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Research Article

Stabilities of Cubic Mappings in Fuzzy Normed Spaces

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Rassias (2001) introduced the pioneering cubic functional equation in the history of mathematical analysis: $f(x+2y)-3f(x+y)+3f(x)-f(x-y)=6f(y)$ and solved the pertinent famous Ulam stability problem for this inspiring equation. This Rassias cubic functional equation was the historic transition from the following famous Euler-Lagrange-Rassias quadratic functional equation: $f(x+y)-2f(x)+f(x-y)=2f(y)$ to the cubic functional equations. In this paper, we prove the Ulam-Hyers stability of the cubic functional equation: $f(x+3y)-3f(x+y)+3f(y-x)-f(x-3y)=48f(y)$ in fuzzy normed linear spaces. We use the definition of fuzzy normed linear spaces to establish a fuzzy version of a generalized Hyers-Ulam-Rassias stability for above equation in the fuzzy normed linear space setting. The fuzzy sequentially continuity of the cubic mappings is discussed.

1. Introduction

Studies on fuzzy normed linear spaces are relatively recent in the field of fuzzy functional analysis. The notion of fuzzyness has a wide application in many areas of science. In 1984, Katsaras [1] first introduced a definition of fuzzy norm on a linear space. Later, several notions of fuzzy norm have been introduced and discussed from different points of view [2, 3]. Concepts of sectional fuzzy continuous mappings and strong uniformly convex fuzzy normed linear spaces have been introduced by Bag and Samanta [4]. Bag and Samanta [5] introduced a notion of boundedness of a linear operator between fuzzy normed spaces, and studied the relation between fuzzy continuity and fuzzy boundedness. They studied boundedness of linear operators over fuzzy normed linear spaces such as fuzzy continuity, sequential fuzzy continuity, weakly fuzzy continuity and strongly fuzzy continuity.

The problem of stability of functional equation originated from a question of Ulam [6] concerning the stability of group homomorphism in 1940. Hyers gave a partial affirmative answer to the question of Ulam for Banach spaces in the next year [7]. Let X and Y be Banach spaces. Assume that $f : X \to Y$ satisfies $\|f(x+y)-f(x)-f(y)\| \le \epsilon$ for all $x, y \in X$ and some $\epsilon > 0$. Then, there exists a unique additive mapping $T : X \to Y$ such that $\|f(x) - T(x)\| \le \epsilon$ for all $x \in X$. Hyers′ theorem was generalized by Aoki [8] for additive mappings. In 1978, a generalized solution for approximately linear mappings was given by Th. M. Rassias [9]. He considered a mapping $f : X \to Y$ satisfying the condition

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in X$, where $\epsilon \ge 0$ and $0 \le p < 1$. This result was later extended to all $p \neq 1$.

In 1982, J. M. Rassias [10] gave a further generalization of the result of Hyers and prove the following theorem using weaker conditions controlled by a product of powers of norms. Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon(\|x\|^p \|y\|^p) \tag{1.2}$$

for all $x, y \in E$, where $\epsilon \ge 0$ and $0 \le p < 1/2$. Then there exists a unique additive mapping $L : E \to E'$ which satisfies

$$\|f(x) - L(x)\| \le \frac{\epsilon}{2 - 2^{2p}} \|x\|^{2p} \tag{1.3}$$

for all $x \in E$. The above mentioned stability involving a product of powers of norms is called Ulam–Gavruta–Rassias stability by various authors [11–25].

In 2008, J. M. Rassias [26] generalized even further the above two stabilities via a new stability involving a mixed product-sum of powers of norms, called JMRassias stability by several authors [27–30].

In the last two decades, several form of mixed type functional equation and its Ulam–Hyers stability are dealt in various spaces like Fuzzy normed spaces, Random normed spaces, Quasi–Banach spaces, Quasinormed linear spaces and Banach algebra by various authors like [31–40].

In 1994, Cheng and Mordeson [2] introduced an idea of a fuzzy norm on a linear space whose associated metric is Kramosil and Michálek type [41]. Since then some mathematicians have defined fuzzy metrics and norms on a linear space from various points of view [42–44].

In 2001, J. M. Rassias [45] introduced the pioneering cubic functional equation in history of mathematical analysis, as follows:

$$f(x+2y) - 3f(x+y) + 3f(x) - f(x-y) = 6f(y), \tag{$*$}$$

and solved the famous Ulam stability problem for this inspiring functional equation. Note that this cubic functional equation $(*)$ was the historic transition from the following famous

Euler-Lagrange quadratic functional equation:

$$f(x+y) - 2f(x) + f(x-y) = 2f(y) \tag{1.4}$$

to the cubic functional equation $(*)$.

The notion of fuzzy stability of the functional equations was initiated by Mirmostafaee and Moslehian in [46]. Later, several various fuzzy versions of stability were investigated [47, 48]. Now, let us introduce the following functional equation:

$$f(x+3y) - 3f(x+y) + 3f(x-y) - f(x-3y) = 48f(y). \tag{1.5}$$

Since the cubic function $f(x) = cx^3$ satisfies in this equation, so we promise that (1.5) is called a cubic functional equation and every solution will be called a cubic function. The stability problem for the cubic functional equation was proved by Wiwatwanich and Nakmahachalasint [49] for mapping $f : E_1 \to E_2$, where E_1 and E_2 are real Banach spaces. A number of mathematicians worked on the stability of some types of the cubic equation [45, 50–54]. In [55], Park and Jung introduced a cubic functional equation different from (1.5) as follows:

$$f(x+3y) + f(3y-x) = 3f(x+y) + 3f(x-y) + 48f(y) \tag{1.6}$$

and investigated the generalized Hyers-Ulam-Rassias stability for this equation on abelian groups. They also obtained results in sense of Hyers-Ulam stability and Hyers-Ulam-Rassias stability. A number of results concerning the stability of different functional equations can be found in [23, 56–59].

In this paper, we prove the Hyers-Ulam-Rassias stability of the cubic functional equation (1.5) in fuzzy normed spaces. Later, we will show that there exists a close relationship between the fuzzy sequentially continuity behavior of a cubic function, control function and the unique cubic mapping which approximates the cubic map.

2. Notation and Preliminary Results

In this section some definitions and preliminary results are given which will be used in this paper. Following [48], we give the following notion of a fuzzy norm.

Definition 2.1. Let X be a linear space. A fuzzy subset N of $X \times \mathbb{R}$ into $[0,1]$ is called a fuzzy norm on X if for every $x, y \in X$ and $s, t \in \mathbb{R}$

($N1$) $N(x,t) = 0$ for $t \leq 0$,

($N2$) $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$,

($N3$) $N(cx,t) = N(x, t/|c|)$ if $c \neq 0$,

($N4$) $N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}$,

($N5$) $N(x,\cdot\,)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$.

The pair (X, N) will be referred to as a fuzzy normed linear space. One may regard $N(x,t)$ as the truth value of the statement "the norm of x is less than or equal to the real number r". Let $(X, \|\cdot\|)$ be a normed linear space. One can be easily verify that

$$N(x,t) = \begin{cases} 0, & t \leq \|x\|, \\ 1, & t > \|x\| \end{cases} \tag{2.1}$$

is a fuzzy norm on X. Other examples of fuzzy normed linear spaces are considered in the main text of this paper.

Note that the fuzzy normed linear space (X, N) is exactly a Menger probabilistic normed linear space (X, N, T) where $T(a,b) = \min\{a,b\}$ [60].

Definition 2.2. A sequence $\{x_n\}$ in a fuzzy normed space (X, N) converges to $x \in X$ (one denote $x_n \to x$) if for every $t > 0$ and $\epsilon > 0$, there exists a positive integer k such that $N(x_n - x, t) > 1 - \epsilon$ whenever $n \geq k$.

Recall that, a sequence $\{x_n\}$ in X is called Cauchy if for every $t > 0$ and $\epsilon > 0$, there exists a positive integer k such that for all $n \geq k$ and all $m \in \mathbb{N}$, we have $N(x_{n+m} - x_n, t) > 1 - \epsilon$. It is known that every convergent sequence in a fuzzy normed space is Cauchy. The fuzzy normed space (X, N) is said to be fuzzy Banach space if every Cauchy sequence in X is convergent to a point in X [46].

3. Main Results

We will investigate the generalized Hyers-Ulam type theorem of the functional equation (1.5) in fuzzy normed spaces. In the following theorem, we will show that under special circumstances on the control function Q, every Q-almost cubic mapping f can be approximated by a cubic mapping C.

Theorem 3.1. *Let $\alpha \in (0,27) \cup (27,\infty)$. Let X be a linear space, and let (Z, N') be a fuzzy normed space. Suppose that an even function $Q : X \times X \to Z$ satisfies $Q(3^n x, 3^n y) = \alpha^n Q(x,y)$ for all $x, y \in X$ and for all $n \in \mathbb{N}$. Suppose that (Y, N') is a fuzzy Banach space. If a function $f : X \to Y$ satisfies*

$$N(f(x+3y) - 3f(x+y) + 3f(x-y) - f(x-3y) - 48f(y), t) \geq N'(Q(x,y), t) \tag{3.1}$$

for all $x, y \in X$ and $t > 0$, then there exists a unique cubic function $C : X \to Y$ which satisfies (1.5) *and the inequality*

$$N(f(x) - C(x), t) \geq \begin{cases} \min\left\{N'\left(Q(0,x), \dfrac{(27-\alpha)t}{3}\right), N'\left(Q(0,x), \dfrac{8(27-\alpha)t}{\alpha}\right)\right\}, & 0 < \alpha < 27 \\ \min\left\{N'\left(Q(0,x), \dfrac{(\alpha-27)t}{3}\right), N'\left(Q(0,x), \dfrac{8(\alpha-27)t}{\alpha}\right)\right\}, & \alpha > 27 \end{cases} \tag{3.2}$$

holds for all $x \in X$ and $t > 0$.

Proof. We have the following two cases.

Case 1 ($0 < \alpha < 27$). Replacing y by $-y$ in (3.1) and summing the resulting inequality with (3.1), we get

$$N(f(y) + f(-y), t) \geq N'(Q(x, y), 24t). \tag{3.3}$$

Since (3.1) and (3.3) hold for any x, let us fix $x = 0$ for convenience. By $(N4)$, we have

$$\begin{aligned} &N(2f(3y) - 54f(y), t) \\ &\quad\geq \min\left\{N'\left(Q(0, y), \frac{t}{3}\right), N\left(f(3y) + f(-3y), \frac{t}{3}\right), N\left(f(y) + f(-y), \frac{t}{9}\right)\right\} \\ &\quad\geq \min\left\{N'\left(Q(0, y), \frac{t}{3}\right), N'(Q(0, 3y), 8t), N'\left(Q(0, y), \frac{8t}{3}\right)\right\} \\ &\quad\geq \min\left\{N'\left(Q(0, y), \frac{t}{3}\right), N'\left(Q(0, y), \frac{8t}{\alpha}\right)\right\}. \end{aligned} \tag{3.4}$$

Replacing y by x in (3.4). By $(N3)$, we have

$$N\left(\frac{f(3x)}{27} - f(x), t\right) \geq \min\left\{N'(Q(0, x)\ , 18t), N'\left(Q(0, x), \frac{432t}{\alpha}\right)\right\}. \tag{3.5}$$

Replacing x by $3^n x$ in (3.5), we get

$$\begin{aligned} N\left(\frac{f(3^{n+1}x)}{27^{n+1}} - \frac{f(3^n x)}{27^n}, \frac{t}{27^n}\right) &\geq \min\left\{N'(Q(0, 3^n x), 18t)\ , N'\left(Q(0, 3^n x), \frac{432t}{\alpha}\right)\right\} \\ &\geq \min\left\{N'\left(Q(0, x), \frac{18t}{\alpha^n}\right), N'\left(Q(0, x), \frac{432t}{\alpha^{n+1}}\right)\right\}. \end{aligned} \tag{3.6}$$

It follows from

$$\frac{f(3^n x)}{27^n} - f(x) = \sum_{i=0}^{n-1} \frac{f(3^{i+1}x)}{27^{i+1}} - \frac{f(3^i x)}{27^i}, \tag{3.7}$$

and last inequality that

$$\begin{aligned} N\left(\frac{f(3^n x)}{27^n} - f(x), \sum_{i=0}^{n-1} \frac{\alpha^i t}{27^i}\right) &\geq \min \bigcup_{i=0}^{n-1} \left\{N\left(\frac{f(3^{i+1}x)}{27^{i+1}} - \frac{f(3^i x)}{27^i}, \frac{\alpha^i t}{27^i}\right)\right\} \\ &\geq \min\left\{N'(Q(0, x), 18t), N'\left(Q(0, x), \frac{432t}{\alpha}\right)\right\}. \end{aligned} \tag{3.8}$$

In order to prove convergence of the sequence $\{f(3^n x)/27^n\}$, we replace x by $3^m x$ to find that for $m, n \in \mathbb{N}$,

$$
\begin{aligned}
N\left(\frac{f(3^{n+m}x)}{27^{n+m}} - \frac{f(3^m x)}{27^m}, \sum_{i=0}^{n-1} \frac{\alpha^i t}{27^{i+m}}\right) &\geq \min\left\{N'(Q(0,3^m x),18t), N'\left(Q(0,3^m x),\frac{432t}{\alpha}\right)\right\} \\
&\geq \min\left\{N'\left(Q(0,x),\frac{18t}{\alpha^m}\right), N'\left(Q(0,x),\frac{432t}{\alpha^{m+1}}\right)\right\}.
\end{aligned}
\tag{3.9}
$$

Replacing t by $\alpha^m t$ in last inequality to get

$$
N\left(\frac{f(3^{n+m}x)}{27^{n+m}} - \frac{f(3^m x)}{27^m}, \sum_{i=m}^{n+m-1} \frac{\alpha^i t}{27^{i}}\right) \geq \min\left\{N'(Q(0,x),18t), N'\left(Q(0,x),\frac{432t}{\alpha}\right)\right\}. \tag{3.10}
$$

For every $n \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\}$, we put

$$
a_{mn} = \sum_{i=m}^{n+m-1} \frac{\alpha^i}{27^i}. \tag{3.11}
$$

Replacing t by t/a_{mn} in last inequality, we observe that

$$
N\left(\frac{f(3^{n+m}x)}{27^{n+m}} - \frac{f(3^m x)}{27^m}, t\right) \geq \min\left\{N'\left(Q(0,x),\frac{18t}{a_{mn}}\right), N'\left(Q(0,x),\frac{432t}{\alpha a_{mn}}\right)\right\}. \tag{3.12}
$$

Let $t > 0$ and $\epsilon > 0$ be given. Since $\lim_{t\to\infty} N'(Q(0,x),t) = 1$, there is some $t_1 \geq 0$ such that $N'(Q(0,x),t_2) > 1-\epsilon$ for every $t_2 > t_1$. The convergence of the series $\sum_{i=0}^{\infty}(\alpha^i/27^i)$ gives some m_1 such that $\min\{432t/\alpha a_{mn}, 18t/a_{mn}\} > t_1$ for every $m \geq m_1$ and $n \in \mathbb{N}$. For every $m \geq m_1$ and $n \in \mathbb{N}$, we have

$$
\begin{aligned}
N\left(\frac{f(3^{n+m}x)}{27^{n+m}} - \frac{f(3^m x)}{27^m}, t\right) &\geq \min\left\{N'\left(Q(0,x),\frac{18t}{a_{mn}}\right), N'\left(Q(0,x),\frac{432t}{\alpha a_{mn}}\right)\right\} \\
&\geq \min\{1-\epsilon, 1-\epsilon\} = 1-\epsilon.
\end{aligned}
\tag{3.13}
$$

This shows that $\{f(3^n x)/27^n\}$ is a Cauchy sequence in the fuzzy Banach space (Y, N), therefore this sequence converges to some point $C(x) \in Y$. Fix $x \in X$ and put $m = 0$ in (3.13) to obtain

$$
N\left(\frac{f(3^n x)}{27^n} - f(x), t\right) \geq \min\left\{N'\left(Q(0,x),\frac{18t}{a_{0n}}\right), N'\left(Q(0,x),\frac{432t}{\alpha a_{0n}}\right)\right\}. \tag{3.14}
$$

For every $n \in \mathbb{N}$,

$$
N(C(x) - f(x), t) \geq \min\left\{N\left(C(x) - \frac{f(3^n x)}{27^n}, \frac{t}{2}\right), N\left(\frac{f(3^n x)}{27^n} - f(x), \frac{t}{2}\right)\right\}. \tag{3.15}
$$

The first two terms on the right hand side of the above inequality tend to 1 as $n \to \infty$. Therefore we have

$$\begin{aligned} N(C(x)-f(x),t) &\geq \min\left\{N\left(C(x)-\frac{f(3^nx)}{27^n},\frac{t}{2}\right),N\left(\frac{f(3^nx)}{27^n}-f(x),\frac{t}{2}\right)\right\} \\ &\geq \min\left\{N'\left(Q(0,x),\frac{9t}{a_{0n}}\right),N'\left(Q(0,x),\frac{216t}{\alpha a_{0n}}\right)\right\} \end{aligned} \tag{3.16}$$

for n large enough. By last inequality, we have

$$N(C(x)-f(x),\mathrm{t}) \geq \min\left\{N'\left(Q(0,x),\frac{(27-\alpha)t}{3}\right),N'\left(Q(0,x),\frac{8(27-\alpha)t}{\alpha}\right)\right\}. \tag{3.17}$$

Now, we show that C is cubic. Use inequality (3.1) with x replaced by 3^nx and y by 3^ny to find that

$$\begin{aligned} &N\left(\frac{f(3^n(x+3y))}{27^n}-\frac{3f(3^n(x+y))}{27^n}+\frac{3f(3^n(x-y))}{27^n}-\frac{f(3^n(x-3y))}{27^n}-\frac{48f(3^ny)}{27^n},t\right) \\ &\quad\geq N'(Q(3^nx,3^ny),27^nt) = N'\left(Q(x,y),\frac{27^nt}{\alpha^n}\right). \end{aligned} \tag{3.18}$$

On the other hand $0<\alpha<27$, hence by $(N5)$

$$\lim_{n\to\infty} N'\left(Q(x,y),\frac{27^nt}{\alpha^n}\right)=1. \tag{3.19}$$

We conclude that C fulfills (1.5). It remains to prove the uniqueness assertion. Let C' be another cubic mapping satisfying (3.17). Fix $x \in X$. Obviously

$$C(3^nx)=27^nC(x), \quad C'(3^nx)=27^nC'(x) \tag{3.20}$$

for all $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, we can write

$$\begin{aligned} N(C(x)-C'(x),t) &= N\left(\frac{C(3^nx)}{27^n}-\frac{C'(3^nx)}{27^n},t\right) \\ &\geq \min\left\{N\left(\frac{C(3^nx)}{27^n}-\frac{f(3^nx)}{27^n},\frac{t}{2}\right),N\left(\frac{f(3^nx)}{27^n}-\frac{C'(3^nx)}{27^n},\frac{t}{2}\right)\right\} \\ &\geq \min\left\{N'\left(Q(0,3^nx),\frac{27^{n-1}(27-\alpha)9t}{2}\right),N'\left(Q(0,3^nx),\frac{27^n(27-\alpha)4t}{\alpha}\right)\right\} \\ &\geq \min\left\{N'\left(Q(0,x),\frac{27^{n-1}(27-\alpha)9t}{2\alpha^n}\right),N'\left(Q(0,x),\frac{27^n(27-\alpha)4t}{\alpha^{n+1}}\right)\right\}. \end{aligned} \tag{3.21}$$

Since $0 < \alpha < 27$, we have

$$\lim_{n\to\infty} \mathrm{N}'\left(Q(0,x), \frac{27^{n-1}(27-\alpha)9t}{2\alpha^n}\right) = N'\left(Q(0,x), \frac{27^n(27-\alpha)4t}{\alpha^{n+1}}\right) = 1. \tag{3.22}$$

Therefore $N'(C(x) - C'(x), t) = 1$ for all $t > 0$, whence $C(x) = C'(x)$.

Case 2 ($27 < \alpha$). We can state the proof in the same pattern as we did in the first case. Replace x, t by $x/3$ and $2t$, respectively in (3.4) to get

$$N'\left(f(x) - 27f\left(\frac{x}{3}\right), t\right) \geq \min\left\{N'\left(Q\left(0, \frac{x}{3}\right), \frac{2t}{3}\right), N'\left(Q\left(0, \frac{x}{3}\right), \frac{16t}{\alpha}\right)\right\}. \tag{3.23}$$

We replace y and t by $x/3^n$ and $t/27^n$ in last inequality, respectively, we find that

$$\begin{aligned}
&N\left(27^n f\left(\frac{x}{3^n}\right) - 27^{n+1} f\left(\frac{x}{3^{n+1}}\right), t\right) \\
&\quad\geq \min\left\{N'\left(Q\left(0, \frac{x}{3^{n+1}}\right), \frac{2t}{3\times 27^n}\right), N'\left(Q\left(0, \frac{x}{3^{n+1}}\right), \frac{16t}{27^n \alpha}\right)\right\} \\
&\quad\geq \min\left\{N'\left(Q(0,x), \frac{2\alpha^{n+1} t}{3\times 27^n}\right), N'\left(Q(0,x), \frac{16\alpha^n t}{27^n}\right)\right\}.
\end{aligned} \tag{3.24}$$

For each $n \in \mathbb{N}$, one can deduce

$$N\left(27^n f\left(\frac{x}{3^n}\right) - f(x), t\right) \geq \min\left\{N'\left(Q(0,x), \frac{2\alpha t}{3b_{0n}}\right), N'\left(Q(0,x), \frac{16t}{b_{0n}}\right)\right\} \tag{3.25}$$

where $b_{0n} = \sum_{i=0}^{n-1}(27^i/\alpha^i)$. It is easy to see that $\{27^n f(x/3^n)\}$ is a Cauchy sequence in (Y, N). Since (Y, N) is a fuzzy Banach space, this sequence converges to some point $C(x) \in Y$, that is,

$$C(x) = \lim_{n\to\infty} 27^n f\left(\frac{x}{3^n}\right). \tag{3.26}$$

Moreover, C satisfies (1.5) and

$$N(f(x) - C(x), t) \geq \min\left\{N'\left(Q(0,x), \frac{(\alpha-27)t}{3}\right), N'\left(Q(0,x), \frac{8(\alpha-27)t}{\alpha}\right)\right\}. \tag{3.27}$$

The proof for uniqueness of C for this case proceeds similarly to that in the previous case, hence it is omitted. □

We note that α need not be equal to 27. But we do not guarantee whether the cubic equation is stable in the sense of Hyers, Ulam and Rassias if $\alpha = 27$ is assumed in Theorem 3.1.

Remark 3.2. Let $0 < \alpha < 27$. Suppose that the mapping $t \mapsto N(Q(x) - f(x), \cdot)$ from $(0, \infty)$ into $[0,1]$ is right continuous. Then we get a fuzzy approximation better than (3.17) as follows.

For every $s, t > 0$, we have

$$\begin{aligned} N(C(x) - f(x), s + t) &\geq \min\left\{N\left(C(x) - \frac{f(3^n x)}{27^n}, s\right), N\left(\frac{f(3^n x)}{27^n} - f(x), t\right)\right\} \\ &\geq \min\left\{N'\left(Q(0,x), \frac{18t}{a_{0n}}\right), N'\left(Q(0,x), \frac{432t}{\alpha a_{0n}}\right)\right\} \end{aligned} \tag{3.28}$$

for large enough n. It follows that

$$N(C(x) - f(x), s + t) \geq \min\left\{N'\left(Q(0,x), \frac{2(27-\alpha)t}{3}\right), N'\left(Q(0,x), \frac{16(27-\alpha)t}{\alpha}\right)\right\}. \tag{3.29}$$

Tending s to zero we infer

$$N(C(x) - f(x), t) \geq \min\left\{N'\left(Q(0,x), \frac{2(27-\alpha)t}{3}\right), N'\left(Q(0,x), \frac{16(27-\alpha)t}{\alpha}\right)\right\}. \tag{3.30}$$

From Theorem 3.1, we obtain the following corollary concerning the stability of (1.5) in the sense of the JMRassias stability of functional equations controlled by the mixed product-sum of powers of norms introduced by J. M. Rassias [26] and called JMRassias stability by several authors [27–30].

Corollary 3.3. *Let X be a Banach space and let $\epsilon > 0$ be a real number. Suppose that a function $f : X \to X$ satisfies*

$$\left\| f(x+3y) - 3f(x+y) + 3f(x-y) - f(x-3y) - 48f(y) \right\| \leq \epsilon\left(\|x\|^p\|y\|^p + \|x\|^{2p} + \|y\|^{2p}\right) \tag{3.31}$$

for all $x, y \in X$ where $0 \leq p < 1/2$. Then there exists a unique cubic function $C : X \to X$ which satisfying (1.5) *and the inequality*

$$\left\|C(x) - f(x)\right\| \leq \frac{\|x\|^p \epsilon}{8} \tag{3.32}$$

for all $x \in X$. The function $C : X \to X$ is given by $C(x) = \lim_{n\to\infty} f(3^n x)/27^n$ for all $x \in X$.

Proof. Define $N : X \times \mathbb{R} \to [0,1]$ by

$$N(x,t) = \begin{cases} \dfrac{t}{t + \|x\|}, & t > 0, \\ 0, & t \leq 0. \end{cases} \tag{3.33}$$

It is easy to see that (X, N) is a fuzzy Banach space. Denote by $Q : X \times X \to \mathbb{R}$ the map sending each (x, y) to $\epsilon(\|x\|^p\|y\|^p + \|x\|^{2p} + \|y\|^{2p})$. By assumption,

$$N(f(x+3y) - 3f(x+y) + 3f(x-y) - f(x-3y) - 48f(y), t) \geq N'(Q(x,y), t). \quad (3.34)$$

Note that $N' : \mathbb{R} \times \mathbb{R} \to [0,1]$ given by

$$N'(x,t) = \begin{cases} \dfrac{t}{t + \|x\|}, & t > 0, \\ 0, & t \leq 0 \end{cases} \quad (3.35)$$

is a fuzzy norm on $\mathbb{R}$. By Theorem 3.1, there exists a unique cubic function $C : X \to X$ satisfies (1.5) and inequality

$$\begin{aligned} \frac{t}{t + \|f(x) - C(x)\|} &= N(f(x) - C(x), t) \\ &\geq \min\{N'(Q(0,x), 8t), N'(Q(0,x), 64t)\} \\ &= \min\left\{\frac{8t}{8t + \epsilon\|x\|}, \frac{64t}{64t + \epsilon\|x\|}\right\} \\ &= \frac{8t}{8t + \epsilon\|x\|} \end{aligned} \quad (3.36)$$

for all $x \in X$ and $t > 0$. Consequently, $8\|f(x) - C(x)\| \leq \|x\|\epsilon$. □

Definition 3.4. Let $f : (X, N) \to (Y, N')$ be a mapping where (X, N) and (Y, N') are fuzzy normed spaces. f is said to be sequentially fuzzy continuous at $x \in X$ if for any $x_n \in X$ satisfying $x_n \to x$ implies $f(x_n) \to f(x)$. If f is sequentially fuzzy continuous at each point of X, then f is said to be sequentially fuzzy continuous on X.

For the various definitions of continuity and also defining a topology on a fuzzy normed space we refer the interested reader to [61, 62]. Now we examine some conditions under which the cubic mapping found in Theorem 3.1 to be continuous. In the following theorem, we investigate fuzzy sequentially continuity of cubic mappings in fuzzy normed spaces. Indeed, we will show that under some extra conditions on Theorem 3.1, the cubic mapping $r \mapsto Q(rx)$ is fuzzy sequentially continuous.

Theorem 3.5. *Denote N_1 the fuzzy norm obtained as Corollary 3.3 on $\mathbb{R}$. Suppose that conditions of Theorem 3.1 hold. If for every $x \in X$ the mappings $r \mapsto f(rx)$ (from $(\mathbb{R}, N_1)$ into (Y, N)) and $r \mapsto Q(0, rx)$ (from $(\mathbb{R}, N_1)$ into (Z, N')) are sequentially fuzzy continuous, then the mapping $r \mapsto C(rx)$ is sequentially continuous and $C(rx) = r^3 C(x)$ for all $r \in \mathbb{R}$.*

Proof. We have the following case.

Case 1 ($0 < \alpha < 27$). Let $\{r_n\}$ be a sequence in $\mathbb{R}$ that converges to some $r \in \mathbb{R}$, and let $t > 0$. Let $\epsilon > 0$ be given. Since $0 < \alpha < 27$,

$$\lim_{n\to\infty} \frac{(27-\alpha)27^n t}{18\alpha^{\mathrm{n}}} = \lim_{n\to\infty} \frac{8(27-\alpha)27^n t}{6\alpha^{n+1}} = \infty, \tag{3.37}$$

there is $m \in \mathbb{N}$ such that

$$\min\left\{ N'\left(Q(0,rx), \frac{(27-\alpha)27^m t}{18\alpha^m}\right), N'\left(Q(0,rx), \frac{8(27-\alpha)27^m t}{6\alpha^{m+1}}\right)\right\} > 1-\epsilon. \tag{3.38}$$

It follows form (3.17) and (3.38) that

$$N\left(\frac{f(3^m rx)}{27^m} - \frac{C(3^m rx)}{27^m}, \frac{t}{3}\right) > 1-\epsilon. \tag{3.39}$$

By the sequentially fuzzy continuity of maps $r \mapsto Q(0,rx)$ and $r \mapsto f(rx)$, we can find some $k \in \mathbb{N}$ such that for any $n \ge k$,

$$N\left(\frac{f(3^m r_n x)}{27^m} - \frac{f(3^m rx)}{27^m}, \frac{t}{3}\right) > 1-\epsilon, \tag{3.40}$$

and

$$\min\left\{ N'\left(Q(0,r_n x) - Q(0,rx), \frac{(27-\alpha)27^m t}{18\alpha^m}\right), N'\left(Q(0,r_n x) - Q(0,rx), \frac{8(27-\alpha)27^m t}{6\alpha^{m+1}}\right)\right\} > 1-\epsilon. \tag{3.41}$$

Hence by last inequality and (3.38), we get

$$\min\left\{ N'\left(Q(0,r_n x), \frac{(27-\alpha)27^m t}{9\alpha^m}\right), N'\left(Q(0,r_n x), \frac{8(27-\alpha)27^m t}{3\alpha^{m+1}}\right)\right\} > 1-\epsilon. \tag{3.42}$$

On the other hand,

$$\begin{aligned} N\left(C(r_n x) - \frac{f(3^m r_n x)}{27^m}, \frac{t}{27^m}\right) &= N\left(\frac{C(3^m r_n x)}{27^m} - \frac{f(3^m r_n x)}{27^m}, \frac{t}{27^m}\right) \\ &\ge \min\left\{ N'\left(Q(0,r_n x), \frac{(27-\alpha)t}{3\alpha^m}\right), N'\left(Q(0,r_n x), \frac{8(27-\alpha)t}{\alpha^{m+1}}\right)\right\}. \end{aligned} \tag{3.43}$$

Hence by last inequality and (3.42), we obtain

$$N\left(C(r_n x) - \frac{f(3^m r_n x)}{27^m}, \frac{t}{3}\right) > 1 - \epsilon. \tag{3.44}$$

Therefore it follows from (3.44), (3.40) and (3.39) that for every $n \geq k$,

$$N(C(r_n x) - C(rx), t) > 1 - \epsilon. \tag{3.45}$$

Therefore for every choice $x \in X$, $t > 0$ and $\epsilon > 0$, we can find some $k \in \mathbb{N}$ such that $N(C(r_n x) - C(rx), t) > 1 - \epsilon$ for every $n \geq k$. This shows that $C(r_n x) \to C(rx)$.

The proof for $p > 27$ proceeds similarly to that in the previous case.

It is not hard to see that $C(rx) = r^3 C(x)$ for every rational number r. Since C is a fuzzy sequentially continuous map, by the same reasoning as the proof of [46], the cubic function $C : X \to X$ satisfies $C(rx) = r^3 C(x)$ for every $r \in \mathbb{R}$. □

The following corollary is the Hyers-Ulam stability [7] of (1.5).

Corollary 3.6. *Let X be a Banach space, and let $\epsilon > 0$ be a real number. Suppose that a function $f : X \to X$ satisfies*

$$\|f(x+3y) - 3f(x+y) + 3f(x-y) - f(x-3y) - 48f(y)\| \leq \epsilon \tag{3.46}$$

for all $x, y \in X$. Then there exists a unique cubic function $C : X \to X$ which satisfies (1.5) *and the inequality*

$$\|C(x) - f(x)\| \leq \frac{3\epsilon}{26} \tag{3.47}$$

for all $x \in X$. Moreover, if for each fixed $x \in X$ the mapping $t \to f(tx)$ from $\mathbb{R}$ to X is fuzzy sequentially continuous, then $C(rx) = r^3 C(x)$ for all $r \in \mathbb{R}$.

Proof. Denote N and N' the fuzzy norms obtained as Corollary 3.3 on X and $\mathbb{R}$, respectively. This time we choose $Q(x,y) = \epsilon$. By Theorem 3.1, there exists a unique cubic function $C : X \to X$ which satisfies the inequality

$$N(f(x) - C(x), t) \geq \min\left\{N'\left(\epsilon, \frac{26t}{3}\right), N'(\epsilon, 8 \times 26t)\right\} = N'\left(\epsilon, \frac{26t}{3}\right) \tag{3.48}$$

for all $x \in X$. It follows that $\|f(x) - C(x)\| \leq 3\epsilon/26$. The rest of proof is an immediate consequence of Theorem 3.5. □

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