

Assume that we have the following normalisation: $\widetilde{y}(0) = 1$. A result analogous to that in Theorem 2.5 is obtained.

**Lemma 3.7.** *As before let $\widetilde{\rho}_n$ denote the norming constants of* (2.5) *and be defined by*

$$\widetilde{\rho}_n = \sum_{j=0}^{m-2}(-\widetilde{c}(j))\left(\frac{\widetilde{y}_n(j)}{\widetilde{y}_n(0)}\right)^2 = \sum_{j=0}^{m-2}(-\widetilde{c}(j))\widetilde{y}_n(j)^2. \tag{3.26}$$

*If $\widehat{\tau}_n$ is defined by*

$$\widehat{\tau}_n = \sum_{j=0}^{m-1}(-\widehat{c}(j))\widehat{y}_n(j)^2, \tag{3.27}$$

*then,*

$$\begin{aligned}(\lambda_n - \lambda_0)\widetilde{\rho}_n = \widehat{\tau}_n &+ \widetilde{c}(-1)\frac{z(0)}{z(-1)}\widetilde{y}_n(-1)^2 - \widetilde{c}(-1)\widetilde{y}_n(-1)\widetilde{y}_n(0) + \widetilde{y}_n(m)^2\widetilde{c}(m-1)\frac{z(m-1)}{z(m)} \\ &- \widetilde{y}_n(m-1)\widetilde{y}_n(m)\widetilde{c}(m-1).\end{aligned} \tag{3.28}$$

*Proof.* Substituting in for $\widehat{c}$ and $\widehat{y}$, we obtain that

$$
\begin{aligned}
\widehat{\tau}_n &= \sum_{j=0}^{m-1}\left[\frac{-z(j-1)\widetilde{c}(j-1)}{z(j)}\widetilde{y}_n(j)^2 + 2\widetilde{y}_n(j-1)\widetilde{y}_n(j)\widetilde{c}(j-1) - \widetilde{y}_n(j-1)^2\frac{z(j)}{z(j-1)}\widetilde{c}(j-1)\right] \\
&= \sum_{j=0}^{m-1}\left[\frac{-z(j-1)\widetilde{c}(j-1)}{z(j)}\widetilde{y}_n(j)^2 + 2\widetilde{y}_n(j-1)\widetilde{y}_n(j)\widetilde{c}(j-1)\right] \\
&\quad - \sum_{j=1}^{m-1}\widetilde{y}_n(j-1)^2\left[\widetilde{b}(j-1) - \widetilde{c}(j-2)\frac{z(j-2)}{z(j-1)} - \widetilde{c}(j-1)\lambda_0\right] - \widetilde{y}_n(-1)^2\frac{z(0)}{z(-1)}\widetilde{c}(-1) \\
&= \sum_{j=1}^{m-1}\left[2\widetilde{y}_n(j-1)\widetilde{y}_n(j)\widetilde{c}(j-1) - \widetilde{y}_n(j-1)^2\left(\widetilde{b}(j-1) - \widetilde{c}(j-1)\lambda_0\right)\right] - \widetilde{y}_n(-1)^2\frac{z(0)}{z(-1)}\widetilde{c}(-1) \\
&\quad - \frac{z(m-2)\widetilde{c}(m-2)}{z(m-1)}\widetilde{y}_n(m-1)^2 + 2\widetilde{y}_n(-1)\widetilde{y}_n(0)\widetilde{c}(-1).
\end{aligned}
\tag{3.29}
$$

Then, using the definition of $\widetilde{\rho}_n$, we obtain that

$$
\begin{aligned}
(\lambda_n - \lambda_0)\widetilde{\rho}_n &= \sum_{j=0}^{m-2}(\lambda_n - \lambda_0)(-\widetilde{c}(j))\widetilde{y}_n(j)\widetilde{y}_n(j) \\
&= \sum_{j=0}^{m-2}\left(\widetilde{c}(j)\widetilde{y}_n(j+1) - \left(\widetilde{b}(j) - \lambda_0\widetilde{c}(j)\right)\widetilde{y}_n(j) + \widetilde{c}(j-1)\widetilde{y}_n(j-1)\right)\widetilde{y}_n(j) \\
&= 2\sum_{j=0}^{m-2}\left[\widetilde{c}(j)\widetilde{y}_n(j+1)\widetilde{y}_n(j)\right] - \sum_{j=0}^{m-2}\left[\left(\widetilde{b}(j) - \lambda_0\widetilde{c}(j)\right)\widetilde{y}_n(j)^2\right] \\
&\quad + \widetilde{c}(-1)\widetilde{y}_n(-1)\widetilde{y}_n(0) - \widetilde{c}(m-2)\widetilde{y}_n(m-2)\widetilde{y}_n(m-1) \\
&= \widehat{\tau}_n + \widetilde{y}_n(-1)^2\frac{z(0)}{z(-1)}\widetilde{c}(-1) + \frac{z(m-2)\widetilde{c}(m-2)}{z(m-1)}\widetilde{y}_n(m-1)^2 - \widetilde{c}(-1)\widetilde{y}_n(-1)\widetilde{y}_n(0) \\
&\quad - \widetilde{c}(m-2)\widetilde{y}_n(m-2)\widetilde{y}_n(m-1).
\end{aligned}
\tag{3.30}
$$

□

**Theorem 3.8.** *If $\widetilde{\rho}_n$, as given in Lemma 3.7, are the norming constants of* (2.5) *with boundary conditions* (2.14) *and*

$$
\widehat{\rho}_n := \sum_{j=0}^{m-1}(-\widehat{c}(j))\left(\frac{\widehat{y}_n(j)}{\widehat{y}_n(0)}\right)^2
\tag{3.31}
$$

*are the norming constants of* (3.3) *with boundary conditions* (3.9), *then*

$$\widetilde{\rho}_n = \frac{1}{(\lambda_n - \lambda_0)}\widehat{\rho}_n. \tag{3.32}$$

*Proof.* Using the boundary conditions (2.14) together with Lemma 3.7, we obtain that

$$(\lambda_n - \lambda_0)\widetilde{\rho}_n = \widehat{\tau}_n. \tag{3.33}$$

Now,

$$\widehat{y}(0) = \widetilde{y}(0) = 1. \tag{3.34}$$

Therefore,

$$(\lambda_n - \lambda_0)\widetilde{\rho}_n = \widehat{y}_n(0)^2\sum_{j=0}^{m-1}(-\widehat{c}(j))\left(\frac{\widehat{y}_n(j)}{\widehat{y}_n(0)}\right)^2 = \sum_{j=0}^{m-1}(-\widehat{c}(j))\left(\frac{\widehat{y}_n(j)}{\widehat{y}_n(0)}\right)^2. \tag{3.35}$$

Thus,

$$\widetilde{\rho}_n = \frac{1}{(\lambda_n - \lambda_0)}\widehat{\rho}_n. \tag{3.36}$$

□

## 4. Conclusion

To conclude, we illustrate how the process may be done the other way around. To do this we start by transforming a second-order difference equation with Dirichlet boundary conditions at both ends to a second-order difference equation of the same type with non-Dirichlet boundary conditions at both ends and then transform this back to the original boundary value problem.

Consider $\widetilde{v}(n)$ such that $\widetilde{v}(n)$ satisfies (2.5) and (2.14). The mapping $\widetilde{v} \mapsto \widehat{v}$, given by

$$\widehat{v}(n) = \widetilde{v}(n) - \widetilde{v}(n-1)\frac{z(n)}{z(n-1)}, \quad n = 0, \dots, m-1, \tag{4.1}$$

can be extended to include $\widehat{v}(-1)$ and $\widehat{v}(m)$ by forcing (3.9). Here $z(n)$ is a solution of (2.5) for $\lambda = \lambda_0 = 0$, with $\lambda_0$ less than the least eigenvalue of (2.5), (2.14) such that $z(n) > 0$ for all $n = -1, \dots, m-1$. The mapping $\widetilde{v} \mapsto \widehat{v}$ then gives that $\widehat{v}(n)$ satisfies (3.3) and (3.9). So (3.3), (3.9) has the same spectrum as (2.5), (2.14) except that one eigenvalue has been added, namely, $\lambda = \lambda_0 = 0$.

Now the mapping $\widehat{v} \mapsto v$ given by

$$v(n) = \widehat{v}(n+1) - \widehat{v}(n)\frac{u_0(n+1)}{u_0(n)}, \quad n = -1, \dots, m-1, \tag{4.2}$$

where $u_0(n)$ is an eigenfunction of (3.3), (3.9) corresponding to the eigenvalue $\lambda_0 = 0$ yields

$$c(n)v(n+1) - b(n)v(n) + c(n-1)v(n-1) = -c(n)\lambda v(n), \quad n = 0, \dots, m-2, \tag{4.3}$$

where

$$\begin{gathered} c(n) = \frac{u_0(n)\widehat{c}(n)}{u_0(n+1)} > 0, \\ b(n) = \left[ \frac{u_0(n)\widehat{c}(n)}{u_0(n+1)\widehat{c}(n+1)} - \frac{\widehat{c}(n-1)u_0(n-1)}{\widehat{c}(n)u_0(n)} + \frac{\widehat{b}(n)}{\widehat{c}(n)} - \lambda_0 \right] \frac{u_0(n)\widehat{c}(n)}{u_0(n+1)}, \end{gathered} \tag{4.4}$$

with boundary conditions

$$v(-1) = 0, \qquad v(m-1) = 0. \tag{4.5}$$

Thus this boundary value problem in $v$ has the same spectrum as that of (3.3), (3.9) but with one eigenvalue removed, namely, $\lambda = \lambda_0 = 0$.

**Lemma 4.1.** *Let $u_0(n) := 1/z(n-1)\widetilde{c}(n-1)$, where $z(n)$ is a solution of (2.5) with $\lambda = \lambda_0 = 0$, where $\lambda_0$ is less than the least eigenvalue of (2.5), (2.14), such that $z(n) > 0$ for all $n = -1, \dots, m-1$. Then $u_0(n)$ is an eigenfunction of (3.3), (3.9) corresponding to the eigenvalue $\lambda_0 = 0$, where we define $u_0(0)$ via $\widehat{l}u_0(1) = 0$ and $u_0(-1) = -u_0(0)/\widehat{h}$.*

*Proof.* By construction, we have that

$$\widehat{h}u_0(-1) + u_0(0) = 0. \tag{4.6}$$

Also

$$\widehat{H} = \frac{\widetilde{b}(m-2)}{\widetilde{c}(m-2)} - \frac{\widehat{b}(m-1)}{\widehat{c}(m-1)} - \frac{z(m-2)}{z(m-1)}\frac{\widehat{c}(m-2)}{\widehat{c}(m-1)}. \tag{4.7}$$

By substituting in for $\widehat{b}(m-1)$, $\widehat{c}(m-1)$ and $\widehat{c}(m-2)$, we obtain

$$\widehat{H} = \frac{\widetilde{b}(m-2)}{\widetilde{c}(m-2)} - \frac{z(m-2)}{z(m-1)}\frac{\widetilde{c}(m-2)}{\widetilde{c}(m-1)} - \frac{z(m-1)}{z(m-2)} - \frac{z(m-3)\widetilde{c}(m-3)}{z(m-2)\widetilde{c}(m-2)}. \tag{4.8}$$

Thus

$$\widehat{H} = -\frac{z(m-2)}{z(m-1)}\frac{\widetilde{c}(m-2)}{\widetilde{c}(m-1)}. \tag{4.9}$$

Hence substituting the expressions for $u_0(m-1)$, $u_0(m)$, and $\widehat{H}$ into the above equation yields

$$\widehat{H}u_0(m-1) + u_0(m) = 0. \tag{4.10}$$

Thus $u_0(n)$ obeys the boundary conditions (3.9).

Next, we show that $u_0(n)$ solves (3.3). Substituting in for $u_0(n)$, we obtain that, for $n = 1, \ldots, m-1$,

$$\widehat{c}(n)u_0(n+1)-\widehat{b}(n)u_0(n)+\widehat{c}(n-1)u_0(n-1) = \frac{\widehat{c}(n)}{z(n)\widetilde{c}(n)}-\frac{\widehat{b}(n)}{z(n-1)\widetilde{c}(n-1)}+\frac{\widehat{c}(n-1)}{z(n-2)\widetilde{c}(n-2)}. \tag{4.11}$$

Using the expressions for $\widehat{c}$ and $\widehat{b}$, we obtain by direct substitution that

$$\widehat{c}(n)u_0(n+1) - \widehat{b}(n)u_0(n) + \widehat{c}(n-1)u_0(n-1) = 0, \quad n = 1, \ldots, m-1. \tag{4.12}$$

Now, if we examine the right-hand side of (3.3), we immediately see that it is equal to 0 for $\lambda_0 = 0$.

We now show that $u_0(n)$ solves (3.3) for $n = 0$ as well, that is $\widehat{l}u_0(0) = 0$. By (3.3) for $\lambda_0 = 0$,

$$\widehat{l}u_0(0) = -\widehat{c}(0)u_0(1) + \widehat{b}(0)u_0(0) - \widehat{c}(-1)u_0(-1). \tag{4.13}$$

Using $u_0(-1) = -\ u_0(0)/\widehat{h}$,

$$\widehat{l}u_0(0) = -\widehat{c}(0)u_0(1) + \left(\widehat{b}(0) + \frac{\widehat{c}(-1)}{\widehat{h}}\right)u_0(0). \tag{4.14}$$

Now, $\widehat{l}u_0(1) = 0$ gives

$$u_0(0) = -\ \frac{\widehat{c}(1)u_0(2) + \widehat{b}(1)u_0(1)}{\widehat{c}(0)}. \tag{4.15}$$

Thus,

$$\widehat{l}u_0(0) = -\widehat{c}(0)u_0(1) + \frac{\widehat{b}(0)}{\widehat{c}(0)}\Big(-\widehat{c}(1)u_0(2) + \widehat{b}(1)u_0(1)\Big) + \frac{\widehat{c}(-1)}{\widehat{h}\widehat{c}(0)}\Big(-\widehat{c}(1)u_0(2) + \widehat{b}(1)u_0(1)\Big). \tag{4.16}$$

Substituting in for $\widehat{c}(n),\ \widehat{b}(n),\ n=0,1$, and $u_0(n),\ n=1,2$, we obtain that

$$\begin{aligned}\widehat{l}u_0(0) &= \left(\frac{z(-1)\widetilde{c}(-1)}{z(0)\widetilde{c}(0)}+\frac{z(0)}{z(-1)}\right)\left[-\frac{z(0)\widetilde{c}(0)}{z(1)}\frac{1}{z(1)\widetilde{c}(1)}+\frac{1}{z(1)}\left(\frac{z(0)\widetilde{c}(0)}{z(1)\widetilde{c}(1)}+\frac{z(1)}{z(0)}\right)\right]\\&\quad+\frac{\widehat{c}(-1)}{\widehat{h}\widehat{c}(0)}\left[-\frac{z(0)\widetilde{c}(0)}{z(1)}\frac{1}{z(1)\widetilde{c}(1)}+\frac{1}{z(1)}\left(\frac{z(0)\widetilde{c}(0)}{z(1)\widetilde{c}(1)}+\frac{z(1)}{z(0)}\right)\right]-\frac{z(-1)\widetilde{c}(-1)}{z(0)}\frac{1}{z(0)\widetilde{c}(0)}.\end{aligned}\tag{4.17}$$

Note that

$$\frac{\widehat{c}(-1)}{\widehat{h}\widehat{c}(0)}=\frac{\widetilde{b}(0)}{\widetilde{c}(0)}-\frac{z(1)}{z(0)}-\frac{\widehat{b}(0)}{\widehat{c}(0)},\tag{4.18}$$

which when we substitute in for $\widehat{b}(0)$ and $\widehat{c}(0)$ becomes

$$\frac{\widehat{c}(-1)}{\widehat{h}\widehat{c}(0)}=\frac{\widetilde{b}(0)}{\widetilde{c}(0)}-\frac{z(1)}{z(0)}-\frac{z(-1)\widetilde{c}(-1)}{z(0)\widetilde{c}(0)}-\frac{z(0)}{z(-1)}.\tag{4.19}$$

Hence,

$$\widehat{l}u_0(0)=-\frac{z(-1)\widetilde{c}(-1)}{z(0)}\frac{1}{z(0)\widetilde{c}(0)}+\left(\frac{\widetilde{b}(0)}{\widetilde{c}(0)}-\frac{z(1)}{z(0)}\right)\left[-\frac{z(0)\widetilde{c}(0)}{z(1)}\frac{1}{z(1)\widetilde{c}(1)}+\frac{z(0)\widetilde{c}(0)}{z(1)\widetilde{c}(1)z(1)}+\frac{1}{z(0)}\right].\tag{4.20}$$

From (2.5) for $n=0$ and $\lambda_0=0$, we have that

$$-\frac{\widetilde{b}(0)}{\widetilde{c}(0)}+\frac{z(1)}{z(0)}+\frac{\widetilde{c}(-1)z(-1)}{\widetilde{c}(0)z(0)}=0.\tag{4.21}$$

Therefore,

$$\widehat{l}u_0(0)=-\frac{z(-1)\widetilde{c}(-1)}{z(0)}\frac{1}{z(0)\widetilde{c}(0)}+\frac{\widetilde{c}(-1)z(-1)}{\widetilde{c}(0)z(0)}\left[-\frac{z(0)\widetilde{c}(0)}{z(1)}\frac{1}{z(1)\widetilde{c}(1)}+\frac{z(0)\widetilde{c}(0)}{z(1)\widetilde{c}(1)z(1)}+\frac{1}{z(0)}\right],\tag{4.22}$$

giving by a straightforward calculation

$$\widehat{l}u_0(0)=0.\tag{4.23}$$

Thus $u_0(n)$ is a solution for (3.3), $n=0,\dots,m-1$ and hence an eigenfunction of (3.3), (3.9) corresponding to the eigenvalue $\lambda_0=0$. □

**Theorem 4.2.** *The boundary value problem* (4.3) *with boundary conditions*

$$v(-1) = 0, \qquad v(m-1) = 0, \tag{4.24}$$

*is the same as the original boundary value problem for $\tilde{v}(n)$, that is,* (2.5) *and* (2.14)*, where $u_0(n)$ is as in Lemma 4.1.*

*Proof.* All we need to show is that $c(n) = \tilde{c}(n)$ and $b(n) = \tilde{b}(n)$. Substituting in for $u_0$ gives directly that

$$c(n) = \tilde{c}(n). \tag{4.25}$$

Now,

$$b(n) = \left[\frac{u_0(n)\hat{c}(n)}{u_0(n+1)\hat{c}(n+1)} - \frac{\hat{c}(n-1)u_0(n-1)}{\hat{c}(n)u_0(n)} + \frac{\hat{b}(n)}{\hat{c}(n)} - \lambda_0\right]\frac{u_0(n)\hat{c}(n)}{u_0(n+1)}, \tag{4.26}$$

which since $c(n) = \tilde{c}(n)$ and $c(n) = u_0(n)\hat{c}(n)/u_0(n+1)$ gives

$$b(n) = \left[\frac{\tilde{c}(n)}{\hat{c}(n+1)} - \frac{\tilde{c}(n-1)}{\hat{c}(n)} + \frac{z(n-1)\tilde{c}(n-1)}{z(n)\tilde{c}(n)} + \frac{z(n)}{z(n-1)} - \lambda_0\right]\tilde{c}(n). \tag{4.27}$$

Using the expression for $\hat{c}$, we obtain that

$$b(n) = \left[\frac{\tilde{c}(n)z(n+1)}{\tilde{c}(n)z(n)} - \frac{\tilde{c}(n-1)z(n)}{\tilde{c}(n-1)z(n-1)} + \frac{z(n-1)\tilde{c}(n-1)}{z(n)\tilde{c}(n)} + \frac{z(n)}{z(n-1)} - \lambda_0\right]\tilde{c}(n). \tag{4.28}$$

But $z(n)$ obeys (2.5) for $\lambda = \lambda_0 = 0$ thus

$$b(n) = \left[\frac{\tilde{b}(n)z(n)}{z(n)\tilde{c}(n)}\right]\tilde{c}(n) = \tilde{b}(n). \tag{4.29}$$

□

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