Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2010, Article ID 765259, 8 pages
doi:10.1155/2010/765259

Research Article

On the Symmetric Properties of Higher-Order Twisted q-Euler Numbers and Polynomials

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Received 14 December 2009; Accepted 19 March 2010

Academic Editor: Panayiotis Siafarikas

In 2009, Kim et al. gave some identities of symmetry for the twisted Euler polynomials of higher-order, recently. In this paper, we extend our result to the higher-order twisted q-Euler numbers and polynomials. The purpose of this paper is to establish various identities concerning higher-order twisted q-Euler numbers and polynomials by the properties of p-adic invariant integral on $\mathbb{Z}_p$. Especially, if $q = 1$, we derive the result of Kim et al. (2009).

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, the symbols $\mathbb{Z}$, $\mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}$, and $\mathbb{C}_p$ will denote the ring of rational integers, the ring of p-adic integers, the field of p-adic rational numbers, the complex number field, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \bigcup \{0\}$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = 1/p$.

When one talks of q-extension, q is variously considered as an indeterminate, a complex $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume that $|q-1|_p < p^{-1/(p-1)}$ so that $q^x = \exp(x \log q)$ for each $x \in \mathbb{Z}_p$. We use the following notation:

$$[x]_q = \frac{1-q^x}{1-q}, \quad [x]_{-q} = \frac{1-(-q)^x}{1+q} \quad \forall x \in \mathbb{Z}_p. \tag{1.1}$$

For a fixed positive integer d with $(p, d) = 1$, set

$$X = X_d = \frac{\lim_{\overleftarrow{n}} \mathbb{Z}}{dp^n\mathbb{Z}}, \qquad X_1 = \mathbb{Z}_p,$$
$$X^* = \bigcup_{\substack{0<a<dp,\\ (a,p)=1}} (a + dp\mathbb{Z}_p), \tag{1.2}$$
$$a + dp^n\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^n}\},$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \leq a < dp^n$. For any $n \in \mathbb{N}$,

$$\mu_q(a + dp^n\mathbb{Z}_p) = \frac{q^a}{[dp^n]_q} \tag{1.3}$$

(see [1–13]) is known to be a distribution on X.

We say that f is a uniformly differentiable function at $a \in \mathbb{Z}_p$ and denote this property by $f \in UD(\mathbb{Z}_p)$ if the difference quotients

$$F_f(x, y) = \frac{f(x) - f(y)}{x - y} \tag{1.4}$$

have a limit $f'(a)$ as $(x, y) \to (a, a)$.

For $f \in UD(\mathbb{Z}_p)$, the fermionic p-adic invariant q-integral on $\mathbb{Z}_p$ is defined as

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = \lim_{n\to\infty} \frac{1}{[p^n]_{-q}} \sum_{x=0}^{p^n-1} f(x)(-q)^x \tag{1.5}$$

(see [14]). Let us define the fermionic p-adic invariant integral on $\mathbb{Z}_p$ as follows:

$$I_{-1}(f) = \lim_{q\to 1} I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = \lim_{n\to\infty} \sum_{x=0}^{p^n-1} f(x)(-1)^x \tag{1.6}$$

(see [1–12, 14–20]). From the definition of q-integral, we have

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0), \quad \text{where } f_1(x) = f(x+1). \tag{1.7}$$

For $n \in \mathbb{N}$, let T_p be the p-adic locally constant space defined by

$$T_p = \bigcup_{n\geq 1} \mathbb{C}_{p^n} = \lim_{n\to\infty} \mathbb{C}_{p^n} = \mathbb{C}_{p^\infty}, \tag{1.8}$$

where $\mathbb{C}_{p^n} = \{\zeta \in \mathbb{C}_p \mid \zeta^{p^n} = 1 \text{ for some } n \geq 0\}$ is the cyclic group of order p^n.

It is well known that the twisted q-Euler polynomials of order k are defined as

$$e^{xt}\left(\frac{2}{e^t\zeta q + 1}\right)^k = \sum_{n=0}^{\infty} E_{n,\zeta,q}^{(k)}(x)\frac{t^n}{n!}, \quad \zeta \in T_p, \tag{1.9}$$

and $E_{n,\zeta,q}^{(k)} = E_{n,\zeta,q}^{(k)}(0)$ are called the twisted q-Euler numbers of order k. When $k = 1$, the polynomials and numbers are called the twisted q-Euler polynomials and numbers, respectively. When $k = 1$ and $q = 1$, the polynomials and numbers are called the twisted Euler polynomials and numbers, respectively. When $k = 1$, $q = 1$, and $\zeta = 1$, the polynomials and numbers are called the ordinary Euler polynomials and numbers, respectively.

In [15], Kim et al. gave some identities of symmetry for the twisted Euler polynomials of higher order, recently. In this paper, we extend our result to the higher-order twisted q-Euler numbers and polynomials.

The purpose of this paper is to establish various identities concerning higher-order twisted q-Euler numbers and polynomials by the properties of p-adic invariant integral on $\mathbb{Z}_p$. Especially, if $q = 1$, we derive the result of [15].

2. Some Identities of the Higher-Order Twisted q-Euler Numbers and Polynomials

Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 (\text{mod } 2)$ and $w_2 \equiv 1 (\text{mod } 2)$.

For $\zeta \in T_p$ and $m \in \mathbb{N}$, we set

$$\begin{aligned} R_q^{(m)}(w_1, w_2 : \zeta) = {} & \frac{\int_{\mathbb{Z}_p^m} e^{(\sum_{i=1}^m x_i + w_2 x)w_1 t} \zeta^{(\sum_{i=1}^m x_i)w_1} q^{(\sum_{i=1}^m x_i)w_1} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_m)}{\int_{\mathbb{Z}_p} e^{w_1 w_2 x t}\zeta^{w_1 w_2 x} q^{w_1 w_2 x} d\mu_{-1}(x)} \\ & \times \int_{\mathbb{Z}_p^m} e^{(\sum_{i=1}^m x_i + w_1 y)w_2 t} \zeta^{(\sum_{i=1}^m x_i)w_2} q^{(\sum_{i=1}^m x_i)w_2} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_m), \end{aligned} \tag{2.1}$$

where

$$\int_{\mathbb{Z}_p^m} f(x_1,\ldots,x_m)d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_m) = \underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{m\text{-times}} f(x_1,\ldots,x_m)d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_m). \tag{2.2}$$

In (2.1), we note that $R_q^{(m)}(w_1, w_2 : \zeta)$ is symmetric in w_1 and w_2.

From (2.1), we derive that

$$
\begin{aligned}
R_q^{(m)}(w_1, w_2 : \zeta) = {}& e^{w_1 w_2 x t} \int_{\mathbb{Z}_p^m} e^{(\sum_{i=1}^m x_i) w_1 t} \zeta^{(\sum_{i=1}^m x_i) w_1} q^{(\sum_{i=1}^m x_i) w_1} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_m) \\
& \times \frac{\int_{\mathbb{Z}_p} e^{w_2 x_m t} \zeta^{w_2 x_m} q^{w_2 x_m} d\mu_{-1}(x_m)}{\int_{\mathbb{Z}_p} e^{w_1 w_2 x t} \zeta^{w_1 w_2 x} q^{w_1 w_2 x} d\mu_{-1}(x)} \\
& \times e^{w_1 w_2 y t} \int_{\mathbb{Z}_{p^{m-1}}} e^{(\sum_{i=1}^{m-1} x_i) w_2 t} \zeta^{(\sum_{i=1}^{m-1} x_i) w_2} q^{(\sum_{i=1}^{m-1} x_i) w_2} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_{m-1}).
\end{aligned}
\tag{2.3}
$$

From the definition of q-integral, we also see that

$$
\begin{aligned}
&\int_{\mathbb{Z}_{p^m}} e^{(\sum_{i=1}^m x_i) w_1 t} \zeta^{(\sum_{i=1}^m x_i) w_1} q^{(\sum_{i=1}^m x_i) w_1} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_m) e^{w_1 w_2 x t} \\
&\quad = \left(\frac{2}{e^{w_1 t} \zeta^{w_1} q^{w_1} + 1} \right)^m e^{w_1 w_2 x t} = \sum_{n=0}^{\infty} E_{n, \zeta^{w_1}, q^{w_1}}^{(m)}(w_2 x) \frac{{w_1}^n t^n}{n!}.
\end{aligned}
\tag{2.4}
$$

It is easy to see that

$$
\frac{\int_{\mathbb{Z}_p} e^{xt} \zeta^x q^x d\mu(x)}{\int_{\mathbb{Z}_p} e^{w_1 x t} \zeta^{w_1 x} q^{w_1 x} d\mu(x)} = \sum_{l=0}^{w_1 - 1} (-1)^l \zeta^l q^l e^{lt} = \sum_{k=0}^{\infty} T_{k,q}(w_1 - 1 : \zeta) \frac{t^k}{k!},
\tag{2.5}
$$

where $T_{k,q}(w_1 - 1 : \zeta) = \sum_{l=0}^{w_1 - 1} (-1)^l \zeta^l q^l l^k$.

From (2.3), (2.4), and (2.5), we can derive

$$
\begin{aligned}
&R_q^{(m)}(w_1, w_2 : \zeta) \\
&= \left(\sum_{l=0}^{\infty} E_{l, \zeta^{w_1}, q^{w_1}}^{(m)}(w_2 x) \frac{w_1^l t^l}{l!} \right) \left(\sum_{k=0}^{\infty} T_{k, q^{w_2}}(w_1 - 1 : \zeta^{w_2}) \frac{w_2^k t^k}{k!} \right) \left(\sum_{i=0}^{\infty} E_{i, \zeta^{w_2}, q^{w_2}}^{(m-1)}(w_1 y) \frac{{w_2}^i t^i}{i!} \right) \\
&= \sum_{n=0}^{\infty} \left\{ \sum_{j=0}^{n} \binom{n}{j} w_2^j w_1^{n-j} E_{n-j, \zeta^{w_1}, q^{w_1}}^{(m)}(w_2 x) \sum_{k=0}^{j} T_{k, q^{w_2}}(w_1 - 1 : \zeta^{w_2}) \binom{j}{k} E_{j-k, \zeta^{w_2}, q^{w_2}}^{(m-1)}(w_1 y) \right\} \frac{t^n}{n!}.
\end{aligned}
\tag{2.6}
$$

From the symmetry of $R_q^{(m)}(w_1, w_2 : \zeta)$ in w_1 and w_2, we also see that

$$
\begin{aligned}
&R_q^{(m)}(w_1, w_2 : \zeta) \\
&= \sum_{n=0}^{\infty} \left\{ \sum_{j=0}^{n} \binom{n}{j} w_1^j w_2^{n-j} E_{n-j, \zeta^{w_2}, q^{w_2}}^{(m)}(w_1 x) \sum_{k=0}^{j} T_{k, q^{w_1}}(w_2 - 1 : \zeta^{w_1}) \binom{j}{k} E_{j-k, \zeta^{w_1}, q^{w_1}}^{(m-1)}(w_2 y) \right\} \frac{t^n}{n!}.
\end{aligned}
\tag{2.7}
$$

Comparing the coefficients on the both sides of (2.6) and (2.7), we obtain an identity for the twisted q-Euler polynomials of higher order as follows.

Theorem 2.1. *Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1(\bmod 2)$ and $w_2 \equiv 1(\bmod 2)$.*
For $n \in \mathbb{Z}_+$ and $m \in \mathbb{N}$, we have

$$\begin{aligned}
&\sum_{j=0}^{n}\binom{n}{j} w_2^j w_1^{n-j} E^{(m)}_{n-j,\zeta^{w_1},q^{w_1}}(w_2 x)\sum_{k=0}^{j} T_{k,q^{w_2}}(w_1-1:\zeta^{w_2})\binom{j}{k} E^{(m-1)}_{j-k,\zeta^{w_2},q^{w_2}}(w_1 y)\\
&\quad=\sum_{j=0}^{n}\binom{n}{j} w_1^j w_2^{n-j} E^{(m)}_{n-j,\zeta^{w_2},q^{w_2}}(w_1 x)\sum_{k=0}^{j} T_{k,q^{w_1}}(w_2-1:\zeta^{w_1})\binom{j}{k} E^{(m-1)}_{j-k,\zeta^{w_1},q^{w_1}}(w_2 y).
\end{aligned}\tag{2.8}$$

Remark 2.2. Taking $m=1$ and $y=0$ in Theorem 2.1, we can derive the following identity:

$$\begin{aligned}
&\sum_{j=0}^{n}\binom{n}{j} w_2^j w_1^{n-j} E_{n-j,\zeta^{w_1},q^{w_1}}(w_2 x)\sum_{k=0}^{j} T_{k,q^{w_2}}(w_1-1:\zeta^{w_2})\binom{j}{k}\\
&\quad=\sum_{j=0}^{n}\binom{n}{j} w_1^j w_2^{n-j} E_{n-j,\zeta^{w_2},q^{w_2}}(w_1 x)\sum_{k=0}^{j} T_{k,q^{w_1}}(w_2-1:\zeta^{w_1})\binom{j}{k}.
\end{aligned}\tag{2.9}$$

Moreover, if we take $x=0$ and $y=0$ in Theorem 2.1, then we have the following identity for the twisted q-Euler numbers of higher order.

Corollary 2.3. *Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1(\bmod 2)$ and $w_2 \equiv 1(\bmod 2)$. For $n \in \mathbb{Z}_+$ and $m \in \mathbb{N}$, we have*

$$\begin{aligned}
&\sum_{j=0}^{n}\binom{n}{j} w_2^j w_1^{n-j} E^{(m)}_{n-j,\zeta^{w_1},q^{w_1}}\sum_{k=0}^{j} T_{k,q^{w_2}}(w_1-1:\zeta^{w_2})\binom{j}{k} E^{(m-1)}_{j-k,\zeta^{w_2},q^{w_2}}\\
&\quad=\sum_{j=0}^{n}\binom{n}{j} w_1^j w_2^{n-j} E^{(m)}_{n-j,\zeta^{w_2},q^{w_2}}\sum_{k=0}^{j} T_{k,q^{w_1}}(w_2-1:\zeta^{w_1})\binom{j}{k} E^{(m-1)}_{j-k,\zeta^{w_1},q^{w_1}}.
\end{aligned}\tag{2.10}$$

We also note that taking $m=1$ in Corollary 1 shows the following identity:

$$\begin{aligned}
&\sum_{j=0}^{n}\binom{n}{j} w_2^j w_1^{n-j} E_{n-j,\zeta^{w_1},q^{w_1}}\sum_{k=0}^{j} T_{k,q^{w_2}}(w_1-1:\zeta^{w_2})\binom{j}{k}\\
&\quad=\sum_{j=0}^{n}\binom{n}{j} w_1^j w_2^{n-j} E_{n-j,\zeta^{w_2},q^{w_2}}\sum_{k=0}^{j} T_{k,q^{w_1}}(w_2-1:\zeta^{w_1})\binom{j}{k}.
\end{aligned}\tag{2.11}$$

Now we will derive another interesting identities for the twisted q-Euler numbers and polynomials of higher order. From (2.3), we can derive that

$$\begin{aligned}
&R_q^{(m)}(w_1,w_2:\zeta)\\
&=\left\{\sum_{i=0}^{w_1-1}(-1)^i q^{w_2 i}\zeta^{w_2 i}\right\}\left\{\sum_{k=0}^{\infty}\left(E_{k,\zeta^{w_1},q^{w_1}}^{(m)}\left(\frac{w_2}{w_1}i+w_2x\right)w_1^k\frac{t^k}{k!}\right\}\left\{\sum_{l=0}^{\infty}\left(E_{l,\zeta^{w_2},q^{w_2}}^{(m-1)}(w_1y)w_2^l\right)\frac{t^l}{l!}\right\}\\
&=\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\binom{n}{k}w_1^k w_2^{n-k}E_{n-k,\zeta^{w_2},q^{w_2}}^{(m-1)}(w_1y)\sum_{i=0}^{w_1-1}(-1)^i\zeta^{w_2 i}q^{w_2 i}E_{k,\zeta^{w_1},q^{w_1}}^{(m)}\left(w_2x+\frac{w_2}{w_1}i\right)\right\}\frac{t^n}{n!}.
\end{aligned}\tag{2.12}$$

From the symmetry of $R_q^{(m)}(w_1,w_2:\zeta)$ in w_1 and w_2, we see that

$$\begin{aligned}
&R_q^{(m)}(w_1,w_2:\zeta)\\
&=\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\binom{n}{k}w_2^k w_1^{n-k}E_{n-k,\zeta^{w_1},q^{w_1}}^{(m-1)}(w_2y)\sum_{i=0}^{w_2-1}(-1)^i\zeta^{w_1 i}q^{w_1 i}E_{k,\zeta^{w_2},q^{w_2}}^{(m)}\left(w_1x+\frac{w_1}{w_2}i\right)\right\}\frac{t^n}{n!}.
\end{aligned}\tag{2.13}$$

Comparing the coefficients on the both sides of (2.12) and (2.13), we obtain the following theorem which shows the relationship between the power sums and the twisted q-Euler polynomials.

Theorem 2.4. *Let $w_1,w_2\in\mathbb{N}$ with $w_1\equiv 1(\mathrm{mod}\,2)$ and $w_2\equiv 1(\mathrm{mod}\,2)$. For $n\in\mathbb{Z}_+$ and $m\in\mathbb{N}$, we have*

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}w_1^k w_2^{n-k}E_{n-k,\zeta^{w_2},q^{w_2}}^{(m-1)}(w_1y)\sum_{i=0}^{w_1-1}(-1)^i\zeta^{w_2 i}q^{w_2 i}E_{k,\zeta^{w_1},q^{w_1}}^{(m)}\left(w_2x+\frac{w_2}{w_1}i\right)\\
&\quad=\sum_{k=0}^{n}\binom{n}{k}w_2^k w_1^{n-k}E_{n-k,\zeta^{w_1},q^{w_1}}^{(m-1)}(w_2y)\sum_{i=0}^{w_2-1}(-1)^i\zeta^{w_1 i}q^{w_1 i}E_{k,\zeta^{w_2},q^{w_2}}^{(m)}\left(w_1x+\frac{w_1}{w_2}i\right).
\end{aligned}\tag{2.14}$$

Remark 2.5. Let $m=1$ and $y=0$ in Theorem 2. Then it follows that

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}w_1^k w_2^{n-k}\sum_{i=0}^{w_1-1}(-1)^i\zeta^{w_2 i}q^{w_2 i}E_{k,\zeta^{w_1},q^{w_1}}\left(w_2x+\frac{w_2}{w_1}i\right)\\
&\quad=\sum_{k=0}^{n}\binom{n}{k}w_2^k w_1^{n-k}\sum_{i=0}^{w_2-1}(-1)^i\zeta^{w_1 i}q^{w_1 i}E_{k,\zeta^{w_2},q^{w_2}}\left(w_1x+\frac{w_1}{w_2}i\right).
\end{aligned}\tag{2.15}$$

Moreover, if we take $x=0$ and $y=0$ in Theorem 2.4, then we have the following identity for the twisted q-Euler numbers of higher order.

Corollary 2.6. *Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1(\mathrm{mod}\,2), w_2 \equiv 1(\mathrm{mod}\,2)$. For $n \in \mathbb{Z}_+$ and $m \in \mathbb{N}$, we have*

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k} w_1^k w_2^{n-k} E^{(m-1)}_{n-k,\zeta^{w_2},q^{w_2}} \sum_{i=0}^{w_1-1}(-1)^i \zeta^{w_2 i} q^{w_2 i} E^{(m)}_{k,\zeta^{w_1},q^{w_1}}\left(\frac{w_2}{w_1}i\right)\\
&\quad= \sum_{k=0}^{n}\binom{n}{k} w_2^k w_1^{n-k} E^{(m-1)}_{n-k,\zeta^{w_1},q^{w_1}} \sum_{i=0}^{w_2-1}(-1)^i \zeta^{w_1 i} q^{w_1 i} E^{(m)}_{k,\zeta^{w_2},q^{w_2}}\left(\frac{w_1}{w_2}i\right).
\end{aligned} \tag{2.16}$$

If we take $m = 1$ in Corollary 2.3, we derive the following identity for the twisted q-Euler polynomials: for $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1(\mathrm{mod}\,2), w_2 \equiv 1(\mathrm{mod}\,2)$, and $n \in \mathbb{Z}_+$,

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k} w_1^k w_2^{n-k} \sum_{i=0}^{w_1-1}(-1)^i \zeta^{w_2 i} q^{w_2 i} E_{k,\zeta^{w_1},q^{w_1}}\left(\frac{w_2}{w_1}i\right)\\
&\quad= \sum_{k=0}^{n}\binom{n}{k} w_2^k w_1^{n-k} \sum_{i=0}^{w_2-1}(-1)^i \zeta^{w_1 i} q^{w_1 i} E_{k,\zeta^{w_2},q^{w_2}}\left(\frac{w_1}{w_2}i\right).
\end{aligned} \tag{2.17}$$

Remark 2.7. If $q = 1$, we can observe the result of [15].

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